Journal of Zhejiang University SCIENCE A
ISSN 1673-565X (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Some stationary weak solutions to inhomogeneous Landau-Lifshitz equations in three dimensions*

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Received Oct. 7, 2006; revision accepted Jan. 17, 2007

Abstract: In this paper, we describe several stationary conditions on weak solutions to the inhomogeneous Landau-Lifshitz equation, which ensure the partial regularity. For certain class of proper stationary weak solutions, a compactness result of the solutions, a finite Hausdorff measure result of the *t*-slice energy concentration sets and an asymptotic limit result of the Radon measures are proved. We also present a subtle rectifiability result for the energy concentration set of certain sequence of strong stationary weak solutions.

Key words: Landau-Lifshitz equation, Stationary condition, Hausdorff measure
doi:10.1631/jzus.2007.A0949 **Document code:** A **CLC number:** O175; O48

INTRODUCTION

Let $\Omega\subset\mathbb{R}^m$ be an open set and $0<T<\infty$. We consider the solutions $u=(u_1, u_2, u_3):\Omega\times(0,T)\to S^2$ of the inhomogeneous Landau-Lifshitz equation (Landau and Lifshitz, 1935; Daniel *et al.*, 1994; Tang, 2001; Li and Wang, 2006)

$$\begin{aligned}\partial_t u = &-\alpha u\wedge[u\wedge(f\Delta u+\nabla f\cdot\nabla u)]\\ &-\beta u\wedge(f\Delta u+\nabla f\cdot\nabla u),\end{aligned} \tag{1}$$

where $\alpha>0$ and $\beta\in\mathbb{R}$ are given constants and f is a real coupling function on Ω. Here S^2 is the unit 2-sphere, and '$\wedge$' (resp. '$\cdot$') denotes the exterior (resp. inner) product in $\mathbb{R}^3$ (resp. $\mathbb{R}^m$). Throughout this paper, we assume that $m=3$, $\alpha^2+\beta^2=1$, and that f is smooth and positive on the closure of Ω. We also introduce a convenient abbreviation as follows: For $p\in S^2$, denote by $R_p:T_pS^2\to T_pS^2$ the rotation $R_pv=\alpha v+\beta p\wedge v$. It is easy to see that, for classical solutions, Eq.(1) is equivalent to

$$R_u\partial_t u = f\Delta u + f|\nabla u|^2 u + \nabla f\cdot\nabla u. \tag{2}$$

In this paper, we are interested in the weak solutions to Eq.(2) (Alouges and Soyeur, 1992). Define

$$\begin{aligned}&W^1(\Omega\times(0,T),S^2)\\ &=\{u\in H^1_{\text{loc}}(\Omega\times(0,T),\mathbb{R}^3):\partial_t u\in L^2(\Omega\times(0,T)),\\ &\quad\nabla u\in L_t^\infty L_x^2(\Omega\times(0,T))\text{ and }|u|=1\text{ a.e.}\}.\end{aligned}$$

We say a map $u\in W^1(\Omega\times(0,T),S^2)$ is a weak solution of Eq.(2), if the following hold:

(1) For any $\phi\in C_0^\infty(\Omega\times(0,T),\mathbb{R}^3)$,

$$\int_0^T\int_\Omega\left\langle R_u\partial_t u - f|\nabla u|^2u,\phi\right\rangle+\left\langle f\partial_i u,\partial_i\phi\right\rangle\mathrm{d}x\mathrm{d}t=0$$

holds, where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbb{R}^3$ w.r.t. the target space indices. In this paper we sum over repeated Greek indices from 1 to m.

(2) For any $t>0$, the global energy inequality holds,

$$\frac{1}{2}\int_{\Omega\times\{t\}}f|\nabla u|^2\,\mathrm{d}x+\alpha\int_0^t\int_\Omega|\partial_t u|^2\,\mathrm{d}z\le\frac{1}{2}\int_{\Omega\times\{0\}}f|\nabla u|^2\,\mathrm{d}x. \tag{3}$$

* Project supported by the National Natural Science Foundation of China (No. 10571158) and the Natural Science Foundation of Zhejiang Province, China (No. Y605076)

Now, we will present some symbols: C is used to denote some "irrelevant" positive constant, and $c(X)$ to denote "relevant" positive constant only depending on X. We denote by $z=(x,t)$ a point in $\mathbb{R}^m\times\mathbb{R}$. For $z_0=(x_0,t_0)$, $r>0$, let

$$B_r(x_0)=\{x\in\mathbb{R}^m:|x-x_0|<r\},$$
$$P_r(z_0)=\{z=(x,t):|x-x_0|<r,|t-t_0|<r^2\},$$
$$P_r^+(z_0)=\{z=(x,t):|x-x_0|<r,0<t-t_0<r^2\},$$

and, for simplicity, $X_r=X_r(0)$. We also denote the f-energy at time t by

$$E_f(u,t)=\frac{1}{2}\int_{\Omega\times\{t\}}f|\nabla u|^2\,\mathrm{d}x.$$

For $B_r(x_0)\subset\Omega$ and $P_r(z_0)\subset\Omega\times(0,T)$, we introduce

$$\mathcal{E}_{x_0}^b(u,t,r)=r^{2-m}\int_{B_r(x_0)\times\{t\}}f|\nabla u|^2\,\mathrm{d}x,$$
$$\mathcal{E}_{z_0}^p(u,r)=r^{-m}\int_{P_r(z_0)}f|\nabla u|^2\,\mathrm{d}z,$$
$$\mathcal{I}_{x_0}^b(u,t,r)=r^{4-m}\int_{B_r(x_0)\times\{t\}}|\partial_t u|^2\,\mathrm{d}x,$$
$$\mathcal{I}_{z_0}^p(u,r)=r^{2-m}\int_{P_r(z_0)}|\partial_t u|^2\,\mathrm{d}z,$$
$$I_{x_0}(u,t,r)=r^{2-m}\int_{B_r(x_0)\times\{t\}}|\partial_t u|^2\,\mathrm{d}x,$$
$$E_{z_0}(u,r)=r^{2-m}\int_{P_r(z_0)}f|\nabla u|^2\,\mathrm{d}z.$$

