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Projectively flat arctangent Finsler metric*

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Abstract: In this work, we study a class of special Finsler metrics F called arctangent Finsler metric, which is a special (α,β)-metric, where α is a Riemannian metric and β is a 1-form. We obtain a sufficient and necessary condition that F is locally projectively flat if and only if α and β satisfy two special equations. Furthermore we give the non-trivial solutions for F to be locally projectively flat. Moreover, we prove that such projectively flat Finsler metrics with constant flag curvature must be locally Minkowskian.

Key words: Arctangent Finsler metric, Projectively flat, (α,β)-metric, Flag curvature
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INTRODUCTION

In recent years, significant progress has been made in the study of Finsler metris which have straight lines in local coordinates (Hamel, 1903). Finsler metrics with this property are called locally projectively flat metrics. An important fact about these metrics is that locally projectively flat Finsler metrics have scalar flag curvature, a natural extension of the sectional curvature in Riemannian geometry. In general, it also depends on the direction (flag pole) in the section (flag), from this it gets the name "flag curvature". It is one of the fundamental problems in Riemann-Finsler geometry to study and characterize Finsler metrics of scalar (or constant) flag curvature. Unlike the Riemannian case, the local metric structure of Finsler metrics of scalar flag curvature is far from being understood. There is an important class of Finsler metrics defined in terms of a Riemannian metric $\alpha=\sqrt{a_{ij}y^iy^j}$ and a 1-form $\beta=b_iy^i$ (thus called (α,β)-metric). It is relatively easy to carry out the computation on the geometric quantities of (α,β)-metric. Thus one would like to investigate locally projectively flat (α,β)-metric.

Randers metrics are the simplest non-Riemannian (α,β)-metric. It has been proved that a Randers metric $F=\alpha+\beta$ is locally projectively flat if and only if $d\beta=0$ and α is locally projectively flat (or equivalently, α is of constant sectional curvature by the Beltrami theorem). Moreover, the local structure of projectively flat Randers metrics of constant flag curvature can be completely determined. See (Shen, 2003) for more details and related references (Bao and Robles, 2003; Bryant, 2002).

Shen and Civi Yildirim (2005) studied a class of special (α,β)-metrics $F=\alpha\phi(s)$, $s=\beta/\alpha$, where $\phi=\phi(s)$ satisfies

$$\phi(s)-s\phi'(s)=(p+rs^2)\phi'(s), \qquad (1)$$

where p and r are constants. They found a sufficient condition for $F=\alpha\phi(\beta/\alpha)$ to be projectively flat in a local coordinate system (x^i), that is, the covariant derivatives $b_{i|j}$ of $\beta=b_iy^i$ with respect to $\alpha=\sqrt{a_{ij}y^iy^j}$ and the spray coefficients of G^i_α of α satisfy

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$$b_{i|j}=2\tau\{(p+b^2)a_{ij}+(r-1)b_ib_j\},\quad (2)$$
$$G^i_\alpha=\theta y^i-\tau\alpha^2b^i,\quad (3)$$

where $b=\sqrt{a_{ij}(x)b^i(x)b^j(x)}$, $\tau=\tau(x)$ is a scalar function and $\theta=t_i(x)y^i$ is a 1-form. A natural question arises: Are these conditions necessary?

In this work, we study the above problem for the class of (α,β)-metrics whose defining function $\phi=\phi(s)$ satisfies Eq.(1) with p=1/2 and r=1/2. In this case, ϕ= 1+εs+$s$arctan(s), where ε is an arbitrary constant. The corresponding metric is expressed in the following form

$$F=\alpha+\varepsilon\beta+\beta\arctan(\beta/\alpha).\quad (4)$$

We call this metric an arctangent metric. We can show that the above conditions are also necessary for arctangent metrics to be projectively flat. More precisely, we prove the following:

