Journal of Zhejiang University SCIENCE A
ISSN 1673-565X (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Congruences for finite triple harmonic sums*

FU Xu-dan[†1,2], ZHOU Xia[1], CAI Tian-xin[1]

([1]Department of Mathematics, Zhejiang University, Hangzhou 310028, China)

([2]Hangzhou Foreign Language School, Hangzhou 310023, China)

[†]E-mail: fuxudan@126.com

Received Sept. 19, 2006; revision accepted Jan. 18, 2007

Abstract: Zhao (2003a) first established a congruence for any odd prime $p>3$, $S(1,1,1;p)\equiv-2B_{p-3}$ (mod p), which holds when $p=3$ evidently. In this paper, we consider finite triple harmonic sum $S(\alpha,\beta,\gamma;p)$ (mod p) is considered for all positive integers α,β,γ. We refer to $w=\alpha+\beta+\gamma$ as the weight of the sum, and show that if w is even, $S(\alpha,\beta,\gamma;p)\equiv0$ (mod p) for $p\geq w+3$; if w is odd, $S(\alpha,\beta,\gamma;p)\equiv rB_{p-w}$ (mod p) for $p\geq w$, here r is an explicit rational number independent of p. A congruence of Catalan number is obtained as a special case.

Key words: Finite triple harmonic sums, Recursive relation, Bernoulli numbers, Catalan numbers
doi:10.1631/jzus.2007.A0946 **Document code:** A **CLC number:** O156

INTRODUCTION

The Bernoulli numbers B_k are defined by the recursive relation (Graham *et al.*, 1994):

$$\sum_{i=0}^{n}\binom{n+1}{i}B_i=0,\quad n\geq1.$$

It is well known that $B_{2k+1}=0$ for $k\geq1$, and $B_0=1$, $B_1=-1/2$, $B_2=1/6$, $B_4=-1/30$, etc. Several researchers studied the congruences of Bernoulli numbers. Using partial sum of multiple zeta value series, Zhao (2003a) first established the congruence for any odd prime $p>3$,

$$\sum_{\substack{i+j+k=p\\ i,j,k>0}}\frac{1}{ijk}\equiv-2B_{p-3}\pmod{p},\qquad(1)$$

which holds when $p=3$ evidently, and remarked that it would be very interesting to find a direct proof of it. An elementary proof of this congruence was given by Ji (2005). Naturally, we define and study the following finite triple harmonic sum

$$S(\alpha,\beta,\gamma;n)=\sum_{\substack{i+j+k=n\\ i,j,k>0}}\frac{1}{i^{\alpha}j^{\beta}k^{\gamma}},$$

where $n\geq3$ and $\alpha,\beta,\gamma\in\mathbb{N}^*$. In this paper, we consider $n=p$ to be prime, and assume $\alpha\leq\beta\leq\gamma$ since they are symmetric. Unfortunately, we could not use Ji's method directly. By dealing with finite triple harmonic sums with some elementary techniques, i.e., the properties of Bernoulli numbers and the recursive method, we get the main theorem as follows:

Theorem 1 Let p be prime.

(1) If w is even and $p\geq w+3$, then

$$S(\alpha,\beta,\gamma;p)\equiv0\pmod{p}.$$

(2) If w is odd and $p\geq w$, then

$$S(\alpha,\beta,\gamma;p)\equiv rB_{p-w}\pmod{p},\qquad(2)$$

where

$$r=r(\alpha,\beta,\gamma)=\frac{1}{w}\left\{(-1)^{\alpha}\sum_{i=0}^{\beta-1}\binom{-\alpha}{i}\binom{w}{\beta-i}+(-1)^{\beta}\sum_{i=0}^{\alpha-1}\binom{-\beta}{i}\binom{w}{\alpha-i}\right\}.$$

* Project (No. 10371107) supported by the National Natural Science Foundation of China

Corollary Let n be any positive integer, then

$$S(2n-1,2n,2n;p)\equiv 0 \pmod p \text{ for prime } p\geq 6n-1,$$

and for prime $p\geq 6n+1$,

$$S(2n,2n,2n+1;p)\equiv C_{2n}B_{p-6n-1} \pmod p,$$

where $C_k=\frac{1}{k+1}\binom{2k}{k}$ is the kth Catalan number.