One of our main results, generalizing that of (Moser, 2002), is the following:

Theorem 1 For $m=3$, let $u\in W^1(\Omega\times(0,T),S^2)$ be a weak solution of Eq.(2) satisfying the stationary condition (Definition 1). There exists a universal constant $\varepsilon_0>0$, such that, if we write

$$\mathcal{S}=\{z_0:\liminf_{r\to 0}\mathcal{I}_{z_0}^p(u,r)\geq\varepsilon_0\},\text{ and}$$
$$\mathcal{R}=\Omega\times(0,T)\backslash\mathcal{S},$$

then $\mathcal{S}$ is a relatively closed subset with vanishing m-dimensional parabolic Hausdorff measure, and $u\in C^\infty(\mathcal{R},S^2)$, that is, $\mathcal{R}=\mathrm{reg}\,u$.

For certain sequence of weakly converging proper stationary weak solutions, we have the following result, which is related to the "H^1-norm" compactness problem. The set $W_\Lambda\subset W^1(\Omega\times(0,T),S^2)$ for $\Lambda>0$ is defined in Section 3.

Theorem 2 (Weak closure of W_Λ) For $m=3$ and $\Lambda>0$, suppose $\{u_i\}_{i=1}^\infty\subset W_\Lambda$ is a sequence of weak solutions of Eq.(2). Then there exist a subsequence u_{i_k} and a map $u\in W^1(\Omega\times(0,T),S^2)$ such that

(1) $u_{i_k}\to\infty$ weakly in $W^1(\Omega\times(0,T),S^2)$ as $i_k\to\infty$ (Such a weak convergence is defined in Section 3).

(2) u solves Eq.(2) in the classical sense on a dense open set $\mathcal{R}_0\subset\Omega\times(0,T)$ whose complement, the energy concentration set Σ defined by Eq.(10), has locally finite m-dimensional parabolic Hausdorff measure. Moreover, u is smooth on $\mathcal{R}_0$.

(3) The t-slice set Σ^t of Σ is a closed subset of Ω, with finite $(m-2)$-dimensional Hausdorff measure. Moreover, $\mathcal{H}^{m-2}(\Sigma^t)\leq c(\Lambda)$ for all $0<t<T$.

Statement (3) in Theorem 2 suggests that the energy concentration set has a better measurable property on spatial direction than on time direction.

For the subsequence (still denoted by $\{u_i\}$) in Theorem 2, we introduce the Radon measures $\mu_i=f|\nabla u_i|^2\mathrm{d}z$, $\nu_i=|\partial_t u_i|^2\mathrm{d}z$ on $\Omega\times(0,T)$, and $\mu_i^t=f|\nabla u_i|^2(t)\mathrm{d}x$ on $\Omega\times\{t\}$. The next theorem describes the asymptotic limits of these measures.

Theorem 3 (Structure of Radon measures) Suppose μ_i, ν_i and μ_i^t are defined as above. Then there exist a subsequence i_k of i and nonnegative Radon measures μ, ν, η, ν_* on $\Omega\times(0,T)$ and the time slices μ^t, η^t on $\Omega\times\{t\}$ such that

(1) $\mu_{i_k}\to\mu$, $\nu_{i_k}\to\nu$ and $\mu_{i_k}^t\to\mu^t$, for all $t>0$, weakly in the sense of Radon measures as $i_k\to\infty$.

(2) The limits of measures have the following structures:

$$\mu=f|\nabla u|^2\,\mathrm{d}z+\eta,$$
$$\nu=|\partial_t u|^2\,\mathrm{d}z+\nu_*,$$
$$\mu^t=f|\nabla u|^2(t)\,\mathrm{d}x+\eta^t,$$

where u is the limit obtained in Theorem 2. Exactly, $\mu=\mu^t\mathrm{d}t$ and $\eta=\eta^t\mathrm{d}t$.

(3) The energy concentration set Σ has the following properties:

$$\Sigma=\bigcap_{r>0}\{z_0:r^{-m}\mu(P_r(z_0))\geq\varepsilon_0\}=\mathrm{sing}\,u\cup\mathrm{supp}\,\eta,$$
$$\Sigma'\subset\mathrm{sing}\,u\cup\mathrm{supp}\,\nu_*\subset\Sigma,$$

where Σ' is defined in Proposition 1. Moreover, for the t-slice set Σ^t, sing $u(\cdot,t)\cup$supp $\eta^t\subset\Sigma^t$.

For certain weakly converging sequence of strong stationary weak solutions, we have a rectifiability result as following, where the set $T_\gamma^\infty(t)$ is defined in Section 4.

Theorem 4 Suppose I is a subset of $(0,T)$ with nonzero 1D Lebesgue measure, $\{u_i\}$ is a sequence in W_A satisfying the strong stationary condition, and $u_i\to u$ weakly in $W^1(\Omega\times(0,T),S^2)$ as $i\to\infty$, as in Theorem 2 and Theorem 3. If for almost all $t\in I$, $\mathcal{H}^{m-2}(T_\gamma^\infty(t))=0$, then both $\eta^t\llcorner\Sigma^t$ and Σ^t are $\mathcal{H}^{m-2}$-rectifiable for almost all $t\in I$.

