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Conversion matrix between two bases of the algebraic hyperbolic space*

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Abstract: This paper presents the matrix representation for the hyperbolic polynomial B-spline basis and the algebraic hyperbolic Bézier basis in a recursive way, which are both generated over the space Ω_n=span{sinh$t$, cosh$t$, t^{n-3}, …, t, 1} in which n is an arbitrary integer larger than or equal to 3. The conversion matrix from the hyperbolic polynomial B-spline basis of arbitrary order to the algebraic hyperbolic Bézier basis of the same order is also given by a recursive approach. As examples, the specific expressions of the matrix representation for the hyperbolic polynomial B-spline basis of order 4 and the algebraic hyperbolic Bézier basis of order 4 are given, and we also construct the conversion matrix between the two bases of order 4 by the method proposed in the paper. The results in this paper are useful for the evaluation and conversion of the curves and surfaces constructed by the two bases.

Key words: Matrix representation, Hyperbolic polynomial B-spline basis, Algebraic hyperbolic Bézier basis, Conversion matrix
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INTRODUCTION

The Bézier basis and B-spline basis are two important bases over the space span{t^n, t^{n-1}, …, t, 1} that are widely used to construct freeform curves and surfaces. However, there still exist several limitations of the NURBS model that are shown in (Mainar *et al*., 2001). These limitations motivate the research of several new spline curve and surface schemes for geometric modelling in CAGD. Two new bases over the space Ω_n=span{sinh$t$, cosh$t$, t^{n-3}, …, t, 1}, $n\geq 3$ were proposed in (Lü *et al*., 2002; Li and Wang, 2005) respectively. The hyperbolic polynomial (HP) B-spline basis (Lü *et al*., 2002) and the algebraic hyperbolic (AH) Bézier basis (Li and Wang, 2005) are both defined by the integral approach. In CAGD, it is both convenient and practical to describe curves and surfaces by matrix representation having advantages of efficient evaluation using Horner's schema and easy conversion between different shape representations as shown in (Grabowski and Li, 1992). In this paper, we will present the matrix representation for the two bases and the conversion matrix from the HP B-spline basis to the AH Bézier basis.

MATRIX REPRESENTATION FOR THE AH BÉZIER BASIS

In (Li and Wang, 2005) the AH Bézier basis functions are constructed in a recursive way, starting with the two initial functions:

$$B_{0,1}(t)=\frac{\sinh(\alpha-t)}{\sinh\alpha},\ B_{1,1}(t)=\frac{\sinh t}{\sinh\alpha},\ t\in[0,\alpha],\ \alpha\in[0,\infty].$$

For n>1, the AH Bézier basis functions {$B_{0,n}(t)$,

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$B_{1,n}(t)$, …, $B_{n,n}(t)$} of the space Ω_{n+1}=span{sinh$t$, cosh$t$, t^{n-2}, …, t, 1} are defined recursively by:

$$B_{0,n}(t)=1-\int_0^t \delta_{0,n-1}B_{0,n-1}(s)\mathrm{d}s,$$

$$B_{i,n}(t)=\int_0^t (\delta_{i-1,n-1}B_{i-1,n-1}(s)-\delta_{i,n-1}B_{i,n-1}(s))\mathrm{d}s,$$

$$B_{n,n}(t)=\int_0^t \delta_{n-1,n-1}B_{n-1,n-1}(s)\mathrm{d}s,$$

where $\delta_{i,n}=1\Big/\int_0^\alpha B_{i,n}(t)\mathrm{d}t$, $0<i<n$. So the definition of the AH Bézier basis can be described as follows:

Definition 1 The AH Bézier basis over the space Ω_{n+1}=span{sinh$t$, cosh$t$, t^{n-2}, …, t, 1} is given by the above functions {$B_{0,n}(t)$, $B_{1,n}(t)$, …, $B_{n,n}(t)$}.