Theorem 1 Let $F=\alpha+\varepsilon\beta+\beta\arctan(\beta/\alpha)$ be an arctangent metric on an open subet $\boldsymbol{U}$ in $\mathbb{R}^n$, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ and $\beta=b_i(x)y^i$. F is locally projectively flat in $\boldsymbol{U}$ if and only if α and β satisfy Eq.(2) and Eq.(3) with $p=r=1/2$, i.e.,

$$b_{i|j}=\tau\{(1+2b^2)a_{ij}-b_ib_j\},\quad (5)$$
$$G^i_\alpha=\theta y^i-\tau\alpha^2b^i,\quad (6)$$

where $b=\sqrt{a_{ij}(x)b^i(x)b^j(x)}$, $\tau=\tau(x)$ is a scalar function and $\theta=t_i(x)y^i$ is a 1-form.

We also determine the metric structure of locally projectively arctangent metrics with constant flag curvature. They must be locally Minkowskian (see Proposition 1 below).

(α,β)-METRICS

Finsler metric under our consideration is special (α,β)-metric, expressed in the following form:

$$F=\alpha\phi(s),\ s=\beta/\alpha,\quad (7)$$

where $\alpha=\sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta=b_iy^i$ is a 1-form. $\phi=\phi(s)$ is a C^∞ positive function on an open interval $(-b_o, b_o)$ satisfying

$$\phi(s)-s\phi'(s)+(b^2-s^2)\phi''(s)>0,\ |s|\le b\le b_o.\quad (8)$$

It is known that F is a Finsler metric if and only if $\|\beta_x\|_\alpha<b_o$ for any $x\in\boldsymbol{M}$. Let G^i and G^i_α denote the spray coefficients of F and α, respectively, given by

$$G^i=\frac{g^{ij}}{4}\{[F^2]_{x^ky^j}y^k-[F^2]_{x^j}\},$$
$$G^i_\alpha=\frac{a^{ij}}{4}\{[\alpha^2]_{x^ky^j}y^k-[\alpha^2]_{x^j}\},\quad (9)$$

where $(g^{ij}):=(g_{ij})^{-1}$, $(g_{ij}):=([F^2]_{y^iy^j}/2)$ and $(a^{ij}):=(a_{ij})^{-1}$.

We have the following:

Lemma 1 (Chern and Shen, 2005; Shen, 2004) The geodesic coefficients G^i are related to G^i_α by

$$G^i=G^i_\alpha+\alpha Qs^i_o+J\{-2Q\alpha s_o+r_{oo}\}\frac{y^i}{\alpha}$$
$$+H\{-2Q\alpha s_o+r_{oo}\}\left\{b^i-s\frac{y^i}{\alpha}\right\},\quad (10)$$

where

$$Q:=\frac{\phi'}{\phi-s\phi'},$$
$$J:=\frac{\phi'(\phi-s\phi')}{2\phi((\phi-s\phi')+(b^2-s^2)\phi'')},$$
$$H:=\frac{\phi''}{2((\phi-s\phi')+(b^2-s^2)\phi'')},$$
$$s=\beta/\alpha,\quad b:=\|\beta_x\|_\alpha.$$

Lemma 2 (Shen and Civi Yildirim, 2005) An (α,β)-metric $F=\alpha\phi(s)$, where $s=\beta/\alpha$, is projectively flat on an open subset $\boldsymbol{U}\subset\mathbb{R}^n$ if and only if

$$(a_{ml}\alpha^2-y_my_l)G^m_\alpha+\alpha^3Qs_{lo}$$
$$+H\alpha\{-2Q\alpha s_o+r_{oo}\}\{b_l\alpha-sy_l\}=0,\quad (11)$$

where $y_m=a_{mi}y^i$.

