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Generalization of 3D Mandelbrot and Julia sets*

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Abstract: In order to further enrich the form of 3D Mandelbrot and Julia sets, this paper first presents two methods of generating 3D fractal sets by utilizing discrete modifications of the standard quaternion algebra and analyzes the limitations in them. To overcome these limitations, a novel method for generating 3D fractal sets based on a 3D number system named ternary algebra is proposed. Both theoretical analyses and experimental results demonstrate that the ternary-algebra-based method is superior to any one of the quad-algebra-based methods, including the first two methods presented in this paper, because it is more intuitive, less time consuming and can completely control the geometric structure of the resulting sets. A ray-casting algorithm based on period checking is developed with the goal of obtaining high-quality fractal images and is used to render all the fractal sets generated in our experiments. It is hoped that the investigations conducted in this paper would result in new perspectives for the generalization of 3D Mandelbrot and Julia sets and for the generation of other deterministic 3D fractals as well.

Key words: Mandelbrot set, Julia set, Fractal, Ray-casting, Quad algebra, Ternary algebra
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INTRODUCTION

Mandelbrot and Julia sets have been extensively investigated in great detail in terms of their aesthetically pleasing geometrical shapes, infinite detail, self-similarity, periodicity and many other characteristics. The beauty and intricacy of these sets in the complex plane stimulate the scholars' enthusiasm for viewing sets that occupy more than two dimensions. However, investigations on the generalizations of fractal sets to higher-dimensional spatial space have come up against a series of problems due to the limitations of mathematical theory and visualization techniques needed to explore such fractals.

Most of the existing researches in this field were conducted using Hamilton quaternion for the generation of Mandelbrot and Julia sets, and mapping three of the four quaternionic components to 3D Cartesian space. Norton first discovered Julia sets of quaternion functions (Norton, 1982; 1989). He visualized these structures using a boundary-tracking algorithm, which provided a global view of never-before-seen 3D fractal sets but required a large amount of memory to operate because it required the efficient storage of all previously generated points to verify that neighboring points had not been previously tested. In the meantime, a quaternion inverse iteration algorithm was proposed by Holbrook for generating sparse point cloud representations of quaternion Julia sets (Holbrook, 1983; 1987), and it was developed by Hart for the purpose of achieving interactive visualization of quaternion Julia sets, which required much less memory than boundary tracking and could quickly visualize the global shapes of Julia sets, yet the resulting images were much less than satisfactory (Hart *et al*., 1990). To obtain a more realistic inspection of the fractal surfaces, John developed a ray-tracing algorithm that could produce high quality images of 3D fractals at various levels of detail (Hart *et al*., 1989), which required as little memory as the

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inverse iteration but produced significantly finer details of fractal sets. Consequently, it was by far the best algorithm when high-quality fractal images were desired. Other than the above works devoted to computational and graphical aspects of the quaternionic quadratic map, Gomatam *et al*.(1995) systematically analyzed the stability of cycles and analytically characterized the associated, generalized fractal sets while Bogush *et al*.(2000) investigated the algebraic and geometrical properties of quaternionic analogs of Julia sets by means of theory group analysis and argued that the 3D part of quaternionic Julia sets could be restored by rotation of some arbitrary 2D Julia subsets around certain axis due to the intrinsic quaternionic symmetries.

However, it has been established that there is no interesting dynamics for the above approach of generating 3D fractal sets from Hamilton quaternion because these sets are locally a rotation of classical fractals in the complex plane (Bedding and Briggs, 1995). Hence, researchers began approaching 3D fractal sets by new quad algebras. Rochon (2000) novelly utilized a commutative bicomplex number to explore fractal sets of more than two dimension and proved that their 4D Mandelbrot sets are connected. He also proposed a bicomplex version of the Fatou-Julia theorem and proved that bicomplex dynamics is an interesting way to generalize the classical Mandelbrot sets in 3D (Rochon, 2003). Martineau and Rochon (2005) presented several distance estimation formulas that can be used to ray trace slices of the bicomplex fractal sets in 3D space. Besides bicomplex numbers, complexified quaternion algebra (CQUAT) (Gintz, 2002) was introduced to iteratively compute the boundaries of fractal sets to the effect that fractal sets with new structures could be obtained.