From the recursive definition of the AH Bézier basis, we can easily obtain two properties of the basis:

$$\begin{cases} B'_{0,n}(t)=-\delta_{0,n-1}B_{0,n-1}(t), \\ B'_{i,n}(t)=\delta_{i-1,n-1}B_{i-1,n-1}(t)-\delta_{i,n-1}B_{i,n-1}(t),\ 1\le i\le n-1 \\ B'_{n,n}(t)=\delta_{n-1,n-1}B_{n-1,n-1}(t), \end{cases} \tag{1}$$

$$(B_{0,n}(0), B_{1,n}(0),\ \ldots, B_{n,n}(0))=(1,\ 0,\ \ldots,\ 0). \tag{2}$$

From the definition of the AH Bézier basis and the two above properties, we can obtain the matrix representation of this basis. Now supposing that we have got the matrix $(e^n_{i,j})_{n\times n}$ representing the AH Bézier basis over the space Ω_n, we will derive the matrix $(e^{n+1}_{i,j})_{(n+1)\times(n+1)}$ that is the matrix representation of the AH Bézier basis over the space Ω_{n+1} on the basis of $(e^n_{i,j})_{n\times n}$, which can be described by the following theorem:

Theorem 1 Suppose ($B_{0,n}(t)$, $B_{1,n}(t)$, …, $B_{n,n}(t)$)= (sinh$t$, cosh$t$, t^{n-2}, …, t, 1) $(e^{n+1}_{i,j})_{(n+1)\times(n+1)}$ and the matrix $(e^n_{i,j})_{n\times n}$ is known, then

$$(e^{n+1}_{i,j})_{(n+1)\times(n+1)}=\begin{pmatrix}(p_{i,j})_{n\times(n+1)}\\ (w_j)_{1\times(n+1)}\end{pmatrix}, \tag{3}$$

where

$$(p_{i,j})_{n\times(n+1)}=\boldsymbol{M}_{n\times n}(e^n_{i,j})_{n\times n}\boldsymbol{T}_{n\times(n+1)},$$

$$\boldsymbol{M}_{n\times n}=\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0\\ 1 & 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & \dfrac{1}{n-2} & 0 & \cdots & 0\\ 0 & 0 & 0 & \dfrac{1}{n-3} & \cdots & 0\\ \vdots & \vdots & \vdots & & \ddots & 0\\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

$$\boldsymbol{T}_{n\times(n+1)}=\begin{pmatrix} -\delta_{0,n-1} & \delta_{0,n-1} & 0 & \cdots & 0\\ 0 & -\delta_{1,n-1} & \delta_{1,n-1} & & \\ \vdots & & \ddots & \ddots & 0\\ 0 & & 0 & -\delta_{n-1,n-1} & \delta_{n-1,n-1} \end{pmatrix}_{n\times(n+1)},$$

$$(w_j)_{1\times(n+1)}=(1-p_{2,1},-p_{2,2},\ldots,-p_{2,n+1}). \tag{4}$$

Proof Considering the derivatives of the AH Bézier basis functions, we can easily get the following result:

$$\begin{aligned}&(B'_{0,n}(t),B'_{1,n}(t),\ldots,B'_{n,n}(t))\\ &=(\cosh t,\sinh t,(n-2)t^{n-3},\ldots,1,0)(e^{n+1}_{i,j})_{(n+1)\times(n+1)}\\ &=(\cosh t,\sinh t,t^{n-3},\ldots,1)\boldsymbol{U}^{-1}_{n\times n}(p_{i,j})_{n\times(n+1)},\end{aligned}$$

where

$$\boldsymbol{U}_{n\times n}=\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & 0 & \cdots & 0\\ 0 & 0 & \dfrac{1}{n-2} & 0 & \cdots & 0\\ 0 & 0 & 0 & \dfrac{1}{n-3} & \cdots & 0\\ \vdots & \vdots & \vdots & & \ddots & 0\\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

And from Eq.(1), we can derive

$$\begin{aligned}&(B'_{0,n}(t),B'_{1,n}(t),\ldots,B'_{n,n}(t))\\ &=(-\delta_{0,n-1}B_{0,n-1}(t),\delta_{0,n-1}B_{0,n-1}(t)-\delta_{1,n-1}B_{1,n-1}(t),\\ &\quad\ldots,\delta_{n-1,n-1}B_{n-1,n-1}(t))\\ &=(B_{0,n-1}(t),B_{1,n-1}(t),\ldots,B_{n-1,n-1}(t))\boldsymbol{T}_{n\times(n+1)}\\ &=(\sinh t,\cosh t,t^{n-3},\ldots,t,1)(e^n_{i,j})_{n\times n}\boldsymbol{T}_{n\times(n+1)}\\ &=(\cosh t,\sinh t,t^{n-3},\ldots,1)\boldsymbol{\eta}_{n\times n}(e^n_{i,j})_{n\times n}\boldsymbol{T}_{n\times(n+1)},\end{aligned}$$

where

$$\boldsymbol{\eta}_{n\times n}=\begin{pmatrix}0&1&0&\cdots&0\\1&0&0&&\\0&0&1&&\\\vdots&&&\ddots&\\0&&&&1\end{pmatrix}_{n\times n}.$$

Comparing the above two equation, Eq.(3) is obvious.

