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3D analytical solution for a rotating transversely isotropic annular plate of functionally graded materials*

CHEN Jiang-ying[†1], CHEN Wei-qiu[2]

([1]*Faculty of Engineering, Ningbo University, Ningbo 315211, China*)

([2]*Department of Civil Engineering, Zhejiang University, Hangzhou 310027, China*)

[†]E-mail: chenjiangying@nbu.edu.cn

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Abstract: The analytical solution for an annular plate rotating at a constant angular velocity is derived by means of direct displacement method from the elasticity equations for axisymmetric problems of functionally graded transversely isotropic media. The displacement components are assumed as a linear combination of certain explicit functions of the radial coordinate, with seven undetermined coefficients being functions of the axial coordinate z. Seven equations governing these z-dependent functions are derived and solved by a progressive integrating scheme. The present solution can be degenerated into the solution of a rotating isotropic functionally graded annular plate. The solution also can be degenerated into that for transversely isotropic or isotropic homogeneous materials. Finally, a special case is considered and the effect of the material gradient index on the elastic field is illustrated numerically.

Key words: Functionally graded materials, Transversely isotropic, Rotating annular plate, Analytical solution

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INTRODUCTION

Rotating annular plate or disk is one of the most important parts in turbo-machinery and flywheel systems. With the emergence of functionally graded materials (FGMs), which possess a continuous variation of material properties in the space, the design of rotating plates may change accordingly. Hence, it is interesting to study the stress distribution in a rotating FGM annular plate.

To simplify the analysis, plane stress state is always assumed for the rotating annular plates or discs (Lekhnitskii, 1968; Timoshenko and Goodier, 1970; Seireg and Surana, 1970; Murthy and Sherbourne, 1970; Leissa and Vagins, 1978; Yeh and Han, 1994). It is interesting to note that an optimization of stress distribution in a rotating disk could be reached by a proper variation of elastic coefficients with the radial direction (Leissa and Vagins, 1978; Jain *et al.*, 1999). Kordkheili and Naghdabadi (2007) studied the thermal stresses in a rotating FGM hollow or solid disk by use of a semi-analytical method; the elastic properties are assumed to vary with the r-coordinate in their analysis. When the elastic properties vary with the hoop coordinate θ, Zhou and Ogawa (2002) derived an analytical solution for a solid rotating disc with cubic anisotropy.

Giving up the plane-stress assumption, Ramu and Iyengar (1974) derived a quasi three-dimensional (3D) solution to investigate the influence of axial stress σ_z on the in-plane stresses in a homogeneous isotropic rotating disk. Lekhnitskii (1968) presented a 3D solution for a rotating ellipse disc of homogeneous anisotropic materials. The 3D analytical solution for an isotropic rotating disc can be found in (Timoshenko and Goodier, 1970). Chen and Lee (2004)

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considered a rotating laminated orthotropic cylindrical shell with two simply-supported ends and investigated the effects of centrifugal and Coriolis forces on the stress distribution. Based on the general solution, Ding *et al.*(2006) derived a 3D analytical solution for a rotating homogeneous transversely isotropic circular disc. Mian and Spencer (1998) proposed an analytical method to study the axisymmetric deformation of functionally graded isotropic material disks with elastic constants being arbitrary functions of the axial coordinate z. Recently, Chen *et al.*(2007) presented a 3D analytical solution for a rotating disc of transversely isotropic FGMs by means of a direct displacement function method.

In this paper, we extend the analysis in (Chen *et al.*, 2007) to a rotating functionally graded transversely isotropic annular plate. The solution is valid for an arbitrary variation of material properties along the thickness coordinate, which makes it possible to optimize the stress distribution in the plate. The results can be reduced to those of a rotating isotropic FGM annular plate as well as those of a transversely isotropic/isotropic homogeneous plate. The solution also can be degenerated into that for the corresponding solid disk. A particular FGM annular plate is considered for numerical calculation. The results show that the variation of material parameters has an obvious effect on the stress distributions, and hence the particular characteristic of FGMs could be utilized for optimization designs of rotating annular plates.