ARCTANGENT FINSLER METRIC

In this section, we consider a special (α,β)-metric in the following form:

$$F=\alpha+\varepsilon\beta+\beta\arctan(\beta/\alpha), \quad (12)$$

where $\alpha=\sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta=b_iy^i$ is a 1-form on $\boldsymbol{M}$, ε is a constant. Let $b_o>0$ be the largest number such that

$$\frac{1-s^2+2b^2}{(1+s^2)^2}>0,\ \ |s|\le b<b_o, \quad (13)$$

where b_0 depends on ε such that $1+\varepsilon s+s\arctan(s)>0$.

Lemma 3 $F=\alpha+\varepsilon\beta+\beta\arctan(\beta/\alpha)$ is a Finsler metric.

Proof If $F=\alpha+\varepsilon\beta+\beta\arctan(\beta/\alpha)$ is a Finsler metric, then

$$\frac{1-s^2+2b^2}{(1+s^2)^2}>0,\ \ |s|\le b<b_o.$$

Let $s=b$, then $\forall b<b_o$, $1+b^2>0$.

By Lemma 1,

$$Q:=\frac{[\varepsilon+\arctan(\beta/\alpha)](\alpha^2+\beta^2)+\alpha\beta}{\alpha^2},$$
$$J:=\frac{\alpha\{[\varepsilon+\arctan(\beta/\alpha)](\alpha^2+\beta^2)+\alpha\beta\}}{2[(1+2b^2)\alpha^2-\beta^2]\{\alpha+[\varepsilon+\arctan(\beta/\alpha)]\beta\}},$$
$$H:=\frac{\alpha^2}{(1+2b^2)\alpha^2-\beta^2}.$$

Eq.(11) is reduced to the following equation:

$$(a_{ml}\alpha^2-y_my_l)G^m_\alpha+\alpha\{[\varepsilon+\arctan(\beta/\alpha)](\alpha^2+\beta^2)+\alpha\beta\}s_{lo}+\left\{-2\alpha^2\frac{[\varepsilon+\arctan(\beta/\alpha)](\alpha^2+\beta^2)+\alpha\beta}{(1+2b^2)\alpha^2-\beta^2}s_o+\frac{\alpha^2 r_{oo}}{(1+2b^2)\alpha^2-\beta^2}\right\}(b_l\alpha-sy_l)=0. \quad (14)$$

Theorem 2 Let $F=\alpha+\varepsilon\beta+\beta\arctan(\beta/\alpha)$ be a Finsler metric on a manifold $\boldsymbol{M}$. F is locally projectively flat if and only if

(a) $b_{i|j}=\tau\{(1+2b^2)a_{ij}-b_ib_j\}$,

(b) $G^i_\alpha=\theta y^i-\tau\alpha^2b^i$,

where $\tau=\tau(x)$ and $\theta=t_i(x)y^i$. In this case,

$$G^i=(\theta+\tau\chi\alpha)y^i, \quad (15)$$

where

$$\chi=\frac{\varepsilon+\arctan(s)}{2\{1+[\varepsilon+\arctan(s)]s\}},\ s=\beta/\alpha.$$

Proof If F is projectively flat, we rewrite Eq.(14) as a polynomial in y^i and α, which is linear in α. This gives

$$[(1+2b^2)\alpha^2-\beta^2](a_{ml}\alpha^2-y_my_l)G^m_\alpha+\alpha[(1+2b^2)\alpha^2-\beta^2]\{[\varepsilon+\arctan(\beta/\alpha)](\alpha^2+\beta^2)+\alpha\beta\}s_{lo}-2\alpha^2\{[\varepsilon+\arctan(\beta/\alpha)](\alpha^2+\beta^2)+\alpha\beta\}(b_l\alpha-sy_l)s_o+\alpha^3r_{oo}(b_l\alpha-sy_l)=0. \quad (16)$$