In order to prove Eq.(4), we let t=0 in the supposition and have

$$(B_{0,n}(0),B_{1,n}(0),...,B_{n,n}(0))=(0,1,...,0,1)(e_{i,j}^{n+1})_{(n+1)\times(n+1)}.$$

Applying Eq.(2), we can get

$$(1,0,...,0)=(p_{2,1}+w_1,p_{2,2}+w_2,...,p_{2,n+1}+w_{n+1}).$$

Hence Eq.(4) holds.

This proves the theorem.

Since Theorem 1 only establishes the relationship between $(e_{i,j}^{n})_{n\times n}$ and $(e_{i,j}^{n+1})_{(n+1)\times(n+1)}$, if we want to obtain the specific expression of $(e_{i,j}^{n+1})_{(n+1)\times(n+1)}$, we still have to know the matrix representation of the AH Bézier basis of the lowest order which is 3. From the definition of the AH Bézier basis, we can easily get the matrix representation formula for the AH Bézier basis of order 3:

Proposition 1 The matrix representation formula for the AH Bézier basis of order 3 is:

$$(B_{0,2}(t),B_{1,2}(t),B_{2,2}(t))=\frac{1}{\cosh\alpha-1}(\sinh t,\cosh t,1)\cdot\begin{pmatrix}-\sinh\alpha&\sinh\alpha&0\\\cosh\alpha&-(\cosh\alpha+1)&1\\-1&\cosh\alpha+1&-1\end{pmatrix}.$$

Now, we can get the matrix representation of the AH Bézier basis of arbitrary order by applying Theorem 1 and Proposition 1 recursively. As an instance, let us construct the matrix representation for the AH Bézier basis of order 4:

Proposition 2 The matrix representation formula for the AH Bézier basis of order 4 is:

$$(B_{0,3}(t),B_{1,3}(t),B_{2,3}(t),B_{3,3}(t))=\frac{1}{\sinh\alpha-\alpha}\cdot(\sinh t,\cosh t,t,1)\begin{pmatrix}-\cosh\alpha&J+\cosh\alpha-1&-J&1\\\sinh\alpha&J(G-\alpha)&JG&0\\1&-J&J&-1\\-\alpha&J(\alpha-G)&-JG&0\end{pmatrix},\tag{5}$$

where $J=\dfrac{\sinh\alpha(\cosh\alpha-1)}{\alpha\cosh\alpha+\alpha-2\sinh\alpha}$, $G=\dfrac{\sinh\alpha-\alpha}{\cosh\alpha-1}$.

Proof From Proposition 1 and the definition of $\delta_{i,n}$, we can get:

$$\boldsymbol{T}_{3\times4}=\begin{pmatrix}-\phi&\phi&0&0\\0&-\varphi&\varphi&0\\0&0&-\phi&\phi\end{pmatrix},$$

where $\phi=\dfrac{\cosh\alpha-1}{\sinh\alpha-\alpha}$, $\varphi=\dfrac{\cosh\alpha-1}{\alpha\cosh\alpha+\alpha-2\sinh\alpha}$.

And we have

$$\boldsymbol{M}_{3\times3}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}.$$

Then, employing Theorem 1, Eq.(5) holds.

This proves the proposition.