AUXILIARY LEMMAS

Lemma 1 (Zhao, 2003b; Hoffman, 2004) Let s, t be positive integers, p be odd prime and $p\geq s+t$, then

$$H(s,t;p-1)\equiv\frac{(-1)^t}{s+t}\binom{s+t}{s}B_{p-s-t} \pmod p,$$

where

$$H(s,t;p-1)=\sum_{1<i<j<p-1}\frac{1}{i^s j^t}.$$

Lemma 2 Let p be prime, α,β,γ be positive integers, then

$$S(\alpha,\beta,\gamma;p)\equiv -S(\alpha-1,\beta,\gamma+1;p)-S(\alpha,\beta-1,\gamma+1;p) \pmod p$$

Proof Noting that $k\equiv-(i+j) \pmod p$ and multiplying the numerator and divisor of the summand by $i+j$, we have

$$S(\alpha,\beta,\gamma;p)\equiv(-1)^\gamma\sum_{\substack{i+j<p\\ i,j>0}}\frac{1}{i^{\alpha-1}j^\beta(i+j)^{\gamma+1}}+(-1)^\gamma\sum_{\substack{i+j<p\\ i,j>0}}\frac{1}{i^\alpha j^{\beta-1}(i+j)^{\gamma+1}} \pmod p.$$

Replacing $i+j$ with $p-i-j=k$, then

$$(-1)^\gamma\sum_{\substack{i+j<p\\ i,j>0}}\frac{1}{i^{\alpha-1}j^\beta(i+j)^{\gamma+1}}\equiv-\sum_{\substack{i+j+k=p\\ i,j,k>0}}\frac{1}{i^{\alpha-1}j^\beta k^{\gamma+1}}=-S(\alpha-1,\beta,\gamma+1;p) \pmod p. \quad (3)$$

Similarly, we have

$$(-1)^\gamma\sum_{\substack{i+j<p\\ i,j>0}}\frac{1}{i^\alpha j^{\beta-1}(i+j)^{\gamma+1}}=-S(\alpha,\beta-1,\gamma+1;p) \pmod p. \quad (4)$$

Combining Eq.(3) with Eq.(4) we derive Lemma 2.

PROOF OF THEOREM 1

We prove it by induction.

(1) If $\alpha=1$, using the technique in the proof of Lemma 2, we have

$$S(1,\beta,\gamma;p)\equiv(-1)^\gamma\sum_{\substack{i+j<p\\ i,j>0}}\frac{1}{j^\beta(i+j)^{\gamma+1}}-S(1,\beta-1,\gamma+1;p) \pmod p. \quad (5)$$

Substituting $i+j$ by k, we have

$$\sum_{\substack{i+j<p\\ i,j>0}}\frac{1}{j^\beta(i+j)^{\gamma+1}}=\sum_{1\leq j<k\leq p-1}\frac{1}{j^\beta k^{\gamma+1}}=H(\beta,\gamma+1;p-1).$$

By Lemma 1, if $p\geq\beta+\gamma+4$ (here $\beta+\gamma+1$ is even), we have

$$H(\beta,\gamma+1;p-1)\equiv 0 \pmod p.$$

Since $B_{p-\beta-\gamma-1}=0$, then Eq.(5) becomes

$$S(1,\beta,\gamma;p)\equiv -S(1,\beta-1,\gamma+1;p) \pmod p.$$

Repeating the steps, one has

$$\begin{aligned}S(1,\beta,\gamma;p)&\equiv -S(1,\beta-1,\gamma+1;p)\\ &\equiv S(1,\beta-2,\gamma+2;p)\equiv\ldots\\ &\equiv(-1)^{\beta-1}S(1,1,\beta+\gamma-1;p)\\ &\equiv 2(-1)^{2\beta+\gamma-2}H(1,\beta+\gamma;p-1)\\ &\equiv 0 \pmod p.\end{aligned}$$

This indicates that (1) is true when $\alpha=1$. Now we assume (1) holds for smaller α, then by Lemma 2 we derive

$$S(\alpha,\beta,\gamma;p)\equiv -S(\alpha-1,\beta,\gamma+1;p)-S(\alpha,\beta-1,\gamma+1;p) \pmod p$$

By the inductive assumption, one has

$$\begin{aligned} S(\alpha,\beta,\gamma;p) &\equiv -S(\alpha,\beta-1,\gamma+1;p) \equiv ... \\ &\equiv (-1)^{\beta-1} S(\alpha,1,\beta+\gamma-1;p) \\ &\equiv 2(-1)^{2\beta+\gamma-2} H(\alpha,\beta+\gamma;p-1) \\ &\equiv 0 \ (\text{mod}\ p). \end{aligned}$$

This completes the proof of (1).