Case 1 Assume that $\varepsilon\neq0$. Replace y with $-y$, to get

$$[(1+2b^2)\alpha^2-\beta^2](a_{ml}\alpha^2-y_my_l)G^m_\alpha-\alpha[(1+2b^2)\alpha^2-\beta^2]\{[\varepsilon-\arctan(\beta/\alpha)](\alpha^2+\beta^2)-\alpha\beta\}s_{lo}+2\alpha^2\{[\varepsilon-\arctan(\beta/\alpha)](\alpha^2+\beta^2)-\alpha\beta\}(b_l\alpha-sy_l)s_o+\alpha^3r_{oo}(b_l\alpha-sy_l)=0. \quad (17)$$

From Eqs.(16) and (17), we get

$$2\varepsilon\alpha[(1+2b^2)\alpha^2-\beta^2](\alpha^2+\beta^2)s_{lo}-4\varepsilon\alpha^2(\alpha^2+\beta^2)(b_l\alpha-sy_l)s_o=0, \quad (18)$$

namely

$$[(1+2b^2)\alpha^2-\beta^2]s_{lo}-2s_o(b_l\alpha^2-\beta y_l)=0. \quad (19)$$

Contracting Eq.(19) with b^l yields

$$(\alpha^2+\beta^2)s_o=0, \quad (20)$$

we get $s_o=0$. Then it follows from Eq.(19) that

$$s_{lo}=0. \quad (21)$$

Thus β is closed.

Case 2 ε=0, Eq.(16) becomes

$$[(1+2b^2)\alpha^2-\beta^2](a_{ml}\alpha^2-y_my_l)G_\alpha^m+\alpha[(1+2b^2)\alpha^2 -\beta^2][(\alpha^2+\beta^2)\arctan(\beta/\alpha)+\alpha\beta]s_{lo} -2\alpha^2[(\alpha^2+\beta^2)\arctan(\beta/\alpha)+\alpha\beta](b_l\alpha-sy_l)s_o +\alpha^3r_{oo}(b_l\alpha-sy_l)=0. \quad (22)$$

Contracting Eq.(22) with b^l yields

$$[(1+2b^2)\alpha^2-\beta^2](b_m\alpha^2-\beta y_m)G_\alpha^m +\alpha^2r_{oo}(b^2\alpha^2-\beta^2)-\alpha^2\beta(\alpha^2+\beta^2)s_o =\alpha(\alpha^2+\beta^2)^2s_o\arctan(\beta/\alpha). \quad (23)$$

Using Taylor expansion of arctan(β/α), we can find that the left side of Eq.(23) is an integral expression in y and the right side of Eq.(23) is a fractional expression in y, we get s_o=0.

Then it follows from Eq.(22) that

$$s_{lo}=0. \quad (24)$$

Thus β is closed.

By Case 1 and Case 2, we get β is closed.

Now Eq.(16) is reduced to the following

$$[(1+2b^2)\alpha^2-\beta^2](a_{ml}\alpha^2-y_my_l)G_\alpha^m =-\alpha^2r_{oo}(b_l\alpha^2-\beta y_l). \quad (25)$$

Contracting Eq.(25) with b^l yields

$$[(1+2b^2)\alpha^2-\beta^2](b_m\alpha^2-\beta y_m)G_\alpha^m =-\alpha^2r_{oo}(b^2\alpha^2-\beta^2). \quad (26)$$

Note that the polynomial $(1+2b^2)\alpha^2-\beta^2$ is not divisible by α^2 and $b^2\alpha^2-\beta^2$, thus $(b_m\alpha^2-y_m\beta)G_\alpha^m$ is divisible by $\alpha^2r_{oo}(b^2\alpha^2-\beta^2)$. Therefore, there is a scalar function $\tau=\tau(x)$ such that

$$r_{oo}=\tau[(1+2b^2)\alpha^2-\beta^2]. \quad (27)$$

By Eqs.(24) and (27), Eq.(10) for G^i can be simplified to

$$G^i=G_\alpha^i+\tau\chi\alpha y^i+\tau\alpha^2b^i, \quad (28)$$

where

$$\chi=\frac{\varepsilon+\arctan(s)}{2\{1+[\varepsilon+\arctan(s)]s\}},\ s=\beta/\alpha.$$

We know that F is projectively flat if and only if

$$G^i=Py^i.$$

By Eq.(28), this is equivalent to the following

$$G_\alpha^i=\theta y^i-\tau\alpha^2b^i,$$

where $\theta=t_i(x)y^i$ is a 1-form. This proves Theorem 2.