MATRIX REPRESENTATION FOR THE HYPERBOLIC POLYNOMIAL (HP) B-SPLINE BASIS

The HP B-spline basis functions are well-defined for arbitrary real numbers as shown in (Lü *et al*., 2002). But in order to conveniently acquire the matrix representation of the HP B-spline basis we just consider an interval (0, α). And through the local support property of the HP B-spline basis proposed in (Lü *et al*., 2002), we know that the nonzero HP B-spline functions of order n+1 on the interval (0, α) are $N_{-n,n+1}(t)$, $N_{1-n,n+1}(t)$, …, $N_{0,n+1}(t)$. Now let us introduce how the HP B-spline is defined recursively in (Lü *et al*., 2002). Beginning with the following functions:

$$N_{0,2}(t)=\begin{cases}\dfrac{\alpha}{2(\cosh\alpha-1)}\sinh t, & 0\le t\le\alpha,\\ \dfrac{\alpha}{2(\cosh\alpha-1)}\sinh(2\alpha-t), & \alpha\le t\le 2\alpha,\\ 0, & \text{elsewhere},\end{cases}$$

and

$$N_{i,2}(t)=N_{0,2}(t-i\alpha),\ (i=0,\ \pm1,\ \pm2,\ ...).$$

For $n\geq3$ let

$$N_{i,n}(t)=\frac{1}{\alpha}\int_{t-\alpha}^{t}N_{i,n-1}(s)\mathrm{d}s,\ (i=0,\ \pm1,\ \pm2,\ ...).$$

Hence the definition of the HP B-spline basis is:

Definition 2 The hyperbolic polynomial B-spline basis of order n is $N_{i,n+1}(t)$ (i=0, ±1, ±2, …).

Two basic and useful properties of the HP B-spline basis that will be used later are listed below:

(1) $N'_{i,n}(t)=\frac{1}{\alpha}(N_{i,n-1}(t)-N_{i+1,n-1}(t))$,

(2) $(N_{-n,n+1}(0), N_{1-n,n+1}(0), \dots, N_{0,n+1}(0))=(\sigma_{1-n,n}, \sigma_{2-n,n}, \dots, \sigma_{0,n}, 0)$,

where $\sigma_{i,n}=\frac{1}{\alpha}\int_0^{\alpha}N_{i,n}(s)\mathrm{d}s$.

We also construct the matrix representation of the HP B-spline basis through a recursive approach, and now present a theorem describing the relationship between matrix representation of the HP B-spline basis of the higher order and that of the lower order.

Theorem 2 Suppose $(N_{-n,n+1}(t), N_{1-n,n+1}(t), \dots, N_{0,n+1}(t))=(\sinh t, \cosh t, t^{n-2}, \dots, t, 1)\,(f_{i,j}^{n+1})_{(n+1)\times(n+1)}$ and the matrix representation $(f_{i,j}^{n})_{n\times n}$ for the HP B-spline basis of order n is known, then

$$(f_{i,j}^{n+1})_{(n+1)\times(n+1)}=\begin{pmatrix}(h_{i,j})_{n\times(n+1)}\\(r_j)_{1\times(n+1)}\end{pmatrix},$$

where $(h_{i,j})_{n\times(n+1)}=\boldsymbol{M}_{n\times n}(f_{i,j}^{n})_{n\times n}\boldsymbol{Y}_{n\times(n+1)}$, $\boldsymbol{M}_{n\times n}$ is the matrix shown in Theorem 1, and

$$\boldsymbol{Y}_{n\times(n+1)}=\frac{1}{\alpha}\begin{pmatrix}-1&1&0&\cdots&0\\0&-1&1&&\\\vdots&&\ddots&\ddots&0\\0&&0&-1&1\end{pmatrix}_{n\times(n+1)},$$

$$(r_j)_{1\times(n+1)}=(\sigma_{1-n,n}-h_{2,1},\sigma_{2-n,n}-h_{2,2},...,\sigma_{0,n}-h_{2,n},-h_{2,n+1}).$$

Proof The process of the proof is the same as that in Theorem 1. So we omit the details.

This proves the theorem.

To determine the definitive expression of the matrix representation formula for the HP B-spline basis of the arbitrary order, we must know the matrix representation of the HP B-spline basis of the initial order that is 3. Through simple calculation with the definition of the basis, we can get the matrix representation formula for the HP B-spline basis of order 3:

Proposition 3 The matrix representation for the HP B-spline basis of order 3 is

$$(N_{-2,3}(t),N_{-1,3}(t),N_{0,3}(t))=\frac{1}{2(\cosh\alpha-1)}\cdot(\sinh t,\cosh t,1)\begin{pmatrix}-\sinh\alpha&\sinh\alpha&0\\\cosh\alpha&-(\cosh\alpha+1)&1\\-1&2\cosh\alpha&-1\end{pmatrix}.$$

Now the matrix representation of the HP B-spline basis of arbitrary order can be obtained through a recursive application of Theorem 2 and Proposition 3. The following proposition can serve as an example for Theorem 2.