The paper is organized as follows. In Section 2 we gather various notions and facts concerning stationary weak solutions. In addition, we establish several key lemmas concerning the generalized monotonicity inequalities and also prove Theorem 1. Theorem 2 and Theorem 3 are proved in Section 3. Finally, in Section 4, for a sequence of weak solutions in W_A satisfying the strong stationary condition we explore some subtle properties of the energy concentration set and the Radon measures, and then Theorem 4 is obtained by applying Theorem 7.

STATIONARY WEAK SOLUTIONS

In this section, we collect some basic facts on (proper, strong) stationary weak solutions and related notions. For a more detailed discussion, we refer the reader to various articles cited below.

First of all, we impose a stationarity condition on weak solutions [cf. (Moser, 2002; Appendix A)].

Definition 1 Let $u\in W^1(\Omega\times(0,T),S^2)$ be a weak solution of Eq.(2). Consider for $\xi\in C_0^\infty(\Omega\times(0,T),\mathbb{R}^3)$ and $\tau\in C_0^\infty(\Omega\times(0,T),[0,\infty))$, the variation

$$u_\sigma(x,t)=u(x+\sigma\xi(x,t),t+\sigma\tau(x,t)),$$

which consists of maps in $W^1(\Omega\times(0,T),S^2)$ for small $|\sigma|$. We say that u is a stationary weak solution, if for all such ξ and τ, the inequality

$$\int_0^T\int_\Omega\left\langle R_u\partial_t u,\left(\partial u_\sigma/\partial\sigma\right)\big|_{\sigma=0}\right\rangle\mathrm{d}x\mathrm{d}t+\left(\partial_\sigma^+\int_0^T E_f(u_\sigma,t)\,\mathrm{d}t\right)\Big|_{\sigma=0}\le 0$$

holds, where ∂_σ^+ denotes the right hand derivative w.r.t. σ.

Remark 1 A simple integration by parts implies that if a solution u is smooth enough, precisely, if $u\in\{v\in W^1(\Omega\times(0,T),S^2):\nabla^2 v\in L^2(\Omega\times(0,T))\}$, then u is stationary.

From the stationary condition, we may derive the following lemma easily:

Lemma 1 Let u be a stationary weak solution of Eq.(2). Let $\phi\in C_0^\infty(\Omega,\mathbb{R}^3)$ and $\theta\in C_0^\infty(\Omega\times(0,T),[0,\infty))$. Then, for almost all $t\in(0,T)$,

$$\int_{\Omega\times\{t\}}\Big[\left\langle R_u\partial_t u,\partial_i u\right\rangle\phi_i-|\nabla u|^2\,\mathrm{div}(f\phi)/2+f\left\langle\partial_i u,\partial_j u\right\rangle\partial_i\phi_j\Big]\mathrm{d}x=0, \tag{4}$$

and for almost all $0\le t_1\le t_2\le T$,

$$\int_{\Omega\times\{t_2\}}f\theta|\nabla u|^2\,\mathrm{d}x-\int_{\Omega\times\{t_1\}}f\theta|\nabla u|^2\,\mathrm{d}x\le\int_{t_1}^{t_2}\int_\Omega\Big[f\partial_t\theta|\nabla u|^2-2\alpha\theta|\partial_t u|^2-2f\partial_i\theta\left\langle\partial_i u,\partial_t u\right\rangle\Big]\mathrm{d}z. \tag{5}$$

We emphasize here that Eq.(5) contains a clue to a proper stationarity condition and a strong stationarity condition on weak solutions.

Definition 2 Let u be a stationary weak solution. We say that u is properly stationary, if for all $\theta\in C^\infty(\Omega\times(0,T),[0,\infty))$ with $\theta(\cdot,t)\in C_0^\infty(\Omega)$, what follows holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\times\{t\}}f\theta|\nabla u|^2\,\mathrm{d}x\le\int_\Omega\Big[f\partial_t\theta|\nabla u|^2-2\alpha\theta|\partial_t u|^2-2f\partial_i\theta\left\langle\partial_i u,\partial_t u\right\rangle\Big]\mathrm{d}x. \tag{6}$$

Let u be a stationary weak solution. u is called strong stationary, if the equality in Eq.(5) [or Eq.(6)] holds.

From Eq.(4), we can prove three types of generalized monotonicity inequalities. The first one is the generalization of Moser's type and the others are of Ding's type (Ding, 1990; Liu, manuscript).

Lemma 2 Let u be a stationary weak solution of Eq.(2). Suppose $B_{r_0}(x_0)\subset\Omega$. If denoting $c_1=\max_\Omega(|\nabla f|/f)$ and $c_2=\max_\Omega f^{-1/2}$, the following statements are true:

(1) When $m=3$, for almost every $t\in(0,T)$,

$$\mathcal{E}^b_{x_0}(u,t,s)\le 10\mathcal{E}^b_{x_0}(u,t,r)+c(f)\mathcal{I}^b_{x_0}(u,t,r) \quad (7)$$

for $0<s\le r\le\min\{r_0,r_1\}$, where r_1 is a radius such that $r_1\max_\Omega|\nabla f|/\min_\Omega f$ is small, and $c(f)=c/\min_\Omega f$.

(2) If $I_{x_0}(u,t,r)\le(c_1+1)c_2^{-2}Kr^{-\gamma}$, for any $r\in(0,r_0)$ and some $K>0$, $0<\gamma<1$,

$$\exp[2(c_1+1)r]\left(\mathcal{E}^b_{x_0}(u,t,r)+Kr^{1-\gamma}/(1-\gamma)\right)$$

is a strictly increasing function of $r\in(0,r_0)$.