BASIC EQUATIONS

For an axisymmetric problem, the equilibrium equations are

$$\begin{aligned} &\sigma_{r,r} + \tau_{rz,z} + r^{-1}(\sigma_r - \sigma_\theta) + F_r = 0, \\ &\tau_{rz,r} + r^{-1}\tau_{rz} + \sigma_{z,z} + F_z = 0, \end{aligned} \tag{1}$$

where σ_r, σ_θ, σ_z and τ_{rz} are the stress components; F_r and F_z are the body force components in r- and z-directions, respectively; and the comma denotes differentiation with respect to the followed coordinate variable.

The constitutive relations for a transversely isotropic material are

$$\begin{aligned} &\sigma_r = c_{11}u_{,r} + c_{12}r^{-1}u + c_{13}w_{,z}, \\ &\sigma_\theta = c_{12}u_{,r} + c_{11}r^{-1}u + c_{13}w_{,z}, \\ &\sigma_z = c_{13}(u_{,r} + r^{-1}u) + c_{33}w_{,z}, \\ &\tau_{rz} = c_{44}(u_{,z} + w_{,r}), \end{aligned} \tag{2}$$

where $c_{ij}=c_{ij}(z)$ (i,j=1,2,3,4) are the elastic coefficients, u and w are the displacement components in r- and z-directions, respectively.

ANALYSIS PROCEDURE AND THE SOLUTION

Consider an FGM annular plate with inner radius b, outer radius a and height h, as shown in Fig.1. The plate rotates about the axisymmetric axis z at a uniform angular velocity ω. Thus, we have $F_r=\rho r\omega^2$ and F_z=0 in Eq.(1), where $\rho=\rho(z)$ is the density of material. The boundary conditions can be expressed as

$$z = \pm h/2: \sigma_z = 0, \tau_{rz} = 0, \tag{3}$$

$$\begin{aligned} &r = a, b: N = \int_{-h/2}^{h/2} \sigma_r \mathrm{d}z = 0, \\ &M = \int_{-h/2}^{h/2} z\sigma_r \mathrm{d}z = 0,\ Q = \int_{-h/2}^{h/2} \tau_{rz} \mathrm{d}z = 0. \end{aligned} \tag{4}$$

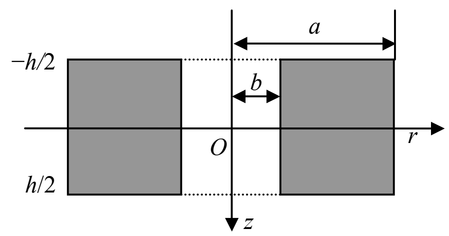

Fig.1 Cylindrical coordinate system for axisymmetric problem

We assume that the displacements take the forms of

$$\begin{aligned} &u = ru_1(z) + r^3u_3(z) + r^{-1}A(z), \\ &w = w_0(z) + r^2w_2(z) + r^4w_4(z) + B(z)\ln r/a, \end{aligned} \tag{5}$$

where u_i, w_i, A and B are undetermined displacement functions of z. Substitution of Eq.(5) into Eq.(2) gives

$$\begin{aligned} \sigma_r = &(c_{11} + c_{12})u_1 + c_{13}w_0' + [(3c_{11} + c_{12})u_3 + c_{13}w_2']r^2 \\ &+ c_{13}w_4'r^4 - (c_{11} - c_{12})Ar^{-2} + c_{13}B'\ln r/a, \end{aligned} \tag{6}$$

$$\sigma_\theta=(c_{11}+c_{12})u_1+c_{13}w_0'+[(3c_{12}+c_{11})u_3+c_{13}w_2']r^2 +c_{13}w_4'r^4+(c_{11}-c_{12})Ar^{-2}+c_{13}B'\ln r/a, \quad (7)$$

$$\sigma_z=2c_{13}u_1+c_{33}w_0'+(4c_{13}u_3+c_{33}w_2')r^2 +c_{33}w_4'r^4+c_{33}B'\ln r/a, \quad (8)$$

$$\tau_{rz}=c_{44}(u_1'+2w_2)r+c_{44}(u_3'+4w_4)r^3+c_{44}(A'+B)r^{-1}, \quad (9)$$

