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Conservation laws for energy and momentum in curved spaces

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Abstract: In arbitrary Riemannian 4-spaces, continuity equations are constructed which could be interpreted as conservation laws for the energy and momentum of the gravitational field. Special attention is given to general relativity to obtain, of natural manner, the pseudotensors of Einstein, Landau-Lifshitz, Möller, Goldberg and Stachel, and also the conservation equations of Komar, Trautman, DuPlessis and Moss.

Key words: Noether theorem, Lagrangians in curved spaces, Energy and momentum in Riemannian spaces, Rund-Lovelock identities
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INTRODUCTION

As general relativity is a physical theory, the question of energy and momentum arises in a natural way, and has been the subject of considerable investigation dating back almost to the inception of the theory itself. Attempts at identifying an energy-momentum density for gravity, however, led only to various energy-momentum complexes which are not covariant objects (pseudotensors) because they inherently depend on the reference frame, and thus by their very nature cannot provide a truly physical local gravitational energy-momentum density. Indeed any such quantity is precluded by the equivalence principle itself, since a gravitational field should not be detectable at a point.

The principal aim of this work is to use the Noether theorem to obtain, in a unified form, several important results on gravitational energy-momentum quantities scattered in the literature which were deduced through different methods. Other conservation laws, as the spin for example, are not considered although they represent an interesting tool in the general treatment of the topic.

For Lagrangians-based theories we exploit from the very beginning the transformation properties of fields. In this paper, we consider gravitational Lagrangians:

$$L = L(g_{ab}; g_{ab,c}; g_{ab,cd}), \tag{1}$$

where g_{ij} is the metric tensor and $a=\partial/\partial x^a$, L is a scalar density of weight one (Lovelock and Rund, 1975) under an arbitrary coordinate transformation $\overline{x}^i = \overline{x}^i(x^j)$:

$$L = J\left(\frac{\overline{x}}{x}\right)\overline{L}. \tag{2}$$

Here we shall employ the property (2) to obtain (in general relativity) in natural manner the energy-momentum pseudotensors of Einstein (Einstein, 1916; Trautman, 1962; Adler *et al*., 1965; Davis, 1974; Dirac, 1975; Persides, 1979; Palmer, 1980), Landau-Lifshitz (Landau and Lifshitz, 1955; Anderson, 1967; Misner *et al*., 1973; Synge, 1976), Möller (Möller, 1958; Laurent, 1959; Florides, 1962; Shah, 1967), Goldberg (1958) and Stachel (1977), and also the continuity equations of Komar (1959), Trautman (1964), Du Plessis (1969) and Moss (1972).

In Section 2, we indicate the notation and quantities we will employ throughout the paper. Section 3 is dedicated to the analysis of conservation laws originated from Eqs.(1), (2) and the Hilbert's variational principle (Rund, 1966a; Rund and Lovelock, 1972; Misner *et al.*, 1973; Lovelock and Rund, 1975):

$$\delta\int_{V_4} L\sqrt{-g}\mathrm{d}^4x = 0. \tag{3}$$

We also mention applications of L in the case of general relativity (Adler *et al.*, 1965; Anderson, 1967; Schild, 1967; Davis, 1974; Dirac, 1975).

RUND-LOVELOCK EXPRESSIONS

Rund and Lovelock in their study of variational principles with Lagrangians verifying Eqs.(1) and (2) showed the importance of the derivatives of L with respect to its arguments (Rund, 1966a; Rund and Lovelock, 1972; Lovelock and Rund, 1975):

$$\left.\begin{aligned} &A^{ij} = A^{ji} \equiv \partial L/\partial g_{ij},\\ &A^{ij,h} = A^{ji,h} \equiv \partial L/\partial g_{ij,h},\\ &A^{ij,hk} = A^{ji,hk} = A^{ij,kh} \equiv \partial L/\partial g_{ij,hk},\end{aligned}\right\} \tag{4}$$

and, in general, only $A^{ij,hk}$ has tensorial character. These quantities have the following properties:

$$\left.\begin{aligned} &A^{ij} = \frac{L}{2}g^{ij} + \frac{4}{3}A^{hk,jr}R^i_{khr} - \Gamma^i_{rn}\Gamma^r_{km}A^{km,rh},\\ &A^{ij,h} = \Gamma^i_{rk}A^{rk,jh} + \Gamma^j_{rk}A^{rk,ih} - \Gamma^h_{rk}A^{rk,ij},\\ &A^{ij,hk} = A^{hk,ij},\ A^{i,jh}_{r\ \ hji} = 0,\\ &A^{ij,hk} + A^{ih,kj} + A^{ik,jh} = 0,\end{aligned}\right\} \tag{5}$$

with the convention of sum performed over repeated suffixes.