(3) If $\mathcal{I}^p_{z_0}(u,r)\le(c_1+1)c_2^{-2}Kr^{-\gamma}$ for any $r\in(0,r_0)$ and some $K>0$, $0<\gamma<1$, then

$$\exp[2(c_1+1)r]\left(E_{z_0}(u,r)+\frac{K}{1-\gamma}r^{1-\gamma}\right)$$

is a strictly increasing function of $r\in(0,r_0)$.

Proof Without loss of generality, we assume that $x_0=0$ and $z_0=0$. From Eq.(4),

$$\int_{B_\rho}\left[2\left\langle R_u\partial_t u,x\cdot\nabla u\right\rangle-(m-2)f|\nabla u|^2-x\cdot\nabla f|\nabla u|^2\right]\mathrm{d}x$$
$$=\int_{\partial B_\rho}\left[-f\rho|\nabla u|^2+2f\rho^{-1}|x\cdot\nabla u|^2\right]\mathrm{d}w, \quad (8)$$

where $\mathrm{d}w$ is the $(m-1)$-dimensional area element in ∂B_ρ. Mainly along the same line of Moser's arguments, (1) follows immediately from Eq.(8) for $m=3$.

Next we prove (2) and (3) for general dimensions m. From Eq.(8), we can complete the proof by checking

$$\frac{\mathrm{d}}{\mathrm{d}r}\left\{\exp[2(c_1+1)r]\left(\mathcal{E}^b_0(u,t,r)+\frac{K}{1-\gamma}r^{1-\gamma}\right)\right\}>0,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}r}\left\{\exp[2(c_1+1)r]\left(E_0(u,r)+\frac{K}{1-\gamma}r^{1-\gamma}\right)\right\}>0.$$

As a consequence of Eq.(5) and Cauchy's inequality, we have the following local energy inequalities: For $\varphi\in C_0^\infty(\Omega\times(0,T))$ and a.e. $0<t_1\le t_2<T$,

$$\int_{\Omega\times\{t_2\}}f\varphi^2|\nabla u|^2\,\mathrm{d}x+\alpha\int_{t_1}^{t_2}\int_\Omega|\partial_t u|^2\varphi^2\mathrm{d}z$$
$$\le c(\alpha,f)\int_{t_1}^{t_2}\int_\Omega f(|\nabla\varphi|^2+|\partial_t\varphi^2|)|\nabla u|^2\,\mathrm{d}z \quad (9)$$
$$+\int_{\Omega\times\{t_1\}}f\varphi^2|\nabla u|^2\,\mathrm{d}x.$$

From the proper stationary condition, it reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\times\{t\}}f\varphi^2|\nabla u|^2\,\mathrm{d}x+\alpha\int_\Omega|\partial_t u|^2\varphi^2\mathrm{d}x$$
$$\le c(\alpha,f)\int_\Omega f(|\nabla\varphi|^2+|\partial_t\varphi^2|)|\nabla u|^2\,\mathrm{d}x, \quad (10)$$

for all $\varphi\in C^\infty(\Omega\times(0,T))$ with $\varphi(\cdot,t)\in C_0^\infty(\Omega)$. Together with Eqs.(3) and (10), a routine approximation argument shows the semi-decreasing property.

Corollary 1 Let $\varphi(\cdot)$ be a bounded Lipschitz function on Ω, vanishing on $\partial\Omega$, and u be a weak solution satisfying the proper stationary condition (6). Then we have, for any $t\in(0,T)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\times\{t\}}f\varphi^2|\nabla u|^2\,\mathrm{d}x\le c(\alpha,f)\|\nabla\varphi\|^2_{L^\infty}\,E_f(u,0).$$

According to Eq.(9) and via Fubini's theorem, we can partially control $\partial_t u$ (Chen *et al.*, 1995).

Lemma 3 There exists a constant $c=c(\alpha,f)>0$ such that, for $P_r(z_0)\subset\Omega\times(0,T)$,

$$\mathcal{I}^p_{z_0}(u,r/2)\le c(\alpha,f)\mathcal{E}^p_{z_0}(u,r).$$

Based on Eq.(7) in Lemma 1, proof of Theorem 1 can be derived in the same manner as that in (Moser, 2002), since $f>0$, f is smooth and the term $\nabla f\cdot\nabla u$ is under natural growth. We skip the details here. Now, we state an important consequence of Theorem 1, that is, when $m=3$, for u satisfying the stationary condition the regular set can be characterized in terms of the local quantity

$$\liminf_{r\to 0}\mathcal{E}^p_{z_0}(u,r).$$

For a map $u\in W^1(\Omega',S^2)$, we define the regular and singular sets, reg u and sing u, by reg $u=\{z\in\Omega':u\in C^\infty$ in a neighborhood of $z\}$ and sing $u=\Omega'\backslash$reg u. Notice that by definition reg u is open, and hence sing u is relatively closed in Ω'.

ENERGY CONCENTRATION SET AND RADON MEASURES

In this section, we consider a weakly converging sequence of certain weak solutions (Lin, 1999; Li and Tian, 2000). We will first prove Theorem 2, and then analyze some elementary properties of the energy concentration set and the Radon measures, which show Theorem 3.

Let W_Λ be the set of all weak solutions u in $W^1(\Omega\times(0,T),S^2)$ such that

(1) $E_f(u,0)\le\Lambda$, for some $\Lambda>0$;

(2) u satisfies the proper stationary condition.