FLAG CURVATURE

In this section, we shall study the following metric with constant flag curvature $K=\lambda$,

$$F=\alpha+\varepsilon\beta+\beta\arctan(\beta/\alpha),$$

where ε is constant. We assume F is locally projectively flat so that in a local coordinate system the spray coefficients of F are in the form Eq.(15). It is known that if the spray coefficients of F are in the form $G^i=Py^i$, then F has scalar curvature with flag curvature

$$K=(P^2-P_{x^k}y^k)/F^2.$$

Then

$$K=[\theta^2-\theta_{x^k}y^k+\tau^2\chi^2\alpha^2-\chi\tau_o\alpha -\tau^2(s^2+1)\chi'\alpha^2+2s\tau^2\chi\alpha^2]/F^2. \quad (29)$$

Observe that

$$s_{x^k}y^k=\tau(s^2+1)\alpha,\ \alpha_{x^k}y^k=2(\theta-\tau\beta)\alpha.$$

Lemma 4 Suppose that $F=\alpha+\varepsilon\beta+\beta\arctan(\beta/\alpha)$ is projectively flat with constant flag curvature $K=\lambda$, then λ=0.

Proof First by Eq.(29), the $K=\lambda$ multiplied by $4F^4$ becomes

$$(\theta^2-\theta_{x^k}y^k)[8\varepsilon\alpha+8\alpha\arctan(\beta/\alpha)$$
$$+4\beta\arctan^2(\beta/\alpha)]\beta+6\tau^2\alpha^4\varepsilon\arctan(\beta/\alpha)$$
$$-2\varepsilon^2\tau_o\alpha^2\beta-2\tau_o\alpha^2\beta\arctan^2(\beta/\alpha)+6\varepsilon^2\tau^2\alpha^2\beta^2$$
$$+6\tau^2\alpha^2\beta^2\arctan^2(\beta/\alpha)+4\tau^2\alpha^3\beta\varepsilon$$
$$+4\tau^2\alpha^3\beta\arctan(\beta/\alpha)+4(\theta^2-\theta_{x^k}y^k)\varepsilon^2\beta^2$$
$$+3\tau^2\alpha^4\varepsilon^2+3\tau^2\alpha^4\arctan^2(\beta/\alpha)-2\tau_o\alpha^3\varepsilon$$
$$-2\tau_o\alpha^3\arctan(\beta/\alpha)+4(\theta^2-\theta_{x^k}y^k)\alpha^2-2\tau^2\alpha^4$$
$$+8(\theta^2-\theta_{x^k}y^k)\varepsilon\beta^2\arctan(\beta/\alpha)$$
$$-4\tau_o\alpha^2\beta\varepsilon\arctan(\beta/\alpha)+12\tau^2\alpha^2\beta^2\varepsilon\arctan(\beta/\alpha)$$
$$=4\lambda[\alpha+\varepsilon\beta+\beta\arctan(\beta/\alpha)]^4. \quad (30)$$

