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On some difference sequence spaces defined by a sequence of Orlicz functions

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Abstract: The idea of difference sequence spaces was introduced in (Kızmaz, 1981) and this concept was generalized in (Et and Çolak, 1995). In this paper we define some difference sequence spaces by a sequence of Orlicz functions and establish some inclusion relations.

Key words: Difference sequence, Orlicz function, Sequence of Orlicz functions, Strongly almost convergent
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INTRODUCTION

Let w be the set of all sequences of real or complex numbers and ℓ_∞, c and c_0 be the sequence spaces of bounded, convergent and null sequences $x=(x_k)$, respectively.

A sequence $x\in\ell_\infty$ is said to be almost convergent (Lorentz, 1948) if all Banach limits of x coincide. Lorentz (1948) proved that

$$\hat{c}=\left\{x=(x_k):\lim_n \frac{1}{n}\sum_{k=1}^{n} x_{k+s} \text{ exists, uniformly in } s\right\}.$$

Maddox (1967; 1978) has defined x to be strongly almost convergent to a number L if

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}\left|x_{k+s}-L\right|=0, \text{ uniformly in } s.$$

Let $p=(p_k)$ be a sequence of strictly positive real numbers. Nanda (1984) defined

$$[\hat{c},p]=\left\{x=(x_k):\lim_n \frac{1}{n}\sum_{k=1}^{n}| x_{k+s}-L|^{p_k}=0, \text{ uniformly in } s\right\},$$

$$[\hat{c},p]_0=\left\{x=(x_k):\lim_n \frac{1}{n}\sum_{k=1}^{n}| x_{k+s}|^{p_k}=0, \text{ uniformly in } s\right\},$$

$$[\hat{c},p]_\infty=\left\{x=(x_k):\sup_{s,n} \frac{1}{n}\sum_{k=1}^{n}\left|x_{k+s}\right|^{p_k}<\infty\right\}.$$

Kızmaz (1981) defined the sequence spaces

$$X(\Delta)=\{x=(x_k):(\Delta x_k)\in X\}$$

for $X=\ell_\infty$, c or c_0, where $\Delta x=(\Delta x_k)=(x_k-x_{k+1})$.

After Et and Çolak (1995) generalized the above sequence spaces to the sequence spaces

$$X(\Delta^m)=\{x=(x_k):(\Delta^m x_k)\in X\}$$

for $X=\ell_\infty$, c or c_0, where $m\in\mathbb{N}$, $\Delta^0 x=(x_k)$, $\Delta x=(x_k-x_{k+1})$, $\Delta^m x=(\Delta^m x_k)=(\Delta^{m-1}x_k-\Delta^{m-1}x_{k+1})$ and so that

$$\Delta^m x_k=\sum_{v=0}^{m}(-1)^v\binom{m}{v}x_{k+v}.$$

An Orlicz function is a function M: $[0,\infty)\rightarrow[0,\infty)$ which is continuous, non-decreasing and convex with $M(0)=0$, $M(x)>0$ for $x>0$ and $M(x)\rightarrow\infty$ as

$x \to \infty$.

It is well known that if M is a convex function and $M(0)=0$, then $M(\lambda x) \le \lambda M(x)$ for all λ with $0<\lambda<1$.

Lindenstrauss and Tzafriri (1971) used the idea of Orlicz function to construct the sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M(|x_k|/\rho) < \infty, \text{ for some } \rho>0 \right\}.$$

The space ℓ_M with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M(|x_k|/\rho) \le 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(x)=x^p$ for $1 \le p < \infty$.