From now on, suppose $\{u_i\}_{i=1}^{\infty}\subset W_\Lambda$. According to Eq.(3) and up to a subsequence, we may assume that $u_i\to u$ weakly in $H^1_{\text{loc}}(\Omega\times(0,T),\mathbb{R}^3)$, $\partial_t u_i\to\partial_t u$ weakly in $L^2(\Omega\times(0,T))$, $\nabla u_i\to\nabla u$ weak-* in $L_t^\infty L_x^2(\Omega\times(0,T))$ and $u_i\to u$ a.e. on $\Omega\times(0,T)$ as $i\to\infty$. Immediately, $u\in W^1(\Omega\times(0,T),S^2)$, and we denote all the above convergence by $u_i\to u$ weakly in $W^1(\Omega\times(0,T),S^2)$ as $i\to\infty$. In the remainder of this section, we always define u_i and u as above. It is natural to ask the following question:

Problem Is the limit u also a solution for Eq.(2)?

The main result of this problem is the following:

Theorem 5 For m=3, let u_i and u be as above. Then u solves Eq.(2) in the classical sense on a dense open set $\mathcal{R}_0\subset\Omega\times(0,T)$ whose complement Σ has locally finite m-dimensional parabolic Hausdorff measure. Moreover, u is smooth on $\mathcal{R}_0$.

Proof Define

$$\Sigma=\bigcap_{r>0}\left\{z_0:\liminf_{i\to\infty}\mathcal{E}^p_{z_0}(u_i,r)\ge\varepsilon_0\right\},\qquad(11)$$

where ε_0 is the universal constant in Theorem 1. It is easy to show that Σ is closed [Theorem 6.1 in (Struwe, 1988)]. A standard covering argument shows that Σ has locally finite m-dimensional parabolic Hausdorff measure. From Theorem 1, it follows that a subsequence $u_i\to u$ in $C^2_{\text{loc}}(\mathcal{R}_0,S^2)$ as $i\to\infty$, and u is a smooth solution of Eq.(2) off Σ.

We call the closed set Σ defined by Eq.(11) the energy concentration set, also the blow-up set. We define the t-slice set of Σ by $\Sigma^t=\{x:z=(x,t)\in\Sigma\}$ for all $0<t<T$. We will even sometimes identify Σ^t and $\Sigma^t\times\{t\}$.

Theorem 6 The t-slice set Σ^t is a closed subset of Ω. Moreover, Σ^t has uniformly bounded finite $(m-2)$-dimensional Hausdorff measures for all $0<t<T$.

Proof Note that Σ^t is closed since Σ is closed. We will identify the closed and open ball below, and this may not confuse our proof. It will be sufficient to show that $\mathcal{H}^{m-2}(\Sigma^t\cap Q)\le c(\Lambda)$ for all $0<t<T$ and compact subsets Q of Ω, where $\mathcal{H}^s$ is the s-dimensional Hausdorff measure. For any $0<\delta<\min\{t^{1/2}, (T-t)^{1/2}, dist(Q,\partial\Omega)\}$, let $\mathscr{F}=\{B_r(x)\subset\Omega: x\in\Sigma^t\cap Q, 0<r\le\delta/10\}$ and $\mathscr{F}_r=\{B=B_r(x)\in\mathscr{F}\}$. For any $B_r(x)\in\mathscr{F}$, we have

$$\liminf_{i\to\infty}\int_{t-r^2}^{t+r^2}\int_{B_r(x)}f\,|\nabla u_i|^2\,\mathrm{d}y\mathrm{d}s\ge\varepsilon_0 r^m.$$

Vitali's covering theorem (for closed balls) implies that there exists a countable (in fact finite here!) family $\mathscr{F}_r'$ of disjoint balls in $\mathscr{F}_r$ such that

$$\Sigma^t\cap Q\subset\bigcup_{B\in\mathscr{F}_r}B\subset\bigcup_{B\in\mathscr{F}_r'}\hat{B},$$

where $\hat{B}$ denotes the concentric ball with radius 5 times the radius of B. Therefore we have

$$\sum_{B\in\mathscr{F}_r'}\varepsilon_0 r^m\le\liminf_{i\to\infty}\sum_{B\in\mathscr{F}_r'}\int_{t-r^2}^{t+r^2}\int_B f\,|\nabla u_i|^2\,\mathrm{d}y\mathrm{d}s$$
$$\le\liminf_{i\to\infty}\int_{t-r^2}^{t+r^2}\int_\Omega f\,|\nabla u_i|^2\,\mathrm{d}y\mathrm{d}s\le 2\Lambda r^2,$$

which yields that $\mathcal{H}^{m-2}_{\delta/2}(\Sigma^t\cap Q)\le C\sum_{B\in\mathscr{F}_r'}r^{m-2}\le c(\Lambda)$ for any small $\delta>0$, then that is done.

Until now, we have finished the proof of Theorem 2. Next, we consider three sequences of Radon measures on $\Omega\times(0,T)$ and $\Omega\times\{t\}$, μ_i, ν_i and μ_i^t, defined in Section 1. Without loss of generality, we may assume $u_i\to u$ and $\nu_i\to\nu$ weakly as Radon measures as $i\to\infty$. Moreover, by Fatou's lemma, we write

$$\mu=f\,|\nabla u|^2\,\mathrm{d}z+\eta,$$

and

$$\nu=|\partial_t u|^2\,\mathrm{d}z+\nu_*$$

for some nonnegative Radon measures η and ν_*. Moreover, passing possibly to a further subsequence, we have

Lemma 4 There exists a subsequence (still denoted i) such that $\mu_i^t \to \mu^t$ weakly as $i\to\infty$, where μ^t is a finite Radon measure on Ω for all $t>0$. Moreover, $\mu=\mu^t dt$.