Replacing y with $-y$, we get

$$-(\theta^2-\theta_{x^k}y^k)[8\varepsilon\alpha+8\alpha\arctan(\beta/\alpha)$$
$$-4\beta\arctan^2(\beta/\alpha)]\beta-6\tau^2\alpha^4\varepsilon\arctan(\beta/\alpha)$$
$$-2\varepsilon^2\tau_o\alpha^2\beta+2\tau_o\alpha^2\beta\arctan^2(\beta/\alpha)+6\varepsilon^2\tau^2\alpha^2\beta^2$$
$$+6\tau^2\alpha^2\beta^2\arctan^2(\beta/\alpha)-4\tau^2\alpha^3\beta\varepsilon$$
$$+4\tau^2\alpha^3\beta\arctan(\beta/\alpha)+4(\theta^2-\theta_{x^k}y^k)\varepsilon^2\beta^2+3\tau^2\alpha^4\varepsilon^2$$
$$+3\tau^2\alpha^4\arctan^2(\beta/\alpha)+2\tau_o\alpha^3\varepsilon-2\tau_o\alpha^3\arctan(\beta/\alpha)$$
$$+4(\theta^2-\theta_{x^k}y^k)\alpha^2-8(\theta^2-\theta_{x^k}y^k)\varepsilon\beta^2\arctan(\beta/\alpha)$$
$$-2\tau^2\alpha^4+4\tau_o\alpha^2\beta\varepsilon\arctan(\beta/\alpha)$$
$$-12\tau^2\alpha^2\beta^2\varepsilon\arctan(\beta/\alpha)$$
$$=4\lambda[\alpha-\varepsilon\beta+\beta\arctan(\beta/\alpha)]^4. \quad (31)$$

From Eq.(30)+Eq.(31), we get

$$8\lambda(\beta^4/\alpha)\arctan^3(\beta/\alpha)+24\lambda\beta^3\arctan^2(\beta/\alpha)$$
$$+[6\tau^2\alpha\beta^2+3\tau^2\alpha^3-24\lambda\alpha\beta^2+4(\theta^2-\theta_{x^k}y^k)\beta^2/\alpha$$
$$-2\tau_o\alpha\beta-8\lambda(\beta^4/\alpha)\varepsilon^2]\arctan(\beta/\alpha)$$
$$=4(\theta^2-\theta_{x^k}y^k)\beta+2\tau\alpha^2\beta-\tau_o\alpha^2-8\lambda\alpha^2\beta-8\lambda\beta^3\varepsilon^2. \quad (32)$$

Using Taylor expansion of arctan(β/α), we can find that the right side of Eq.(32) is an integral expression in y and the left side of Eq.(32) is a fractional expression in y, so that we get

$$8\lambda(\beta^4/\alpha)\arctan^3(\beta/\alpha)+24\lambda\beta^3\arctan^2(\beta/\alpha)$$
$$+[6\tau^2\alpha\beta^2+3\tau^2\alpha^3-24\lambda\alpha\beta^2+4(\theta^2-\theta_{x^k}y^k)\beta^2/\alpha$$
$$-2\tau_o\alpha\beta-8\lambda(\beta^4/\alpha)\varepsilon^2]\arctan(\beta/\alpha)=0. \quad (33)$$

From Eq.(33)/arctan(β/α), we get

$$8\lambda(\beta^4/\alpha)\arctan^2(\beta/\alpha)+24\lambda\beta^3\arctan(\beta/\alpha)$$
$$=-6\tau^2\alpha\beta^2-3\tau^2\alpha^3+24\lambda\alpha\beta^2-4(\theta^2-\theta_{x^k}y^k)\beta^2/\alpha$$
$$+2\tau_o\alpha\beta+8\lambda(\beta^4/\alpha)\varepsilon^2. \quad (34)$$

For the same reason,

$$\lambda(\beta/\alpha)\arctan(\beta/\alpha)=-3\lambda,$$

thus λ=0.

Proposition 1 Let $F=\alpha+\varepsilon\beta+\beta\arctan(\beta/\alpha)$ be projectively flat with zero flag curvature, then α is flat metric and β is parallel. In this case, F is locally Minkowshian.