Let M be an Orlicz function and $p=(p_k)$ be any sequence of strictly positive real numbers. Güngör and Et (2003) defined the following sequence spaces

$$[\hat{c}, M, p](\Delta^m) = \left\{ x=(x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} [M(|\Delta^m x_{k+s} - L|/\rho)]^{p_k} = 0, \text{ uniformly in } s, \text{ for some } \rho>0 \text{ and } L>0 \right\},$$

$$[\hat{c}, M, p]_0(\Delta^m) = \left\{ x=(x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} [M(|\Delta^m x_{k+s}|/\rho)]^{p_k} = 0, \text{ uniformly in } s, \text{ for some } \rho>0 \right\},$$

$$[\hat{c}, M, p]_\infty(\Delta^m) = \left\{ x=(x_k) : \sup_{s,n} \frac{1}{n} \sum_{k=1}^{n} [M(|\Delta^m x_{k+s}|/\rho)]^{p_k} < \infty, \text{ for some } \rho > 0 \right\},$$

Let $M=(M_k)$ be a sequence of Orlicz functions. Mursaleen *et al.*(2001) defined the following sequence spaces

$$\ell_\infty(M,\Delta) = \left\{ x=(x_k) : \sup_k (M_k(|\Delta x_k|/\rho)) < \infty \right\},$$

$$c_0(M,\Delta) = \left\{ x=(x_k) : (M_k(|\Delta x_k|/\rho)) \to 0, k \to \infty \right\}.$$

SOME NEW SEQUENCE SPACES DEFINED BY A SEQUENCE OF ORLICZ FUNCTIONS

Definition 1 Let $M=(M_k)$ be a sequence of Orlicz functions and $p=(p_k)$ be any sequence of strictly positive real numbers. We define the following sequence sets

$$[\hat{c}, M, p](\Delta^m) = \left\{ x=(x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} [M_k(|\Delta^m x_{k+s} - L|/\rho)]^{p_k} = 0, \text{ uniformly in } s, \text{ for some } \rho>0 \text{ and } L>0 \right\},$$

$$[\hat{c}, M, p]_0(\Delta^m) = \left\{ x=(x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} [M(|\Delta^m x_{k+s}|/\rho)]^{p_k} = 0, \text{ uniformly in } s, \text{ for some } \rho>0 \right\},$$

$$[\hat{c}, M, p]_\infty(\Delta^m) = \left\{ x=(x_k) : \sup_{s,n} \frac{1}{n} \sum_{k=1}^{n} [M(|\Delta^m x_{k+s}|/\rho)]^{p_k} < \infty, \text{ for some } \rho>0 \right\}.$$

If $M_k(x)=x$ for every k, then $[\hat{c}, M, p](\Delta^m)=[\hat{c}, p](\Delta^m)$, $[\hat{c}, M, p]_0(\Delta^m)=[\hat{c}, p]_0(\Delta^m)$ and $[\hat{c}, M, p]_\infty(\Delta^m)=[\hat{c}, p]_\infty(\Delta^m)$. We denote $[\hat{c}, M, p](\Delta^m)$, $[\hat{c}, M, p]_0(\Delta^m)$ and $[\hat{c}, M, p]_\infty(\Delta^m)$ by $[\hat{c}, M](\Delta^m)$, $[\hat{c}, M]_0(\Delta^m)$ and $[\hat{c}, M]_\infty(\Delta^m)$, respectively, when $p_k=1$ for all k.

Theorem 1 Let $M=(M_k)$ be a sequence of Orlicz functions. Then the following statements are equivalent:

(i) $[\hat{c}, p]_\infty(\Delta^m) \subseteq [\hat{c}, M, p]_\infty(\Delta^m)$;

(ii) $[\hat{c}, p]_0(\Delta^m) \subseteq [\hat{c}, M, p]_\infty(\Delta^m)$;

(iii) $\sup_n \frac{1}{n} \sum_{k=1}^{n} [M_k(t/\rho)]^{p_k} < \infty$ $(t, \rho>0)$.

Proof (i)⇒(ii) is obvious, since $[\hat{c}, p]_0(\Delta^m) \subseteq [\hat{c}, p]_\infty(\Delta^m)$.