In this paper we present three new approaches to generate 3D Mandelbrot and Julia sets based on Jiang's quaternion, Qu's quaternion and ternary algebra, with the goal of exploring as much as possible the extensions of 3D fractal sets. Prior to the description of three new methods, a ray-casting algorithm based on period checking is outlined. Then, methods of creating 3D Mandelbrot sets from Jiang's quaternion and Qu's quaternion are introduced and the drawbacks inherent in all quad-algebra-based methods are pointed out, to overcome which the method of generating 3D Mandelbrot sets based on ternary algebra is developed. Particular attention of this section is paid to the investigation on the properties of the resulting 3D sets and on the advantages of generating 3D sets from ternary algebra. Application of the third method to 3D Julia sets and an analysis of the properties of resulting sets are also included.

RAY-CASTING VOLUME RENDERING ALGORITHM BASED ON PERIOD CHECKING

In order to produce 3D fractal images of high quality, we put forward a ray-casting volume rendering algorithm based on period checking, which borrows the idea from (Welstead and Cromer, 1989) and colors 3D fractal sets according to the periodicities of maps to reveal patterns of finite attracting orbits and chaos. The procedure of our algorithm is summarized as follows:

Step 1: Initialize parameters, including maximum periodicity to be counted, maximum number of iteration, threshold of overflow and the region of algebra space to be observed.

Step 2: Compute the periodicity of each point in the range of given algebra space, and (1) set the periodicity of a point to −1 if overflow happens before its stable orbit is reached; (2) set the periodicity of a point to 0 if we can get its stable orbit but fail to work out its periodicity.

Thus, periodicity is assigned to every point in the given space, which also determines whether a point belongs to a fractal set or not. It is obvious that those points whose periodicities equal −1 do not reside in the fractal set while those points whose periodicities equal 0 are on the boundary of the fractal set.

Step 3: Render 3D fractal set by ray-casting algorithm.

3D fractal surfaces are non-differentiable and have no exact normal defined. However, a shading model must be used which requires the definition of a surface normal so as to realistically render these surfaces. In our algorithm, a finite difference technique is employed to approximate the gradient of a point (x,y,z) on the fractal surface:

$$\left.\begin{aligned} g_x &= \frac{per(x+\mathrm{d}x, y, z) - per(x-\mathrm{d}x, y, z)}{2\mathrm{d}x}, \\ g_y &= \frac{per(x, y+\mathrm{d}y, z) - per(x, y-\mathrm{d}y, z)}{2\mathrm{d}y}, \\ g_z &= \frac{per(x, y, z+\mathrm{d}z) - per(x, y, z-\mathrm{d}z)}{2\mathrm{d}z}, \end{aligned}\right\} \quad (1)$$

where $per(x,y,z)$ represents the periodicity at location (x,y,z) which has been calculated in Step 2, and dx, dy, dz are grid intervals along x, y, z axes respectively such that dx=dy=dz. The resulting vector (g_x,g_y,g_z) is then used as the surface normal at point (x,y,z) and the fractal object can be shaded at this point using a standard shading algorithm.

For convenient observation on the distribution of attractive orbits in a fractal set, we define the color and opacity of every point according to their periodicities. All the points are assigned the opacity of 1 except that those having −1 periodicity are assigned the opacity of 0.

3D MANDELBROT SETS ON THE BASIS OF TWO NEW QUAD ALGEBRAS

Enlightened by Terry's discovery of new exciting fractal images with the introduction of CQUAT, it is natural for us to explore other possible methods to generate 3D fractal sets by utilizing discrete modifications of the standard quaternion algebra to iteratively compute their set boundaries. Considering that any quad algebraic system closed under addition and multiplication can be employed to generate 3D fractal sets since the iterative functions used to generate these sets contain only addition and multiplication, we exploit two new forms of quad algebra, i.e. Jiang's quaternion (Jiang, 1990) and Qu's quaternion (Qu and Zhou, 2001) to model 3D fractal sets hereinafter. A quaternion number $q=x+y\mathrm{i}+z\mathrm{j}+\omega\mathrm{k}$ is a four-tuple of independent real values (x,y,z,ω) assigned to one real axis and three imaginary axes i, j, k. Given two quaternion numbers q_1 and q_2, then we have $q_1+q_2=(x_1+x_2)+(y_1+y_2)\mathrm{i}+(z_1+z_2)\mathrm{j}+(\omega_1+\omega_2)\mathrm{k}$.