On the other hand, the Euler-Lagrange expressions defined by

$$L^{ij} = A^{ij} - A^{ij,h}{}_{,h} + A^{ij,hk}{}_{,hk} \tag{6}$$

can be written using Eq.(5), in the form:

$$L^{ij} = \frac{L}{2}g^{ij} + \frac{2}{3}A^{hk,jr}R^i_{khr} + A^{ij,hk}{}_{;hk}, \tag{7}$$

where ;c represents the covariant derivative. Besides, it is always valid the contracted Bianchi identities:

$$L^{ij}{}_{;j} = 0. \tag{8}$$

As an example, if L is the scalar density of weight one corresponding to general relativity (Adler *et al.*, 1965; Anderson, 1967; Schild, 1967; Davis, 1974; Dirac, 1975):

$$L = \sqrt{-g}R, \quad g = \det(g_{ab}). \tag{9}$$

Assuming $R \equiv R^{ij}{}_{ji}$, the scalar curvature:

$$A^{ij,km} = \frac{1}{2}\sqrt{-g}(2g^{ij}g^{km} - g^{im}g^{jk} - g^{ik}g^{jm}). \tag{10}$$

Thus, Eqs.(7) and (10) imply that the Euler-Lagrange relations are proportional to the Einstein tensor:

$$L^{ij} = -\sqrt{-g}G^{ij}, \tag{11}$$

which satisfies Eq.(8) because $G^{ab}{}_{;b} = 0$.

CONTINUITY EQUATIONS IN RIEMANNIAN SPACES

The variational principle (3) is invariant under general transformations $\overline{x}^i = \overline{x}^i(x^j)$, in particular, we can use infinitesimal coordinate changes

$$\overline{x}^i = x^i + \varepsilon^i\xi^i(x^i) \tag{12}$$

without sum over i, and ε^i denoting small constant parameters. The Noether theorem (Noether, 1918; Weyl, 1935; Rund, 1966b; Trautman, 1967; Kimberling, 1972; Byers, 1996a; 1996b) establishes that each continuous symmetry transformation (Wigner, 1967) which leaves the corresponding variational principle invariant, implies a conservation law; and hence, a constant of motion. Here we employ the Noether theorem via the approach of (Lanczos, 1969; 1970; 1973): we apply Eq.(12) to Eq.(3) but now considering that the ε^i are new variational variables, the Lagrange equations for ε^i lead us to continuity equations of Noether. Thus, it is possible to deduce the following important relations not found explicitly in the literature on gravitational energy-momentum pseudotensors (Synge, 1967; Bak *et al.*, 1994; Chang *et al.*, 1999; Babak and Grishchuk, 2000):

$$\left(B_r{}^i\xi^r - U_r{}^{ij}\xi^r{}_{,j} - \frac{1}{2}A_r{}^{i,jh}\xi^r{}_{,jh}\right)_{,i} = 0, \qquad (13)$$

where

$$U_r{}^{ij} = A_r{}^{i,jh}{}_{;h} + \Gamma_{krh}A^{kj,ih} - \frac{1}{2}\Gamma^i{}_{hk}A_r{}^{j,hk}, \qquad (14)$$

$$B_r{}^i = U_r{}^{ij}{}_{,j} = -\frac{L}{2}\delta_r^i + L_r{}^i + \frac{1}{2}(A^{jh,i} - A^{jh,ki}{}_{,k})g_{jh,r} + \frac{1}{2}A^{hk,ij}g_{kh,jr}. \qquad (15)$$

Besides, with Eqs.(8) and (15), it is easy to obtain the conservation law:

$$B_r{}^i{}_{,i} \equiv L_r{}^i{}_{;i} = 0. \qquad (16)$$

In the case of general relativity theory, L is given by Eq.(9), and a complete study of Eq.(13) when ξ^r is a vectorial field leads to results of Komar (1959) and Du Plessis (1969), and if ξ^r is a killing vector, then it is also possible to deduce the relations of Trautman (1964) and Moss (1972). On the other hand, Eq.(16) allows us to construct the energy-momentum pseudotensors of Landau-Lifshitz (Landau and Lifshitz, 1955; Anderson, 1967; Misner *et al.*, 1973; Synge, 1976), Möller (Möller, 1958; Laurent, 1959; Florides, 1962; Shah, 1967), Goldberg (1958) and Stachel (1977).

Sometimes in the Einstein theory, we use the Langrangian (Adler *et al.*, 1965; Anderson, 1967; Misner *et al.*, 1973; Dirac, 1975):

$$\overline{L} = \sqrt{-g}g^{ab}(\Gamma^i{}_{ab}\Gamma^j{}_{ij} - \Gamma^i{}_{ja}\Gamma^j{}_{ib}), \qquad (17)$$

such that $\sqrt{-g}R = \overline{L} + (\textit{ordinary divergence})$, then the empty field equations are the same for Eqs.(9) and (17), besides $\partial\overline{L}/\partial g_{ij,hk} = 0$. However, $\overline{L}$ satisfies Eq.(2) when ξ^r are constants, therefore Eq.(13) is equivalent to $B^i_{r,i} = 0$ and from Eq.(15):

$$B_r{}^i = 8\pi\sqrt{-g}t_r{}^i = \frac{1}{2}\left(\frac{\partial\overline{L}}{\partial g_{jh,i}}g_{jh,r} - \overline{L}\delta_r^i\right), \qquad (18)$$

where t_r^i is the canonical energy-momentum pseudotensor of Einstein (Einstein, 1916; Trautman, 1962; Adler *et al.*, 1965; Davis, 1974; Dirac, 1975; Persides, 1979; Palmer, 1980). Thus, the conservation law $t^i_{r,i} = 0$ is implied by the translational invariance of $\overline{L}$.

CONCLUSION

Our analysis shows that the Lanczos approach (Lanczos, 1969; 1970; 1973) to the Noether theorem gives, in a unified form, the principal gravitational continuity equations in Riemannian spaces. A possible consequence, in the thematic of conservation laws, of the methodology here exhibited is its application to quantum field theories in curved spacetimes.

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