(ii)⇒(iii). Let $[\hat{c},p]_0(\Delta^m)\subseteq[\hat{c}, M, p]_\infty(\Delta^m)$. Suppose that (iii) is not satisfied. Then for some t, $\rho>0$

$$\sup_n \frac{1}{n}\sum_{k=1}^{n}[M_k(t/\rho)]^{p_k}=\infty,$$

and therefore there is sequence (n_i) of positive integers such that

$$\frac{1}{n_i}\sum_{k=1}^{n_i}[M_k(i^{-1}/\rho)]^{p_k}>i,\ i=1,\ 2,\ ... \tag{1}$$

Define $x=(x_k)$ by

$$x_k=\begin{cases} i^{-1}, & 1\le k\le n_i,\ i=1,\ 2,\ ...;\\ 0, & k>n_i.\end{cases}$$

Then $x\in[\hat{c},p]_0(\Delta^m)$, but by Eq.(1), $x\notin[\hat{c}, M, p]_\infty(\Delta^m)$ which contradicts (ii). Hence (iii) must hold.

(iii)⇒(i). Let (iii) be satisfied and $x\in[\hat{c},p]_\infty(\Delta^m)$. Suppose that $x\notin[\hat{c}, M, p]_\infty(\Delta^m)$. Then

$$\sup_{s,n} \frac{1}{n}\sum_{k=1}^{n}[M_k(|\Delta^m x_{k+s}|/\rho)]^{p_k}=\infty. \tag{2}$$

Let $t=|\Delta^m x_{k+s}|$ for each k and fixed s, then by Eq.(2)

$$\sup_n \frac{1}{n}\sum_{k=1}^{n}[M_k(t/\rho)]^{p_k}=\infty,$$

which contradicts (iii). Hence (i) must hold.

Theorem 2 Let $1\le p_k\le\sup_k p_k<\infty$. Then the following statements are equivalent for a sequence of Orlicz functions $M=(M_k)$:

(I) $[\hat{c}, M, p]_0(\Delta^m)\subseteq[\hat{c},p]_0(\Delta^m)$;

(II) $[\hat{c}, M, p]_0(\Delta^m)\subseteq[\hat{c},p]_\infty(\Delta^m)$;

(II) $\inf_n \frac{1}{n}\sum_{k=1}^{n}[M_k(t/\rho)]^{p_k}>0$ (t, $\rho>0$).

Proof (I)⇒(II) is obvious.

(II)⇒(III). Let $[\hat{c}, M, p]_0(\Delta^m)\subseteq[\hat{c}, p]_\infty(\Delta^m)$. Suppose that (III) does not hold. Then

$$\inf_n \frac{1}{n}\sum_{k=1}^{n}[M_k(t/\rho)]^{p_k}=0\ (t,\ \rho>0). \tag{3}$$

We can choose an index sequence (n_i) such that

$$\frac{1}{n_i}\sum_{k=1}^{n_i}[M_k(i/\rho)]^{p_k}<i^{-1},\ i=1,\ 2,\ ...$$

Define the sequence $x=(x_k)$ by

$$x_k=\begin{cases} i, & 1\le k\le n_i,\ i=1,\ 2,\ ...;\\ 0, & k>n_i.\end{cases}$$

Thus by Eq.(3), $x\in[\hat{c}, M, p]_0(\Delta^m)$ but $x\notin[\hat{c},p]_\infty(\Delta^m)$ which contradicts (II). Hence (III) must hold.

(III)⇒(I). Let (III) hold and $x\in[\hat{c}, M, p]_0(\Delta^m)$, i.e.,

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}[M_k(|\Delta^m x_{k+s}|/\rho)]^{p_k}=0, \text{uniformly in } s. \tag{4}$$

Suppose that $x\notin[\hat{c}, p]_0(\Delta^m)$. Then for some number $\varepsilon_0>0$ and index n_0, we have

$$|\Delta^m x_{k+s}|\ge\varepsilon_0, \text{ for some } s\ge s' \text{ and } 1\le k\le n_0.$$

Therefore

$$[M_k(\varepsilon_0/\rho)]^{p_k}\le[M_k(|\Delta^m x_{k+s}|/\rho)]^{p_k}$$

and consequently by Eq.(4)

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}[M_k(\varepsilon_0/\rho)]^{p_k}=0,$$

which contradicts (III). Hence

$$[\hat{c}, M, p]_0(\Delta^m)\subseteq[\hat{c}, p]_0(\Delta^m).$$

Theorem 3 Let $1\le p_k\le\sup_k p_k<\infty$. The inclusion $[\hat{c}, M, p]_\infty(\Delta^m)\subseteq[\hat{c},p]_0(\Delta^m)$ hold if