Jiang's quaternion and its Mandelbrot set

The product of two Jiang's quaternion numbers q_1 and q_2 is defined as:

$$q_1\cdot q_2=(x_1x_2+y_1y_2+z_1z_2+\omega_1\omega_2)+(x_1y_2+y_1x_2+z_1\omega_2+\omega_1z_2)\mathrm{i}+(x_1z_2+z_1x_2+y_1\omega_2+\omega_1y_2)\mathrm{j}+(x_1\omega_2+\omega_1x_2+y_1z_2+z_1y_2)\mathrm{k}. \quad (2)$$

Let $q_1=q_2=q$, we get

$$q^2=(x^2+y^2+z^2+\omega^2)+2(xy+z\omega)\mathrm{i}+2(xz+y\omega)\mathrm{j}+2(x\omega+yz)\mathrm{k}. \quad (3)$$

A collectivity of all Jiang's quad numbers constitutes Jiang's quaternion number system, which is herein denoted as Q_J for short.

Definition 1 The Mandelbrot sets of Jiang's quad maps f:$q\rightarrow q^m+c$ (q, $c\in Q_\mathrm{J}$, $m\in\mathbb{N}$, $m\geq 2$) are defined as the sets that comprise all the Jiang's quaternion values c not attracted to infinity when the maps are iterated with q_0 set to the critical point of maps. Namely, they are the sets of Jiang's quaternion numbers c such that

$$M_\mathrm{J}=\{c\in Q_\mathrm{J}:\{f^n(q_0)\}_{n=1}^{\infty}\nrightarrow\infty\}, \quad (4)$$

where q_0=0 is the critical point of maps. However, instead of starting with q_0=0, it is simpler to start with $q_0=c$, which yields the same result. Therefore, the computational definition of Jiang's Mandelbrot set can also be expressed as

$$M_\mathrm{J}=\{c\in Q_\mathrm{J}:\{f^n(c)\}_{n=1}^{\infty}\nrightarrow\infty\}. \quad (5)$$

Qu's quaternion and its Mandelbrot set

Multiplication of Qu's quaternion is defined as follows:

$$q_1\cdot q_2=(x_1x_2-y_1\omega_2-z_1z_2-\omega_1y_2)+(x_1y_2+y_1x_2-z_1\omega_2-\omega_1z_2)\mathrm{i}+(x_1z_2+y_1y_2+z_1x_2-\omega_1\omega_2)\mathrm{j}+(x_1\omega_2+y_1z_2+z_1y_2+\omega_1x_2)\mathrm{k}. \quad (6)$$

Squaring a Qu's quaternion number q yields

$$q^2=(x^2-2y\omega-z^2)+2(xy-z\omega)\mathrm{i}+(2xz+y^2-\omega^2)\mathrm{j}+2(x\omega+yz)\mathrm{k}. \quad (7)$$

Qu's quaternion number system, which consists of all Qu's quaternion numbers, is signified as Q_Q for concise sake.

Definition 2 The Mandelbrot sets of Qu's quad maps f:$q\rightarrow q^m+c$ (q, $c\in Q_\mathrm{Q}$, $m\in\mathbb{N}$, $m\geq 2$) can be defined as

$$M_\mathrm{Q}=\{c\in Q_\mathrm{Q}:\{f^n(q_0)\}_{n=1}^{\infty}\nrightarrow\infty\}, \quad (8)$$

where q_0=0 is the critical point of maps. For the same reason as Jiang's Mandelbrot sets, the computational definition of Qu's Mandelbrot sets practically used in our experiments is given by

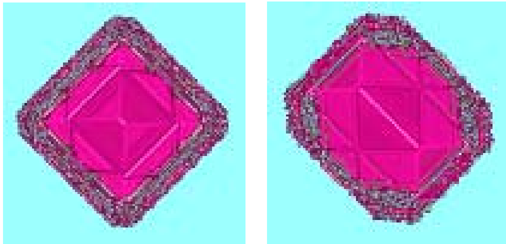
$$M_\mathrm{Q}=\{c\in Q_\mathrm{Q}:\{f^n(c)\}_{n=1}^{\infty}\nrightarrow\infty\}. \quad (9)$$

Experimental results and discussions

For convenience in analysis and comparison, all the fractal images given in this paper are rendered by the period-checking-based ray-casting algorithm within the same region of algebra space, specifically speaking, x, y, $z\in[-1, 1]$, and with the same grid interval 0.02.