Proposition 4 The matrix representation for the HP B-spline basis of order 4 is

$$(N_{-3,4}(t),N_{-2,4}(t),N_{-1,4}(t),N_{0,4}(t))=\frac{1}{2\alpha(\cosh\alpha-1)}\cdot(\sinh t,\cosh t,t,1)\cdot\begin{pmatrix}-\cosh\alpha&1+2\cosh\alpha&-(2+\cosh\alpha)&1\\\sinh\alpha&-2\sinh\alpha&\sinh\alpha&0\\1&-(1+2\cosh\alpha)&1+2\cosh\alpha&-1\\-\alpha&2\alpha\cosh\alpha&-\alpha&0\end{pmatrix}.\tag{6}$$

Proof Employing the proposition 3, we can get

$$(\sigma_{-2,3},\sigma_{-1,3},\sigma_{0,3})=\left(\frac{\sinh\alpha-\alpha}{2\alpha(\cosh\alpha-1)},\frac{\alpha\cosh\alpha-\sinh\alpha}{\alpha(\cosh\alpha-1)},\frac{\sinh\alpha-\alpha}{2\alpha(\cosh\alpha-1)}\right).$$

And we also have

$$\boldsymbol{M}_{3\times3}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},\ \boldsymbol{Y}_{3\times4}=\frac{1}{\alpha}\begin{pmatrix}-1&1&0&0\\0&-1&1&0\\0&0&-1&1\end{pmatrix}.$$

Applying $\sigma_{-2,3}$, $\sigma_{-1,3}$, $\sigma_{0,3}$, $\boldsymbol{M}_{3\times3}$, $\boldsymbol{Y}_{3\times4}$ in Theorem 2, we can obtain the $(f_{i,j}^4)_{4\times4}$ as shown in Eq.(6).

This proves the proposition.

CONVERSION MATRIX FROM THE HP B-SPLINE BASIS TO THE AH BÉZIER BASIS

As two different bases over the same space, we spontaneously want to know the relationship between the two bases which could be useful for exploring the properties of the curves and surfaces constructed by the two bases. From what has been discussed above, we can find that the conversion matrix between the AH basis and the HP basis could be easily obtained by some matrix computation. Now we will construct the conversion matrix from the HP B-spline basis to the AH Bézier basis in a recursive way. If we know the conversion matrix $(a_{i,j}^n)_{n\times n}$ of order n, the following theorem shows the way in which we can derive the conversion matrix $(a_{i,j}^{n+1})_{(n+1)\times(n+1)}$ of order $(n+1)$.

Theorem 3 Suppose $(N_{-n,n+1}(t), N_{1-n,n+1}(t), \ldots, N_{0,n+1}(t))=(B_{0,n}(t), B_{1,n}(t), \ldots, B_{n,n}(t))(a_{i,j}^{n+1})_{(n+1)\times(n+1)}$ and the conversion matrix from the HP B-spline basis of order n to the AH Bézier basis $(a_{i,j}^n)_{n\times n}$ is known, then

$$a_{1,n+1}^{n+1}=0,\ a_{1,j}^{n+1}=\sigma_{j-n,n},\ 1\le j\le n, \tag{7}$$

$$a_{i,1}^{n+1}=a_{i-1,1}^{n+1}-\frac{a_{i-1,1}^n}{\alpha\delta_{i-2,n-1}},\ 2\le i\le n+1, \tag{8}$$

$$a_{i,n+1}^{n+1}=a_{i-1,n+1}^{n+1}+\frac{a_{i-1,n}^n}{\alpha\delta_{i-2,n-1}},\ 2\le i\le n+1, \tag{9}$$

$$a_{i,j}^{n+1}=a_{i-1,j}^{n+1}+\frac{a_{i-1,j-1}^n-a_{i-1,j}^n}{\alpha\delta_{i-2,n-1}},\ 2\le i\le n+1,\ 2\le j\le n, \tag{10}$$

where

$$\sigma_{i,n}=\frac{1}{\alpha}\int_0^\alpha N_{i,n}(s)\mathrm{d}s,\ \delta_{i,n}=\frac{1}{\int_0^\alpha B_{i,n}(t)\mathrm{d}t}.$$