Proof Under the assumption that K=0, use the same method as Lemma 4, we obtain

$$[3\tau^2\alpha^4/2+3\tau^2\alpha^2\beta^2+2(\theta^2-\theta_{x^k}y^k)\beta^2-\tau_o\alpha^2\beta]$$
$$\cdot\arctan^2(\beta/\alpha)+[2\tau^2\alpha^3\beta-\tau_o\alpha^3+4(\theta^2-\theta_{x^k}y^k)\alpha\beta]$$
$$\cdot\arctan(\beta/\alpha)+3\tau^2\alpha^4\varepsilon^2/2+2(\theta^2-\theta_{x^k}y^k)(\varepsilon^2\beta^2+\alpha^2)$$
$$-\varepsilon^2\tau^2\alpha^4-\tau_o\alpha^2\beta+3\varepsilon^2\tau^2\alpha^2\beta^2=0.$$

Using Taylor expansion of arctan(β/α), we get

$$3\tau^2\alpha^4/2+3\tau^2\alpha^2\beta^2-\tau_o\alpha^2\beta+2(\theta^2-\theta_{x^k}y^k)\beta^2=0, \quad (35)$$

$$2\tau^2\alpha^3\beta-\tau_o\alpha^3+4(\theta^2-\theta_{x^k}y^k)\alpha\beta=0. \quad (36)$$

From Eq.(35)×(α/β)−Eq.(36), we get

$$\tau^2(3\alpha^2/2+\beta^2)\alpha^2/\beta=2(\theta^2-\theta_{x^k}y^k)\beta. \quad (37)$$

From Eq.(37)×β, we get

$$\tau^2(3\alpha^2/2+\beta^2)\alpha^2=2(\theta^2-\theta_{x^k}y^k)\beta^2. \quad (38)$$

Note that β^2 is not divisible by $3\alpha^2/2+\beta^2$ and α^2. Thus τ=0. In this case,

$$b_{i|j}=0,\ \ G^i=G^i_\alpha=\theta y^i.$$

By Lemma 4, F has zero flag curvature, so α has zero sectional curvature and α is locally isometric to the Euclidean metric. Namely, α is flat metric and β is parallel. In this case, F is locally Minkowshian.

SOLUTIONS

In this section, we assume $F=\alpha+\varepsilon\beta+\beta\arctan(\beta/\alpha)$ is projectively flat. By Theorem 2,

$$b_{i|j}=\tau\{(1+2b^2)a_{ij}-b_ib_j\}, \quad (39)$$

$$G^i_\alpha=\theta y^i-\tau\alpha^2 b^i, \quad (40)$$

we are going to find special solutions of Eq.(39) and Eq.(40) such that $\tau\beta$ is closed. The idea is similar to that of Mo *et al*.(2006) and Shen (2006). We only consider the case that $\tau\neq0$, since $\tau=0$ is trivial.

Assume that $\tau\beta$ is closed, i.e. locally, there is a scalar function $\rho=\rho(x)$ such that

$$\tau b_i=\rho_{x^i}/2. \quad (41)$$

Since, by Eq.(39), β is closed, we may let $\beta=\sigma_o/2$, where $\sigma=\sigma(x)$ is a scalar function and $\sigma_o:=\sigma_{x^i}y^i$.

Let $\alpha:=e^\rho\bar{\alpha}$, where

$$\bar{\alpha}:=\sqrt{|y|^2+\mu(|x|^2|y|^2-<x,y>^2)}\Big/(1+\mu|x|^2). \quad (42)$$

We have

$$2\tau\beta=\tau\sigma_o=\rho_o, \quad (43)$$

where $\rho_o:=\rho_{x^i}y^i$.

The spray coefficients of α are given by

$$G^i_\alpha=G^i_{\bar{\alpha}}+\rho_o y^i-\bar{a}^{ij}\rho_{x^j}\bar{\alpha}^2/2$$
$$=\{-\mu<x,y>/(1+\mu|x|^2)+\rho_o\}y^i-a^{ij}\rho_{x^j}\alpha^2/2. \quad (44)$$

Thus Eq.(40) is satisfied.