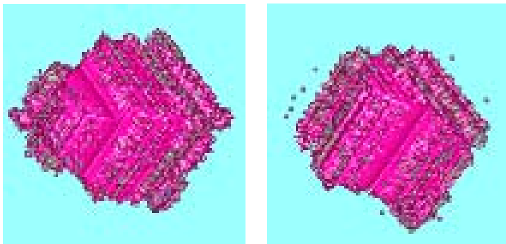
Mandelbrot sets of Jiang's quaternionic quadratic map with the fourth real value ω being 0.022 and 0.220 are shown in Fig.1a and Fig.1b respectively while those of Qu's are illustrated in Fig.2a and Fig.2b accordingly. It can be observed from Figs.1 and 2 that the structures of 3D Mandelbrot sets created from either Jiang's quaternion or Qu's hardly have any comparability with 2D Mandelbrot sets generated by complex maps, therefore, it is difficult for us to have a good insight into the infinite detail of these 3D sets. Furthermore, the maximal dimension that can be easily and comprehensively represented on a computer screen is commonly 3, hence we have to predefine one element of the iterative parameter (ω is predefined in our experiments) in order to project the 4D fractal sets onto 3D Cartesian space in the process of generating 3D fractal sets from these quad algebras. Nevertheless, different orientations of projection might lead to somewhat big diversities among resulting 3D fractal subsets, and some subsets in 3D space may reveal no fractal characteristic at all. As a matter of fact, methods of generating fractal sets by exploiting other quad algebras, including Hamilton quaternion and bicomplex numbers (see Figs.3 and 4), would encounter the same problems mentioned hereinabove as well.

In the final analysis, the intrinsic drawbacks in the creation of 3D fractal sets from quad algebras are caused by the indispensable projective operation in order to visualize 3D subsets of 4D objects obtained from quaternion maps, which would inevitably result in information loss and ultimately lead to the difficulty in controlling over the structure of resulting subsets as well as in achieving expected 3D fractal images. To surmount the obstacles in visualizing 3D fractal sets generated by quad algebras, we explored the possibility of creating 3D sets directly in three spatial dimensions so that 3D fractal images could be produced with no projective operation involved. This motivation led to the third approach of generating 3D fractal sets on the basis of ternary algebra.



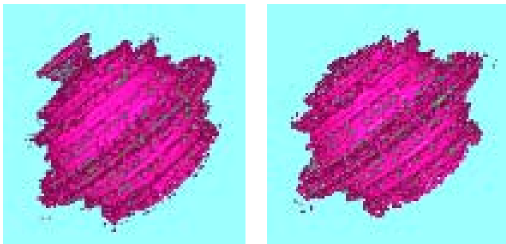
(a) (b)

Fig.1 Mandelbrot sets in Jiang's quaternion
(a) ω=0.022; (b) ω=0.220



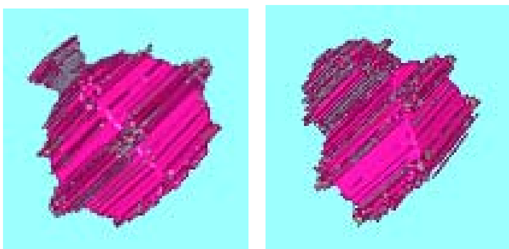
(a) (b)

Fig.2 Mandelbrot sets in Qu's quaternion
(a) ω=0.022; (b) ω=0.220



(a) (b)

Fig.3 Mandelbrot sets in Hamilton quaternion
(a) ω=0.022; (b) ω=0.220



(a) (b)

Fig.4 Mandelbrot sets in bi-complex numbers
(a) ω=0.022; (b) ω=0.220

3D MANDELBROT SET ON THE BASIS OF TERNARY ALGEBRA

Ternary algebra and its Mandelbrot set

Fundamentals of ternary algebra are briefly introduced in Appendix A (Qu, 1994). A ternary number is of the form

$$t = x\mathrm{i} + y\mathrm{j} + z\mathrm{k}, \tag{10}$$

where x, y, z are real numbers and i, j, k are imaginary units. Given two ternary numbers t_1 and t_2, then we have

$$t_1 + t_2 = (x_1 + x_2)\mathrm{i} + (y_1 + y_2)\mathrm{j} + (z_1 + z_2)\mathrm{k}, \tag{11}$$

$$t_1 \cdot t_2 = (x_1x_2 - y_1z_2 - z_1y_2)\mathrm{i} + (x_1y_2 + y_1x_2 - z_1z_2)\mathrm{j} + (x_1z_2 + y_1y_2 + z_1x_2)\mathrm{k}. \tag{12}$$

Squaring of a ternary number t yields

$$t^2 = (x^2 - 2yz)\mathrm{i} + (2xy - z^2)\mathrm{j} + (2xz + y^2)\mathrm{k}. \tag{13}$$

A total of the ternary numbers of the form Eq.(10) constitutes the ternary number system (i.e. ternary algebra), which is marked as T. Ternary algebra extends the concept of number to 3D arithmetic space. And we can certainly generate 3D fractal sets from it.