(2) If $\alpha=\beta=1$, $\gamma\geq1$ is odd, by the recursive relation of Lemma 2, we have

$$\begin{aligned} S(1,1,\gamma;p) &\equiv (-1)^{\gamma} \cdot 2H(1,\gamma+1;p-1) \\ &\equiv (-1)^{\gamma} \cdot 2 \cdot \frac{(-1)^{\gamma+1}}{\gamma+2} \binom{\gamma+2}{1} B_{p-\gamma-2} \\ &= -2B_{p-\gamma-2} \ (\text{mod}\ p), \end{aligned}$$

which shows Eq.(2) is true for $\alpha=\beta=1$. Now we assume Eq.(2) holds for smaller weight w. By the well-known result

$$\binom{-\beta}{i-1} + \binom{-\beta}{i} = \binom{-\beta+1}{i},$$

we have

$$\sum_{i=1}^{\alpha-1} \binom{-\beta}{i} \binom{w}{\alpha-i} = \sum_{i=1}^{\alpha-1} \binom{-\beta+1}{i} \binom{w}{\alpha-i} - \sum_{i=1}^{\alpha-1} \binom{-\beta}{i-1} \binom{w}{\alpha-i}. \quad (6)$$

Since $\binom{-\beta}{0} = 1$, Eq.(6) is equivalent to

$$\sum_{i=0}^{\alpha-1} \binom{-\beta}{i} \binom{w}{\alpha-i} = \sum_{i=0}^{\alpha-1} \binom{-\beta+1}{i} \binom{w}{\alpha-i} - \sum_{i=0}^{\alpha-2} \binom{-\beta}{i} \binom{w}{\alpha-1-i}.$$

Similarly, we could obtain

$$\sum_{i=0}^{\beta-1} \binom{-\alpha}{i} \binom{w}{\beta-i} = \sum_{i=0}^{\beta-1} \binom{-\alpha+1}{i} \binom{w}{\beta-i} - \sum_{i=0}^{\beta-2} \binom{-\alpha}{i} \binom{w}{\beta-1-i}.$$

Hence,

$$-r(\alpha-1,\beta,\gamma+1) - r(\alpha,\beta-1,\gamma+1) = r(\alpha,\beta,\gamma).$$

By the inductive assumption and the recursive relation of Lemma 2, we get

$$\begin{aligned} S(\alpha,\beta,\gamma;p) &\equiv -S(\alpha-1,\beta,\gamma+1;p) - S(\alpha,\beta-1,\gamma+1;p) \\ &= r(\alpha,\beta,\gamma) B_{p-w} \ (\text{mod}\ p). \end{aligned}$$

This completes the proof of Theorem 1.

As for the proof of the corollary, we only need the following identity:

$$\sum_{i=0}^{q} \binom{x}{q-i} \binom{y}{i} = \binom{x+y}{q}$$

for any integers x, y and $q\geq0$.

ACKNOWLEDGEMENT

The authors would like to thank the referees for their constructive reports and valuable suggestions.

References

Graham, R.L., Knuth, D.E., Patashnik, O., 1994. Concrete Mathematics (2nd Ed.). Addison-Wesley.

Hoffman, M.E., 2004. Quasi-symmetric Functions and Mod *p* Multiple Harmonic Sums. Http://arxiv.org/abs/math.NT/0401319

Ji, C.G., 2005. A simple proof of a curious congruence by Zhao. *Proc. Amer. Math. Soc.*, **133**:3469-3472. [doi:10.1090/S0002-9939-05-07939-6]

Zhao, J.Q., 2003a. Bernoulli Numbers, Wolstenholme's Theorem, and p^5 Variations of Lucas' Theorem. Http://arxiv.org/abs/math.NT/0303332, V1.

Zhao, J.Q., 2003b. Partial Sums of Multiple Zeta Value Series I: Generalizations of Wolstenholme's Theorem. Http://arxiv.org/abs/math.NT/0301252, V2.