where a prime indicates differentiation with respect to z. Substitution of Eqs.(6)~(9) into Eq.(1), yields

$$[c_{44}(u_1'+2w_2)]'+8c_{11}u_3+2c_{13}w_2'+\rho\omega^2=0, \quad (10)$$

$$[c_{44}(u_3'+4w_4)]'+4c_{13}w_4'=0, \quad (11)$$

$$[c_{44}(A'+B)]'+c_{13}B'=0, \quad (12)$$

$$(2c_{13}u_1+c_{33}w_0')'+2c_{44}(u_1'+2w_2)=0, \quad (13)$$

$$(4c_{13}u_3+c_{33}w_2')'+4c_{44}(u_3'+4w_4)=0, \quad (14)$$

$$(c_{33}w_4')'=0, \quad (15)$$

$$(c_{33}B')'=0. \quad (16)$$

Substituting Eqs.(8) and (9) into the boundary conditions, i.e. Eq.(3), gives rise to

$$2c_{13}(\pm h/2)u_1(\pm h/2)+c_{33}(\pm h/2)w_0'(\pm h/2)=0, \quad (17)$$

$$4c_{13}(\pm h/2)u_3(\pm h/2)+c_{33}(\pm h/2)w_2'(\pm h/2)=0, \quad (18)$$

$$c_{33}(\pm h/2)w_4'(\pm h/2)=0, \quad (19)$$

$$c_{33}(\pm h/2)B'(\pm h/2)=0, \quad (20)$$

$$c_{44}(\pm h/2)[u_1'(\pm h/2)+2w_2(\pm h/2)]=0, \quad (21)$$

$$c_{44}(\pm h/2)[u_3'(\pm h/2)+4w_4(\pm h/2)]=0, \quad (22)$$

$$c_{44}(\pm h/2)[A'(\pm h/2)+B(\pm h/2)]=0. \quad (23)$$

Integrating Eqs.(15) and (16) with respect to z, and noticing Eqs.(19) and (20), we have

$$w_4=b_6, \quad (24)$$

$$B=b_2, \quad (25)$$

where b_6 and b_2 are integral constants. Integrating Eqs.(11) and (12), and making use of Eqs.(22)~(25), we obtain

$$u_3=-4b_6(z+h/2)+b_5, \quad (26)$$

$$A=-b_2(z+h/2)+b_1, \quad (27)$$

where b_5 and b_1 are integral constants. We can obtain B=0 and A=0 from Eqs.(25) and (27) by setting b_2=0 and b_1=0. In this case, Eq.(5) can be employed for a solid disc (Chen *et al*., 2007). Integrating Eq.(14), and making use of Eqs.(18), (24) and (26), we derive

$$w_2=16b_6f_1(z)-4b_5f_0(z)+b_4, \quad (28)$$

where b_4 is an integral constant, and

$$f_n(z)=\int_{-h/2}^{z}(\varsigma+h/2)^n c_{13}/c_{33}\mathrm{d}\varsigma, (n=0,1). \quad (29)$$

Integrating Eq.(10), and noticing Eqs.(21), (26) and (28), we have

$$c_{44}(u_1'+2w_2)=32b_6g_1(z)-8b_5g_0(z)-\omega^2R_0(z), \quad (30)$$

$$32b_6g_1(h/2)-8b_5g_0(h/2)-\omega^2R_0(h/2)=0, \quad (31)$$

where

$$g_n(z)=\int_{-h/2}^{z}(\varsigma+h/2)^n(c_{11}-c_{13}^2/c_{33})\mathrm{d}\varsigma, \quad R_0(z)=\int_{-h/2}^{z}\rho(\varsigma)\mathrm{d}\varsigma, (n=0,1). \quad (32)$$

Integrating Eq.(30), and making use of Eq.(28), we obtain

$$u_1=-32b_6F_1(z)+8b_5F_0(z) -2b_4(z+h/2)+b_3-\omega^2R_2(z), \quad (33)$$

where b_3 is an integral constant, and

$$F_n(z)=\int_{-h/2}^{z}[f_n(\varsigma)-g_n(\varsigma)/c_{44}]\mathrm{d}\varsigma, \quad R_2(z)=\int_{-h/2}^{z}R_0(\varsigma)/c_{44}\mathrm{d}\varsigma, (n=0,1). \quad (34)$$