Definition 3 The Mandelbrot sets of ternary maps f: $t \to t^m + c$ ($t, c \in T$, $m \in \mathbb{N}$, $m \geq 2$) are defined by

$$M_T = \{c \in T : \{f^n(t_0)\}_{n=1}^{\infty} \not\to \infty\}, \tag{14}$$

where $t_0 = 0$ is the critical point of ternary maps. Similar to the Mandelbrot sets of quaternion algebras, we can also rewrite Eq.(14) as

$$M_T = \{c \in T : \{f^n(c)\}_{n=1}^{\infty} \not\to \infty\}. \tag{15}$$

The following properties of M_T can be deduced according to the fundamentals of ternary algebra.

Property 1 M_T is symmetrical about the plane defined by $y+z=0$, which is proved in Appendix B.

Property 2 The projection of M_T on the plane $x-y+z=0$ is identical to the Mandelbrot set of complex mapping $z \to z^m + c$ ($z, c \in \mathbb{C}$) with corresponding exponent m, where $\mathbb{C}$ denotes the complex number system.

Property 3 M_T may be reconstructed by moving some arbitrary 2D Mandelbrot subset in the plane $x-y+z=l$ along the line $x=-y=z$, where l is a real number.

Based on Properties 2, 3 as well as the characteristics of 2D Mandelbrot sets in the complex plane, we can deduce another property of M_T, that is,

Property 4 The bottom surface of M_T parallel to the plane $x-y+z=0$ consists of $m-1$ integral petals.

Experimental results and discussion

The 3D Mandelbrot sets of ternary maps $t \to t^m + c$ ($t, c \in T$) with exponents being 2, 5, 6 and 7 are illustrated in Figs.5a~5d. It can be observed from these images that the Mandelbrot sets are all symmetrical about a plane perpendicular to the screen, which is defined by equation $y+z=0$ in the light of Property 1. The 3D Mandelbrot sets in Figs.5a~5d also manifest somewhat columnar shapes with the most irregular and sophisticated lateral surfaces. And in accordance with Properties 2, 3, all the bottom surfaces (see Fig.6) of these fractal shapes are parallel with the plane $x-y+z=0$ while their generatrices are oriented along the same direction defined by $x=-y=z$.

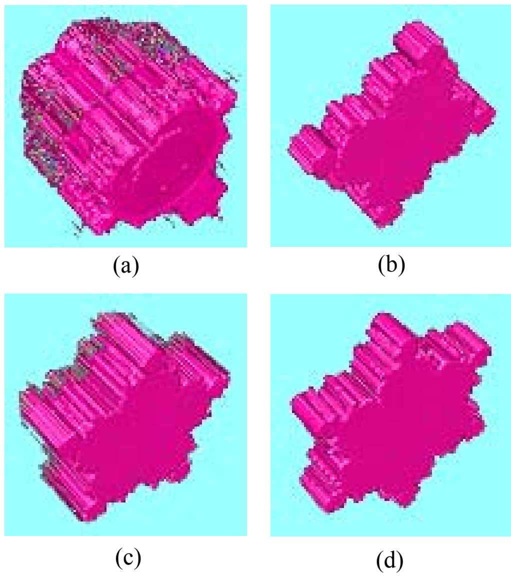


Fig.5 3D Mandelbrot sets in ternary algebra
(a) $m=2$; (b) $m=5$; (c) $m=6$; (d) $m=7$

It is shown in Figs.6a~6d that the bottom surfaces of the 3D Mandelbrot sets generated by ternary maps with the exponents being 2, 5, 6, 7 are made up of 1, 4, 5, 6 integral petals correspondingly, which fully justifies the above-claimed Property 4.

To sum up, the ternary-algebra-based method of generating 3D fractal sets has several advantages over quad-algebra-based ones. Firstly, we could get a better understanding of the complicated structures of these 3D fractal sets by referring to their counterparts in the complex plane. Secondly, fractal sets on the