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}[M_k(t/\rho)]^{p_k}=\infty\ (t,\ \rho>0). \tag{5}$$

Proof Let $[\hat{c}, M, p]_\infty(\Delta^m) \subseteq [\hat{c}, p]_0(\Delta^m)$. Suppose that Eq.(5) does not satisfied. Therefore there is a number $t_0>0$ and an index sequence (n_i) such that

$$\frac{1}{n_i}\sum_{k=1}^{n_i}[M_k(t_0/\rho)]^{p_k} \le N < \infty, \quad i=1, 2, \ldots \tag{6}$$

Define the sequence $x=(x_k)$ by

$$x_k = \begin{cases} t_0, \ 1 \le k \le n_i, \ i=1, 2, \ldots; \\ 0, \ k > n_i. \end{cases}$$

Thus by Eq.(6), $x \in [\hat{c}, M, p]_\infty(\Delta^m)$, but $x \notin [\hat{c}, p]_0(\Delta^m)$. Hence Eq.(5) must hold.

Conversely, let Eq.(5) be satisfied. If $x \in [\hat{c}, M, p]_\infty(\Delta^m)$, then for each s and n

$$\frac{1}{n}\sum_{k=1}^{n}[M_k(|\Delta^m x_{k+s}|/\rho)]^{p_k} \le N < \infty. \tag{7}$$

Suppose that $x \notin [\hat{c}, p]_0(\Delta^m)$. Then for some number $\varepsilon_0>0$ there is a number s_0 and index n_0

$$|\Delta^m x_{k+s}| \ge \varepsilon_0, \text{ for } s \ge s_0.$$

Therefore

$$[M_k(\varepsilon_0/\rho)]^{p_k} \le [M_k(|\Delta^m x_{k+s}|/\rho)]^{p_k},$$

and hence for each k and s we get

$$\frac{1}{n}\sum_{k=1}^{n}[M_k(\varepsilon_0/\rho)]^{p_k} \le N < \infty,$$

for some $N>0$, by Eq.(7) which contradicts Eq.(5). Hence $[\hat{c}, M, p]_\infty(\Delta^m) \subseteq [\hat{c}, p]_0(\Delta^m)$.

Theorem 4 Let $1 \le p_k \le \sup_k p_k < \infty$. Then the inclusion $[\hat{c}, p]_\infty(\Delta^m) \subseteq [\hat{c}, M, p]_0(\Delta^m)$ hold if

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}[M_k(t_0/\rho)]^{p_k} = 0 \ \ (t, \rho>0). \tag{8}$$

Proof Let $[\hat{c}, p]_\infty(\Delta^m) \subseteq [\hat{c}, M, p]_0(\Delta^m)$. Suppose that Eq.(8) does not hold. Then for some $t_0>0$,

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}[M_k(t_0/\rho)]^{p_k} = L \ne 0. \tag{9}$$

Define $x=(x_k)$ by $x_k = t_0\sum_{v=0}^{k-m}(-1)^m \binom{m+k-v-1}{k-v}$ for $k=1, 2, \ldots$ Thus $x \notin [\hat{c}, M, p]_0(\Delta^m)$ by Eq.(9), but $x \in [\hat{c}, p]_\infty(\Delta^m)$. Hence Eq.(8) must hold.

Conversely, suppose that Eq.(8) hold and $x \in [\hat{c}, p]_\infty(\Delta^m)$. Then for every k and s

$$|\Delta^m x_{k+s}| \le N < \infty.$$

Therefore

$$[M_k(|\Delta^m x_{k+s}|/\rho)]^{p_k} \le [M_k(N/\rho)]^{p_k},$$

and

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}[M_k(|\Delta^m x_{k+s}|/\rho)]^{p_k} \le \lim_n \frac{1}{n}\sum_{k=1}^{n}[M_k(N/\rho)]^{p_k} = 0$$

by Eq.(8). Hence $x \in [\hat{c}, M, p]_0(\Delta^m)$.

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