Integrating Eq.(13), and combining with Eqs.(17), (28) and (33), yields

$$2c_{13}u_1+c_{33}w_0'=-64b_6G_1(z)+16b_5G_0(z)+2\omega^2R_1(z), \quad (35)$$

$$32b_6G_1(h/2)-8b_5G_0(h/2)-\omega^2R_1(h/2)=0, \quad (36)$$

where

$$G_n(z)=\int_{-h/2}^{z}g_n(\varsigma)\mathrm{d}\varsigma, \quad R_1(z)=\int_{-h/2}^{z}R_0(\varsigma)\mathrm{d}\varsigma, (n=0,1). \quad (37)$$

In view of Eq.(33), the integration of Eq.(35), leads to

$$w_0 = 64b_6H_1(z) - 16b_5H_0(z) + 4b_4f_1(z) - 2b_3f_0(z) + 2\omega^2R_3(z) + b_0, \quad (38)$$

where b_0 is an integral constant, representing a rigid-body translation, which contributes nothing to the stress field, and

$$H_n(z) = \int_{-h/2}^{z}[c_{13}F_n(\varsigma) - G_n(\varsigma)]/c_{33}\mathrm{d}\varsigma,$$
$$R_3(z) = \int [R_1(\varsigma) + R_2(\varsigma)c_{13}]/c_{33}\mathrm{d}\varsigma, (n=0,1). \quad (39)$$

Substituting $u_1(z)$, $u_3(z)$, $w_0(z)$, $w_2(z)$, $w_3(z)$, $A(z)$ and $B(z)$ into Eqs.(5)~(9) leads to the expressions for displacements and stresses, in which $\tau_{rz}=c_{44}(u_1'+2w_2)r$. It can be shown that Eq.(36) also can be derived from the conditions $Q(a)=0$ and $Q(b)=0$, by virtue of the expressions for shear stress as well as Eqs.(30) and (37).

From Eq.(4), the resultant force $N(r)$ and the bending moment $M(r)$ can be written as

$$N(r) = \int_{-h/2}^{h/2}\sigma_r\mathrm{d}z = -4b_6(8N_{01} + M_{01}^0r^2) + b_5(8N_{00} + M_{00}^0r^2) - 2b_4M_{01}^1 + b_3M_{00}^1 + (b_2J_{01} - b_1J_{00})r^{-2} - \omega^2L_0, \quad (40)$$

$$M(r) = \int_{-h/2}^{h/2}z\sigma_r\mathrm{d}z = -4b_6(8N_{11} + M_{11}^0r^2) + b_5(8N_{10} + M_{10}^0r^2) - 2b_4M_{11}^1 + b_3M_{10}^1 + (b_2J_{11} - b_1J_{10})r^{-2} - \omega^2L_1, \quad (41)$$

where

$$N_{mn} = \int_{-h/2}^{h/2}z^m[c_1(z)F_n(z) + 2c_2(z)G_n(z)]\mathrm{d}z,$$
$$M_{mn}^l = \int_{-h/2}^{h/2}c_l(z)z^m(z+h/2)^n\mathrm{d}z,$$
$$M_{mn}^l = \int_{-h/2}^{h/2}c_l(z)z^m(z+h/2)^n\mathrm{d}z,$$
$$J_{mn} = \int_{-h/2}^{h/2}z^m(c_{11} - c_{12})(z+h/2)^n\mathrm{d}z,$$
$$L_m = \int_{-h/2}^{h/2}z^m[c_1(z)R_2(z) - 2c_2(z)R_1(z)]\mathrm{d}z,$$
$$c_2(z) = c_{13}/c_{33},\ c_1(z) = c_{12} + c_{11} - 2c_{13}^2/c_{33},$$
$$c_0(z) = 2c_1 + c_{11} - c_{12},\ (l,m,n=0,1). \quad (42)$$