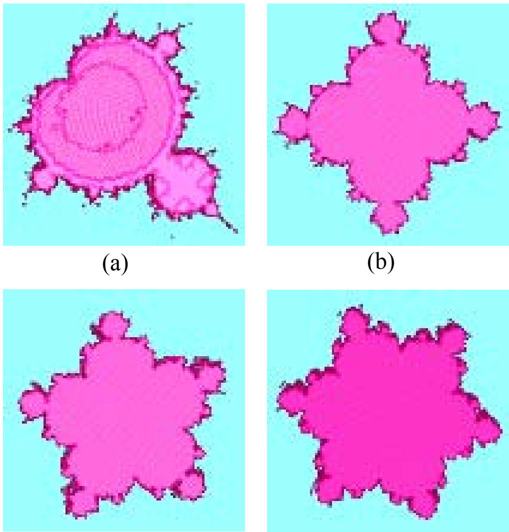


Fig.6 The bottom surfaces of Mandelbrot sets displayed in Fig.5. (a) *m*=2; (b) *m*=5; (c) *m*=6; (d) *m*=7

basis of ternary algebra could be rendered directly in 3D space without any projective operation due to the fact that the three components of a ternary number can be exactly mapped to 3D Cartesian space. As a result, both information loss and randomness in the structures of resulting objects could be completely avoided. Finally, creating 3D fractal sets from ternary algebra is more efficient than from quaternion algebras, in respect that the iterative operations involved in calculating the periodicities of discrete points during the process of generating 3D sets can be greatly reduced. Specifically speaking, the implementation frequency of the key operation in the period-checking procedure has been decreased from $n\cdot 4^m$ to $n\cdot 3^m$ by substituting quaternion algebras with ternary algebra, where n is the number of iteration needed to distinguish whether a point in 3D space belongs to a fractal set and m is the exponent of the map. And it is evident that the larger the map's exponent is, the more predominant the ternary-algebra-based method is.

APPLICATION TO 3D JULIA SETS

As we can see, the method of creating 3D fractal sets by utilizing ternary algebra has many advantages such as intuition, good controllability, expeditiousness and so on. Consequently, we should like to discuss in this section the application of ternary algebra to the generation of 3D Julia sets.

Definition 4 The general Julia sets of ternary maps f: $t\to t^m+c$ (t, $c\in T$, $m\in\mathbb{N}$, $m\geq 2$) contain all the initial points t_0 whose orbits under these maps remain bounded, which can be simplified as

$$J_T=\{t_0\in T:\{f_c^n(t_0)\}_{n=1}^{\infty}\not\to\infty\}. \tag{16}$$

The following properties of J_T in Eq.(16) can be derived from the fundamentals of ternary algebra:

Property 5 The projection of J_T on the plane $x-y+z=0$ is consistent with the 2D Julia set of complex map $z\to z^m+c$ (z, $c\in\mathbb{C}$) with the same exponent m, where $\mathbb{C}$ indicates the complex number system.

Property 6 J_T can be reconstructed through the movement of 2D Julia subset in the plane $x-y+z=l$ along the direction defined by $x=-y=z$, where l is a real number.

Based on Properties 5, 6 as well as the properties of the Julia sets generated by complex maps $z\to z^m+c$ (z, $c\in\mathbb{C}$, $m\in\mathbb{N}$), we can deduce another property of J_T, namely,

Property 7 The bottom surface of J_T, parallel to the plane $x-y+z=0$, consists of m major leaves situated symmetrically around the center of the surface.

Illustrated in Figs.7 and 8 are some examples of 3D general Julia sets in ternary algebra, which have

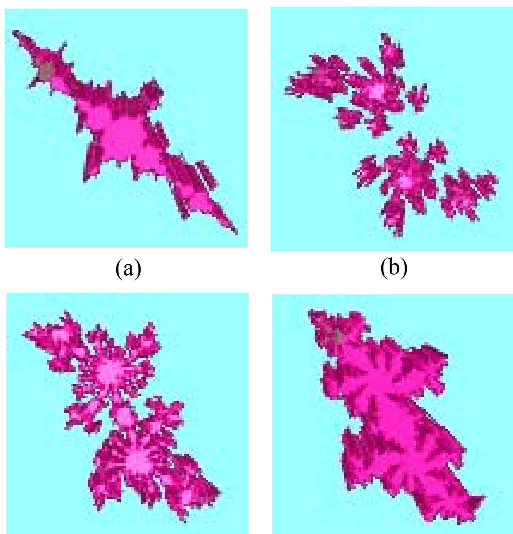
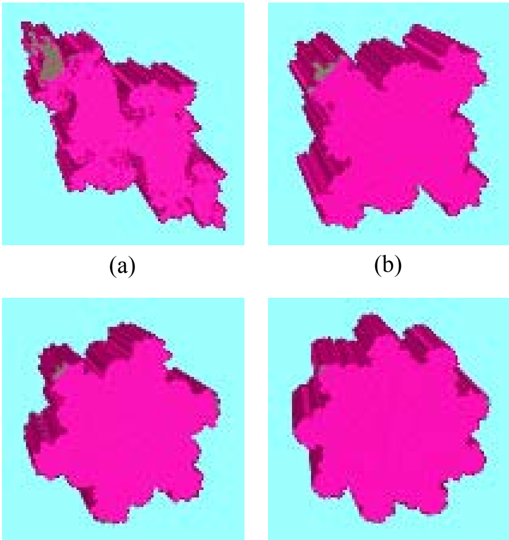