The proof in (Ilmanen, 1994; Li and Tian, 2000) involves just a little change for word, using a classical semi-decreasing property (see Corollary 1). We choose such a subsequence i, and identify Ω with $\Omega\times\{t\}$ sometimes. Consequently, we have $\mu^t= f|\nabla u|^2(t)dx+\eta^t$, and $\eta=\eta^t dt$, for some nonnegative Radon measure η^t on Ω. Some properties of the concentration sets Σ and Σ^t can be characterized by the following

Proposition 1

(1) $\Sigma=\bigcap_{r>0}\{z_0: r^{-m}\mu(P_r(z_0))\geq\varepsilon_0\}$;

(2) $\Sigma=\mathrm{sing}\,u\cup\mathrm{supp}\,\eta$;

(3) $\Sigma'\subset\mathrm{sing}\,u\cup\mathrm{supp}\,v_*\subset\Sigma$.

Where $\Sigma'=\bigcap_{r>0}\{z_0: \liminf_{i\to\infty}\mathcal{I}_{z_0}^p(u_i,r/2)\geq c(\alpha,f)\varepsilon_0\}$, and $c(\alpha,f)$ is the constant in Lemma 3.

Proof For any $z\in\Omega\times(0,T)$, set $T_z=\{r>0:\mu(\partial P_r(z))=0\}$. Since μ is Radon, T_z^c is an at most countable set of $\mathbb{R}$. For any $r\in T_z$, we have $\int_{P_r(z)} f|\nabla u_i|^2\,dz\to\mu(P_r(z))$ as $i\to\infty$, thus

$$\liminf_{i\to\infty}\mathcal{E}_z^p(u_i,r)=\lim_{i\to\infty}\mathcal{E}_z^p(u_i,r)=r^{-m}\mu(P_r(z)). \quad (12)$$

If $z\in\Sigma$, then Eq.(12) implies that $r^{-m}\mu(P_r(z))\geq\varepsilon_0$ for any $r\in T_z$. Taking $r_0\in T_z^c$, and choosing $r_k\in T_z$ with $r_k\nearrow r_0$ as $k\to\infty$, we have

$$\begin{aligned} & r_0^{-m}\mu(P_{r_0}(z)) \\ & =\lim_{k\to\infty} r_k^{-m}\sup\{\mu(K): K \text{ is compact}, K\subset P_{r_0}(z)\} \\ & \geq\lim_{k\to\infty} r_k^{-m}\limsup_{j\to\infty}\mu(P_{r_j}(z))\geq\varepsilon_0. \end{aligned}$$

On the other hand, if for any $r>0$, there holds $r^{-m}\mu(P_r(z))\geq\varepsilon_0$ at z, then for $r\in T_z$, $\liminf_{i\to\infty}\mathcal{E}_z^p(u_i,r)\geq\varepsilon_0$ by Eq.(12). For any $r_0\in T_z^c$, we have

$$\begin{aligned}\liminf_{i\to\infty}\mathcal{E}_z^p(u_i,r_0) & \geq\liminf_{i\to\infty} r_0^{-m}\int_{P_{r_j}(z)} f|\nabla u_i|^2\,dz \\ & \geq r_0^{-m}\mu(P_{r_j}(z))\end{aligned}$$

for any sequence $r_j\in T_z$ with $r_j\nearrow r_0$. Since $\mu(P_{r_0}(z))=\sup\{\mu(K)$: K is compact, $K\subset P_{r_0}(z)\}$, we conclude that $\liminf_{i\to\infty}\mathcal{E}_z^p(u_i,r_0)\geq r_0^{-m}\mu(P_{r_0}(z))\geq\varepsilon_0$. This proves (1).

We observe the fact: Theorem 5 implies that $u_i\to u$ in $C^2_{\mathrm{loc}}(\mathcal{R}_0)$ as $i\to\infty$, thus $\mu_i\to|\nabla u|^2dz$ and $v_i\to|\partial_t u|^2dz$ locally in $\mathcal{R}_0$ weakly as $i\to\infty$. The later implies $\mathrm{sing}\,u\subset\Sigma$, $\mathrm{supp}\,\eta\subset\Sigma$ and $\mathrm{supp}\,v_*\subset\Sigma$. Suppose now $z_0\in\Sigma$, then for any small $r>0$, $r^{-m}\mu(P_r(z))\geq\varepsilon_0/2$ by (1). If $z_0\notin\mathrm{sing}\,u$, then u is smooth near z_0, and hence $\mathcal{E}_{z_0}^p(u,r)\leq\varepsilon_0/4$ for $r>0$ sufficiently small. Thus, $r^{-m}\eta(P_r(z))>\varepsilon_0/4$, for all positive small r. This is $z_0\in\mathrm{supp}\,\eta$. This proves (2). Note that (3) is a direct consequence of Lemma 3.

By Proposition 1, we can see easily that

$$\mathrm{sing}\,u(\cdot,t)\cup\mathrm{supp}\,\eta^t\subset\Sigma^t. \quad (13)$$

But we cannot say that Σ^t is a subset of $\mathrm{sing}\,u(\cdot,t)\cup\mathrm{supp}\,\eta^t$, since the singularity of u at point (x,t) may occur at t-direction. Here we finish the proof of Theorem 3.

RECTIFIABILITY RESULT

In this section, we will provide a slight rectifiability result when considering a weakly converging sequence of weak solutions satisfying the strong stationary condition. We will see again that the generalized monotonicity formulas play an important role.