Proof Differentiating the HP B-spline basis directly, we have

$$\begin{aligned}&(N'_{-n,n+1}(t),N'_{1-n,n+1}(t),\ldots,N'_{0,n+1}(t))\\&=(N_{1-n,n}(t),N_{2-n,n}(t),\ldots,N_{0,n}(t))\boldsymbol{Y}_{n\times(n+1)}\\&=(B_{0,n-1}(t),B_{1,n-1}(t),\ldots,B_{n-1,n-1}(t))(a_{i,j}^n)_{n\times n}\boldsymbol{Y}_{n\times(n+1)}.\end{aligned} \tag{11}$$

The derivative of the HP B-spline basis can also be expressed as

$$\begin{aligned}&(N'_{-n,n+1}(t),N'_{1-n,n+1}(t),\ldots,N'_{0,n+1}(t))\\&=(B'_{0,n}(t),B'_{i,n}(t),\ldots,B'_{n,n}(t))(a_{i,j}^{n+1})_{(n+1)\times(n+1)}\\&=(B_{0,n-1}(t),B_{1,n-1}(t),\ldots,B_{n-1,n-1}(t))\boldsymbol{T}_{n\times(n+1)}(a_{i,j}^{n+1})_{(n+1)\times(n+1)}.\end{aligned} \tag{12}$$

Comparing Eqs.(11) and (12), we have

$$\boldsymbol{T}_{n\times(n+1)}(a_{i,j}^{n+1})_{(n+1)\times(n+1)}=(a_{i,j}^n)_{n\times n}\boldsymbol{Y}_{n\times(n+1)},$$

that is,

$$\begin{pmatrix}(a_{2,1}^{n+1}-a_{1,1}^{n+1})\delta_{0,n-1}&\cdots&(a_{2,n+1}^{n+1}-a_{1,n+1}^{n+1})\delta_{0,n-1}\\(a_{3,1}^{n+1}-a_{2,1}^{n+1})\delta_{1,n-1}&&(a_{3,n+1}^{n+1}-a_{2,n+1}^{n+1})\delta_{1,n-1}\\\vdots&&\vdots\\(a_{n+1,1}^{n+1}-a_{n,1}^{n+1})\delta_{n-1,n-1}&\cdots&(a_{n+1,n+1}^{n+1}-a_{n,n+1}^{n+1})\delta_{n-1,n-1}\end{pmatrix}_{n\times(n+1)}$$
$$=\frac{1}{\alpha}\begin{pmatrix}-a_{1,1}^n&a_{1,1}^n-a_{1,2}^n&\cdots&a_{1,n-1}^n-a_{1,n}^n&a_{1,n}^n\\-a_{2,1}^n&a_{2,1}^n-a_{2,2}^n&&a_{2,n-1}^n-a_{2,n}^n&a_{2,n}^n\\\vdots&&&&\\-a_{n,1}^n&a_{n,1}^n-a_{n,2}^n&\cdots&a_{n,n-1}^n-a_{n,n}^n&a_{n,n}^n\end{pmatrix}_{n\times(n+1)}. \tag{13}$$

Now we let t=0 in the two bases, then we have

$$\begin{aligned}&(N_{-n,n+1}(0),N_{1-n,n+1}(0),\ldots,N_{0,n+1}(0))\\&\quad=(B_{0,n}(0),B_{1,n}(0),\ldots,B_{n,n}(0))(a_{i,j}^{n+1})_{(n+1)\times(n+1)},\end{aligned}$$

that is,

$$(\sigma_{1-n,n},\sigma_{2-n,n},\ldots,\sigma_{0,n},0)=(1,0,\ldots,0)(a_{i,j}^{n+1})_{(n+1)\times(n+1)}.$$

Hence Eq.(7) holds.

And equalling corresponding components of the two matrices in Eq.(13) , we can easily figure out that Eqs.(8), (9) and (10) hold.

This proves the theorem.

The specific expression of $(a_{i,j}^{n+1})_{(n+1)\times(n+1)}$ is determinated by the initial conversion matrix indicating the relation between the HP B-spline basis of order 3 and the AH Bézier basis of the same order. Through Proposition 1 and Proposition 3, we can get the initial conversion matrix as shown in the following proposition.