Fig.7 Julia sets of quadric ternary map. (a) *c*=(−0.84, −0.42, 0.42); (b) *c*=(0.07354, −0.35027, −0.42381); (c) *c*=(−0.12933, 0.24924, 0.50791); (d) *c*=(−0.39054, −0.48679, 0.0)



(a) (b) (c) (d)

Fig.8 General Julia sets of ternary maps with *c*=(−0.49695, −0.18323, 0.31372). (a) *m*=2; (b) *m*=4; (c) *m*=6; (d) *m*=9

been oriented for the convenience of observation on their bottom surfaces. It can be observed from these figures that all the Julia sets reveal some kind of columnar shapes with extraordinary rough lateral surfaces. The bottom surfaces of the Julia sets in Figs.7a~7d obtained by setting parameter c in ternary map $t \rightarrow t^2+c$ to four different values are all made up of two centro-symmetrical leaves. And the bottom surfaces of the four Julia sets in Figs.8a~8d generated from ternary maps with exponents being four different integers are made up of corresponding number of major leaves symmetrical about their centers. These experimental results agree well with Properties 5~7.

CONCLUSION

The research on this project began as an endeavor to explore as much as possible the extensions of 3D Mandelbrot and Julia sets. Three new approaches for generating 3D fractal sets by utilizing new forms of algebra were put forward. A ray-casting algorithm based on period checking was summarized before the presentation of these new methods.

The first attempt, i.e., two methods based on discrete modifications of standard quaternion algebra, was motivated by Terry's work. However, both experimental results and theoretical analyses indicated that the methods of generating 3D fractal sets from quad algebras inevitably experienced problems such as difficulty in perceiving the infinite detail of resulting sets and incapability of completely controlling their geometrical structures.

Then, the third method based on ternary algebra as a substitute for quad algebras in the generation of 3D Mandelbrot and Julia sets was proposed for the purpose of getting over disadvantages intrinsic to the quad-algebra-based methods. Properties with regard to the structures of the resulting sets, which were of great benefit for understanding their beauty and intricacies, were deduced on the basis of the fundamentals of ternary algebra. It turned out that not only the problems due to projection were completely avoided but also the computational costs were considerably reduced.

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APPENDIX A: FUNDAMENTALS OF TERNARY ALGEBRA

The conjugate of a ternary number $t=x\mathrm{i}+y\mathrm{j}+z\mathrm{k}$ is

$$\overline{t}=x\mathrm{i}-z\mathrm{j}-y\mathrm{k}. \quad \text{(A1)}$$

Given three ternary numbers t_1, t_2 and t_3, then the following equations hold:

$$t_1+t_2=t_2+t_1,\ (t_1+t_2)+t_3=t_1+(t_2+t_3). \quad \text{(A2)}$$

$$t_1\cdot t_2=t_2\cdot t_1,\ (t_1\cdot t_2)\cdot t_3=t_1\cdot(t_2\cdot t_3),\ t_1\cdot(t_2+t_3)=t_1\cdot t_2+t_1\cdot t_3. \quad \text{(A3)}$$

$$\overline{t_1\pm t_2}=\overline{t_1}\pm\overline{t_2},\ \overline{t_1\cdot t_2}=\overline{t_1}\cdot\overline{t_2}. \quad \text{(A4)}$$

For a ternary number $t=x\mathrm{i}+y\mathrm{j}+z\mathrm{k}$, there is an eigenvalue that can be given as

$$\begin{aligned} t_{\mathrm{e}}&=p\mathrm{i}+q\mathrm{j}+r\mathrm{k}\\ &=\left(x+\frac{1}{2}y-\frac{1}{2}z\right)\mathrm{i}+\frac{\sqrt{3}}{2}(y+z)\mathrm{j}+(x-y+z)\mathrm{k}, \end{aligned} \quad \text{(A5)}$$

where the binary number $p\mathrm{i}+q\mathrm{j}$ and the unitary number r are respectively referred to as binary eigenvalue and unitary eigenvalue of the ternary number t.