We set for $0<\gamma<1$, $K>0$, and $\{u_i\}\subset W_A$,

$$T_\gamma^K(t)=\bigcap_n\bigcap_{i_k\to\infty}\left\{x\in\Omega: \limsup_{i_k\to\infty}\sup_{0<r<1/n}\frac{c_2^2 r^\gamma}{c_1+1}I_x(u_{i_k},t,r)\geq K\right\},$$

where i_k is any subsequence of i. Note that $T_\gamma^\infty(t)=\bigcap_K T_\gamma^K(t)$, and

$$[T_\gamma^K(t)]^c=\bigcup_n\bigcup_{i_k\to\infty}\left\{x\in\Omega: \limsup_{i_k\to\infty}\frac{c_2^2 r^\gamma}{c_1+1}I_x(u_{i_k},t,r)<K,\ \text{uniformly for all } r\in(0,1/n)\right\}.$$

It is easy to check that $T_\gamma^K(t)$ is measurable, and so is $T_\gamma^\infty(t)$. We have

Lemma 5 (1) For any $x\in\Omega\setminus T_\gamma^\infty(t)$, there exist $r_0>0$ and $K>0$ such that $\exp[2(c_1+1)r]\cdot\left(r^{2-m}\mu^t(B_r(x))+\frac{K}{1-\gamma}r^{1-\gamma}\right)$ is a monotonically increasing function of $r\in(0,r_0)$;

(2) The nonnegative quantity $\Theta^t(\mu^t,x)=\lim_{r\to0}r^{2-m}\mu^t(B_r(x))$ exists for every $x\in(\Sigma^t\cap T_r^\infty(t))^c$. Moreover, $\Theta^t(\mu^t,x)$ vanishes on $\Omega\backslash\Sigma^t$;

(3) For $\mathcal{H}^{m-2}$-a.e. $x\in(\Sigma^t\cap T_\gamma^\infty(t))^c$, the limit $\Phi^t(\eta^t,x)=\lim_{r\to0}r^{2-m}\eta^t(B_r(x))$ exists, equals $\Theta^t(\mu^t,x)$, and also vanishes on $\Omega\backslash\Sigma^t$;

(4) Suppose $\Sigma^t\setminus T_\gamma^\infty(t)$ is nonempty. If $x\in[T_\gamma^\infty(t)]^c$ with $\Theta^t(\mu^t,x)>0$, then $x\in\Sigma^t\cap T_\gamma^\infty(t)$.

Proof Applying Lemma 2 (2) we have the statements (1) and (2). By Eq.(13), $\Theta^t(\mu^t,x)$ vanishes on $\Omega\backslash\Sigma^t$. Parts (3) and (4) follows from the fact that u is $\mathcal{H}^{m-2}$-a.e. smooth.

Similarly, we set for $0<\gamma<1$ and $K>0$,

$$S_\gamma^K=\bigcap_n\bigcap_{i_k\to\infty}\left\{z\in\Omega\times(0,T):\limsup_{i_k\to\infty}\sup_{0<r<1/n}\frac{c_2^2r^\gamma}{c_1+1}\mathcal{I}_z^p(u_{i_k},r)\geq K\right\}.$$

Lemma 6

(1) For any $z\in\Omega\times(0,T)\setminus S_\gamma^\infty$, there exist $r_0>0$ and $K>0$ s.t. $\exp[2(c_1+1)r]\left(r^{2-m}\mu(P_r(z))+Kr^{1-\gamma}(1-\gamma)^{-1}\right)$ is a monotonically increasing function of $r\in(0,r_0)$;

(2) The nonnegative quantity $\Theta(\mu,z)=\lim_{r\to0}r^{2-m}\mu(P_r(z))$ exists for every $z\in(\Sigma\cap S_\gamma^\infty)^c$. Moreover, $\Theta(\mu,z)$ vanishes on $\Omega\times(0,T)\backslash\Sigma$;

(3) For $\mathcal{H}^m$-a.e. $z\in(\Sigma\cap S_\gamma^\infty)^c$, the limit $\Phi(\eta,z)=\lim_{r\to0}r^{2-m}\eta(P_r(z))$ exists, equals $\Theta(\mu,z)$, and also vanishes on $\Omega\times(0,T)\backslash\Sigma$;

(4) Suppose $\Sigma\setminus S_\gamma^\infty$ is nonempty, then for any $z\in(S_\gamma^\infty)^c$ with $\Theta(\mu,z)>0$, we have $z\in\Sigma\setminus S_\gamma^\infty$.

Theorem 7 Suppose $\{u_i\}$ is a sequence in W_Λ satisfying the strong stationary condition, and $u_i\to u$ weakly in $W^1(\Omega\times(0,T),S^2)$, as $i\to\infty$, as in Theorem 2 and Theorem 3. Then $\eta^t\lfloor(\Sigma^t\setminus T_\gamma^\infty(t))$ is $\mathcal{H}^{m-2}$-rectifiable for almost all $0<t<T$, and thus $\Sigma^t\setminus T_\gamma^\infty(t)$ is $\mathcal{H}^{m-2}$-rectifiable for almost all $0<t<T$.