Now, by using the method of Shen (2006), we are going to solve Eq.(39). Since β is closed, $r_{oo}=b_{o|o}$. By Eq.(43) and Eq.(44), we get

$$r_{oo}=\frac{\partial b_i}{\partial x^j}y^iy^j-2b_iG^i_\alpha$$
$$=\sigma_{oo}/2-\sigma_o\{-\mu<x,y>/(1+\mu|x|^2)+\tau\sigma_o\}+2\tau b^2\alpha^2. \quad (45)$$

Then Eq.(39) is equivalent to the following equation:

$$\sigma_{oo}+2\sigma_o\mu<x,y>/(1+\mu|x|^2)=2\tau\alpha^2+3\tau\sigma_o^2/2. \quad (46)$$

We assume that $\rho=\rho(x)$ and $\sigma=\sigma(x)$ are defined by a common function $h=h(x)$ in the following form

$$\rho=\rho(h),\ \ \sigma=\sigma(h). \quad (47)$$

Then Eq.(43) is equivalent to $\tau\sigma'=\rho'$.

Eq.(46) is reduced to

$$h_{oo}+2h_o\mu<x,y>/(1+\mu|x|^2)$$
$$=2\rho' e^{2\rho}\bar{\alpha}^2/(\sigma')^2+(3\rho'/2-\sigma''/\sigma')h_o^2. \quad (48)$$

We need the following lemma:

Lemma 5 (Shen, 2006) Let

$$h=\frac{C_1+<\boldsymbol{a},x>+\theta|x|^2\Big/\left(1+\sqrt{1+\mu|x|^2}\right)}{\sqrt{1+\mu|x|^2}}, \quad (49)$$

where θ and C_1 are any constant and $\boldsymbol{a}\in\mathbb{R}^n$ is a constant vector. Then h satisfies

$$h_{oo}+2h_o\mu<x,y>/(1+\mu|x|^2)=(\theta-\mu h)\bar{\alpha}^2. \quad (50)$$

Comparing Eq.(48) and Eq.(50), we see that if σ satisfies

$$3\rho'/2-\sigma''/\sigma'=0, \quad (51)$$

$$2\rho' e^{2\rho}/(\sigma')^2=\theta-\mu h, \quad (52)$$

then Eq.(48) holds.

From Eq.(51), we have

$$\sigma'=2C_2e^{3\rho/2}, \quad (53)$$

where C_2 is a positive constant. Then, putting Eq.(53) into Eq.(52). We can get

$$\rho' = 2C_2^2(\theta - \mu h)\mathrm{e}^{\rho}. \qquad (54)$$

Solve this equation to get

$$\rho = -\ln[4C_2^2(\mu h^2 - 2\theta h - C_3)], \qquad (55)$$

where C_3 is a positive constant. Thus, we get

Theorem 3 Let $F=\alpha+\varepsilon\beta+\beta\arctan(\beta/\alpha)$ be a Finsler metric, where ε is a constant. Let $\rho:=\rho(x)$ and $h:=h(x)$ be as follow:

$$\rho = -\ln[4C_2^2(\mu h^2 - 2\theta h - C_3)],$$
$$h = \frac{C_1 + \langle \boldsymbol{a}, x\rangle + \theta |x|^2 \Big/ \left(1+\sqrt{1+\mu|x|^2}\right)}{\sqrt{1+\mu|x|^2}},$$

where C_1, $C_2>0$, C_3, μ and θ are constants, and $\boldsymbol{a}\in\mathbb{R}^n$ is a constant vector. Define

$$\alpha := \mathrm{e}^{\rho}\bar{\alpha}, \;\; \beta := C_2\mathrm{e}^{3\rho/2}h_{\mathrm{o}},$$

where

$$\bar{\alpha} := \sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}\,/(1+\mu|x|^2).$$

Then $F=\alpha+\varepsilon\beta+\beta\arctan(\beta/\alpha)$ is projectively flat.

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