In 3D arithmetic space, those points with zero binary eigenvalues constitute a line defined by

$$x=-y=z, \quad \text{(A6)}$$

while those points with zero unitary eigenvalues constitute a plane defined by

$$x-y+z=0. \quad \text{(A7)}$$

Assuming $p_1\mathrm{i}+q_1\mathrm{j}+r_1\mathrm{k}$ and $p_2\mathrm{i}+q_2\mathrm{j}+r_2\mathrm{k}$ to be two eigenvalues of some ternary numbers, then the following operational formulas hold:

$$\begin{aligned}&(p_1\mathrm{i}+q_1\mathrm{j}+r_1\mathrm{k})\pm(p_2\mathrm{i}+q_2\mathrm{j}+r_2\mathrm{k})\\ &\quad=(p_1\pm p_2)\mathrm{i}+(q_1\pm q_2)\mathrm{j}+(r_1\pm r_2)\mathrm{k},\end{aligned} \quad \text{(A8)}$$

$$\begin{aligned}&(p_1\mathrm{i}+q_1\mathrm{j}+r_1\mathrm{k})\cdot(p_2\mathrm{i}+q_2\mathrm{j}+r_2\mathrm{k})\\ &\quad=(p_1p_2-q_1q_2)\mathrm{i}+(p_1q_2+q_1p_2)\mathrm{j}+r_1r_2\mathrm{k},\end{aligned} \quad \text{(A9)}$$

$$\begin{aligned}&(p_1\mathrm{i}+q_1\mathrm{j}+r_1\mathrm{k})\div(p_2\mathrm{i}+q_2\mathrm{j}+r_2\mathrm{k})\\ &=[(p_1p_2+q_1q_2)\mathrm{i}+(q_1p_2-p_1q_2)\mathrm{j}]/(p_2^2+q_2^2)+(r_1/r_2)\mathrm{k}.\end{aligned} \quad \text{(A10)}$$

As for an eigenvalue of the form $p\mathrm{i}+q\mathrm{j}+r\mathrm{k}$, its converse is defined as

$$1/(p\mathrm{i}+q\mathrm{j}+r\mathrm{k})=(p\mathrm{i}-q\mathrm{j})/(p^2-q^2)+(1/r)\mathrm{k}, \quad \text{(A11)}$$

and its conjugate is

$$\overline{p\mathrm{i}+q\mathrm{j}+r\mathrm{k}}=p\mathrm{i}-q\mathrm{j}+r\mathrm{k}. \quad \text{(A12)}$$

Analogous to the absolute value of a complex number, the norm of the eigenvalue $p\mathrm{i}+q\mathrm{j}+r\mathrm{k}$ is defined by

$$|p\mathrm{i}+q\mathrm{j}+r\mathrm{k}|=r(p^2+q^2). \quad \text{(A13)}$$

As we can see from Eqs.(A5)~(A13) that a ternary number $t=x\mathrm{i}+y\mathrm{j}+z\mathrm{k}$ practically represents one binary number $p\mathrm{i}+q\mathrm{j}$ and one unitary number r simultaneously and that the operation of a ternary algebra is equal to the operation of both its binary eigenvalue and its unitary eigenvalue.

APPENDIX B: PROOF OF PROPERTY 1

Given three functions of ternary number t, $f(t)=t^m+c$, $g(t)=t^m+\overline{c}$, $h(t)=\overline{t}$, according to Eq.(A4), we have:

$$h\cdot f(t)=\overline{t^m+c}=\overline{t^m}+\overline{c}=(\overline{t})^m+\overline{c},$$

$$g\cdot h(t)=(\overline{t})^m+\overline{c}.$$

Therefore $h\cdot f(t)=g\cdot h(t)$. That is, if $c\in M_T$, then $\overline{c}\in M_T$.

Let $c=x\mathrm{i}+y\mathrm{j}+z\mathrm{k}$, then according to the definition of conjugate in Eq.(A1), we get $\overline{c}=x\mathrm{i}-z\mathrm{j}-y\mathrm{k}$, so the following equations can be obtained:

$$c+\overline{c}=2x\mathrm{i}+(y-z)\mathrm{j}+(z-y)\mathrm{k},$$

$$c-\overline{c}=(y+z)\mathrm{j}+(z+y)\mathrm{k}.$$

Hence, c and $\overline{c}$ are symmetrical about the plane $y+z=0$. Consequently, the 3D Mandelbrot sets of the ternary maps $t\to t^m+c$ ($t, c\in T$, $m\in\mathbb{N}$, $m\geq 2$) are bound to be symmetrical about the plane $y+z=0$.