Proof The result will be followed when one shows that for $\mathcal{H}^{m-2}$-a.e. x in $\Sigma^t\setminus T_\gamma^\infty(t)$, the quantity $\Phi^t(\eta^t,x)>c>0$, by Theorem 5.6 in (Preiss, 1987), and Lemma 5. We choose $\theta=\varphi^2$ in Eq.(5), where $\varphi\in C_0^\infty(B_r(x))$ is a cutoff function for small r. From the strong stationarity condition and Young's inequality, we obtain

$$\left|r^{2-m}\int_{B_r(x)\times\{t\}}f|\nabla u_i|^2\varphi^2\mathrm{d}y-r^{2-m}\int_{B_r(x)\times\{s\}}f|\nabla u_i|^2\varphi^2\mathrm{d}y\right|$$
$$\leq2\alpha r^{2-m}\int_{P_r(x,t)}|\partial_tu_i|^2\varphi^2\mathrm{d}z+\delta r^{-m}\int_{P_r(x,t)}f|\nabla u_i|^2\varphi^2\mathrm{d}z$$
$$+c(\delta)r^{2-m}\int_{P_r\setminus P_{r/2}(x,t)}|\partial_tu_i|^2\,\mathrm{d}z$$

for almost all $t-r^2<s<t+r^2$. Via Fubini's theorem, we may choose a uniform $c>0$ and an $s_i\in(t-r^2,t+r^2)$ such that

$$2cr^{-2}\int_{P_r(x,t)}f|\nabla u_i|^2\varphi^2\mathrm{d}z\leq\int_{B_r(x)}f|\nabla u_i|^2(y,s_i)\varphi^2(y)\mathrm{d}y.$$

The two estimates above imply that,

$$r^{2-m}\int_{B_r(x)\times\{t\}}f|\nabla u_i|^2\varphi^2\mathrm{d}y\geq\frac{c}{16}\mathcal{E}_z^p(u_i,r/2)-c(\alpha)\mathcal{I}_z^p(u_i,r).\tag{14}$$

Set $S_\tau^t=\bigcup_n\bigcap_{0<r<1/n}\{x\in\Omega:\liminf_{i\to\infty}\mathcal{I}_z^p(u_i,r)\geq\tau\}$ for $\tau>0$. Let $\delta>0$. By Vitali's covering theorem, we can cover S_τ^t by a family of countable balls $\hat{B}_j$ centered at x in S_τ^t, where B_j in Ω are disjoint, with radii $r_j<\delta/10$, and satisfy the following

$$\tau\sum_jr_j^{m-2}\leq\liminf_{i\to\infty}\sum_j\int_{P_{r_j}(x,t)}|\partial_tu_i|^2\,\mathrm{d}z\leq\liminf_{i\to\infty}\int_{t-\delta^2}^{t+\delta^2}\int_\Omega|\partial_tu_i|^2\,\mathrm{d}z.$$

Letting $\delta\to0$, we find that for almost all $0<t<T$,

$$\mathcal{H}^{m-2}(S_\tau^t)\leq(c\tau)^{-1}\limsup_{\delta\to0}\left\{\int_{t-\delta^2}^{t+\delta^2}\int_\Omega|\partial_tu|^2\,\mathrm{d}z+\nu_*\left[\Omega\times(t-\delta^2,t+\delta^2)\right]\right\}=0,$$

because of $\int_0^T\int_\Omega |\partial_t u|^2\,\mathrm{d}z+\nu_*(\Omega\times(0,T))\le\Lambda$. Now for every x in $\Sigma^t\setminus(T_\gamma^\infty(t)\cup S_\tau^t)$, Eq.(14) yields that $r^{2-m}\mu^t(B_r(x))\ge c\varepsilon_0/16-c(\alpha)\tau>0$ for sufficiently small τ>0; thus $\Phi^t(\eta^t,x)$>0, by Lemma 5. We complete the proof.

ACKNOWLEDGEMENT

The authors would like to thank Prof. Fanghua Lin and Prof. Xiangao Liu for their helpful discussions and encouragement when they visited Zhejiang University.

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APPENDIX A

In this appendix, authors will show an interesting trivial phenomenon on some matching-stationary condition.

Definition A1 Let $u\in W^1(\Omega\times(0,T),S^2)$. Suppose u satisfies Eq.(2) with the smooth positive map f. Let ξ, τ and u_σ be as in Definition 1. Consider also the variation $f_\sigma(x,t)=f(x+\sigma\xi(x,t))$, which consists of smooth positive maps on the closure of Ω for small $|\sigma|$. We say that (f,u) is a stationary pair of coupling function and weak solution for the system Eq.(2), if for all such ξ and τ, the following inequality holds

$$\int_0^T\int_\Omega\left\langle R_u\partial_t u,(\partial u_\sigma/\partial\sigma)|_{\sigma=0}\right\rangle \mathrm{d}x\mathrm{d}t +\left(\partial_\sigma^+\int_0^T E_{f_\sigma}(u_\sigma,t)\,\mathrm{d}t\right)\Bigg|_{\sigma=0}\le 0.$$

If (f,u) is a stationary pair, we also say that u is matching-stationary with f.

Suppose u is matching-stationary with f. Let ϕ and θ be as in Lemma 1. A simple calculation gives that

$$\int_{\Omega\times\{t\}}\Big[\left\langle R_u\partial_t u,\partial_i u\right\rangle\phi_i-\tfrac{1}{2}f|\nabla u|^2\,\mathrm{div}\phi +f\left\langle\partial_i u,\partial_j u\right\rangle\partial_i\phi_j\Big]\mathrm{d}x=0 \tag{A1}$$

holds for almost all t in $(0,T)$, and Eq.(5) remains. Moreover, from Eq.(A1), the generalized monotonicity inequalities of Moser's and Ding's types are also true. A similar argument as in Section 2 implies that u is also partially regular. However, from Remark 1, we can get that u satisfies Eq.(4) on the regular set of u; hence, we obtain directly from Eqs.(4) and (A1) that f or u is locally constant on the regular set of u. Here, we say a function w is locally constant on an open set Ω', if there is a nonempty open set V of Ω' such that w is constant on V. Thus, we have the unusual remark as follows:

Remark A1 For a general coupling function f which is not local constant, the matching-stationary weak solution is necessarily a trivial solution.