Proposition 5 The conversion formula from the HP B-spline basis of order 3 to the AH Bézier basis of the same order is

$$(N_{-2,3}(t),N_{-1,3}(t),N_{0,3}(t))$$
$$=(B_{0,2}(t),B_{1,2}(t),B_{2,2}(t))\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ 0 & 1 & 0\\ 0 & \frac{1}{2} & \frac{1}{2}\end{pmatrix}.$$

Now the conversion matrix from the HP B-spline basis of the arbitrary order to the AH Bézier basis can be derived by recursively applying Theorem 3 and Proposition 5. As an application of Theorem 3, the conversion matrix from the HP B-spline basis of order 4 to the AH Bézier basis is presented as follows:

Proposition 6 The conversion formula from the HP B-spline basis of order 3 to the AH Bézier basis of the same order is

$$(N_{-3,4}(t),N_{-2,4}(t),N_{-1,4}(t),N_{0,4}(t))$$
$$=(B_{0,3}(t),B_{1,3}(t),B_{2,3}(t),B_{3,3}(t))\begin{pmatrix}L & Z & L & 0\\ 0 & Z & 2L & 0\\ 0 & 2L & Z & 0\\ 0 & L & Z & L\end{pmatrix}, \quad (14)$$

where

$$L=\frac{\sinh\alpha-\alpha}{2\alpha(\cosh\alpha-1)},\ Z=\frac{\alpha\cosh\alpha-\sinh\alpha}{\alpha(\cosh\alpha-1)}.$$

Proof From the definitions of $\delta_{i,n}$ and $\sigma_{i,n}$, we can get

$$(\delta_{0,2},\delta_{1,2},\delta_{2,2})$$
$$=\left(\frac{\cosh\alpha-1}{\sinh\alpha-\alpha},\frac{\cosh\alpha-1}{\alpha\cosh\alpha+\alpha-2\sinh\alpha},\frac{\cosh\alpha-1}{\sinh\alpha-\alpha}\right),$$

$$(\sigma_{-2,3},\sigma_{-1,3},\sigma_{0,3})$$
$$=\left(\frac{\sinh\alpha-\alpha}{2\alpha(\cosh\alpha-1)},\frac{\alpha\cosh\alpha-\sinh\alpha}{\alpha(\cosh\alpha-1)},\frac{\sinh\alpha-\alpha}{2\alpha(\cosh\alpha-1)}\right),$$

and we have

$$(a_{i,j}^{3})_{3\times3}=\begin{pmatrix}1/2 & 1/2 & 0\\ 0 & 1 & 0\\ 0 & 1/2 & 1/2\end{pmatrix}.$$

Employing Theorem 3, we can easily find that Eq.(14) holds.

This proves the proposition.

CONCLUSION

In this paper, we present the matrix representation for the hyperbolic polynomial B-spline basis and the algebraic hyperbolic Bézier basis over the space Ω_n=span{sinh$t$, cosh$t$, t^{n-3},…, t, 1} ($n\geq3$) and give the explicit expressions of the matrix representation for the two bases of order 4. We also present the conversion matrix from the HP B-spline basis to the AH Bézier basis and show an example of order 4. The matrix forms for curves and surfaces are largely promoted in CAD. So we expect the results can be employed in the CAD/CAM systems.

References

Grabowski, H., Li, X., 1992. Coefficient formula and matrix of nonuniform B-spline functions. *Computer-Aided Design*, **24**(12):637-642. [doi:10.1016/0010-4485(92)90018-6]

Li, Y.J., Wang, G.Z., 2005. Two kinds of B-basis of the algebraic hyperbolic space. *Journal of Zhejiang University SCIENCE*, **6A**(7):750-759. [doi:10.1631/jzus.2005.A0750]

Lü, Y.G., Wang, G.Z., Yang, X.N., 2002. Uniform hyperbolic polynomial B-spline curves. *Computer Aided Geometric Design*, **19**(6):379-393. [doi:10.1016/S0167-8396(02)00092-4]

Mainar, E., Peña, J.M., Sánchez-Reyes, J., 2001. Shape preserving alternatives to the rational Bézier model. *Computer Aided Geometric Design*, **18**(1):37-60. [doi:10.1016/S0167-8396(01)00011-5]