Substituting Eqs.(40) and (41) into Eq.(4), we can derive the following equations to determine b_4, b_3, b_2 and b_1:

$$4b_6[8N_{01}+(a^2+b^2)M_{01}^0] - b_5[8N_{00} + (a^2+b^2)M_{00}^0]+2b_4M_{01}^1 - b_3M_{00}^1+\omega^2L_0=0, \quad (43)$$

$$4b_6[8N_{11} + (a^2+b^2)M_{11}^0] - b_5[8N_{10} + (a^2+b^2)M_{10}^0] + 2b_4M_{11}^1 - b_3M_{10}^1 + \omega^2L_1 = 0, \quad (44)$$

$$a^2b^2(4b_6M_{01}^0 - b_5M_{00}^0) + b_2J_{01} - b_1J_{00} = 0, \quad (45)$$

$$a^2b^2(4b_6M_{11}^0 - b_5M_{10}^0) + b_2J_{11} - b_1J_{10} = 0. \quad (46)$$

First, we can solve for b_5 and b_6 from Eqs.(33) and (36), then b_3 and b_4 from Eqs.(43) and (44), and finally b_1 and b_2 from Eqs.(45) and (46). Thus, the complete analytical solution for a rotating transversely isotropic FGM annular plate is derived.

It is easy to reduce the solution to that for a transversely isotropic FGM solid disc by setting $b=0$, $b_1=0$, and $b_2=0$. In this case, Eq.(45) is just the same as Eq.(46), and we can determine b_6, b_5, b_4 and b_3 from Eqs.(33), (36), (43) and (44).

NUMERICAL EXAMPLE

Consider a particular FGM with $c_{11}=c_{11}^0\mathrm{e}^{\kappa\eta}$, $c_{33}=c_{33}^0\mathrm{e}^{-\kappa\eta}$, $c_{12}=c_{12}^0$, $c_{13}=c_{13}^0$, $c_{44}=c_{44}^0$, $\rho=\rho_0\mathrm{e}^{-2\kappa\eta}$, where c_{ij}^0 and ρ_0 are material constants at $z=-h/2$, $\eta=z/h+1/2$ is a dimensionless thickness coordinate, and κ is the gradient index, which determines the degree of variation of material properties along the thickness. Obviously, $\kappa=0$ corresponds to the homogeneous material.

The parameters used in the numerical example are listed in Table 1. Note that the constant b_0 can be obtained by setting the axial displacement $w=0$ at the point ($r=0.1$, $z=0$).

The following dimensionless quantities are introduced for convenience

$$\xi = r/a,\ \zeta = z/a,\ \Sigma_i = \sigma_i/(\rho_0\omega^2a^2),$$
$$U = uc_{11}^0/(\rho_0\omega^2a^3),\ W = wc_{11}^0/(\rho_0\omega^2a^3). \quad (47)$$

The distributions of the axial displacement, radial displacement, radial stress, and hoop stress at the mid-plane (i.e. $z=0$) are shown in Figs.2~5 for three values of the gradient index ($\kappa=0$, 1 and 2.5). It is shown that gradient index κ has an obvious influence on the displacements and stresses. Stress concentra-

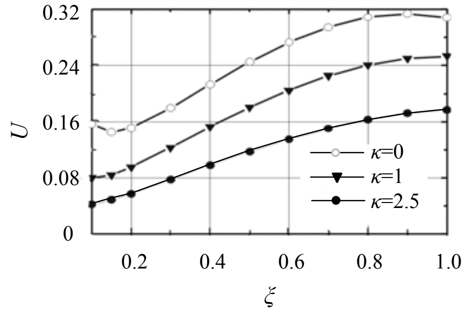

Fig.2 Dimensionless radial displacement at z=0

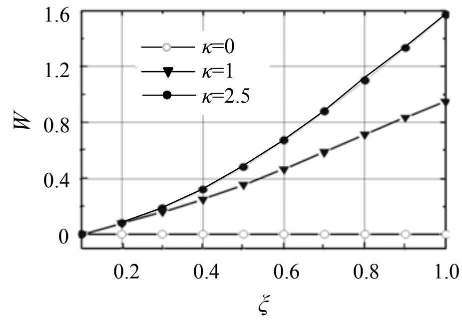

Fig.3 Dimensionless axial displacement at z=0

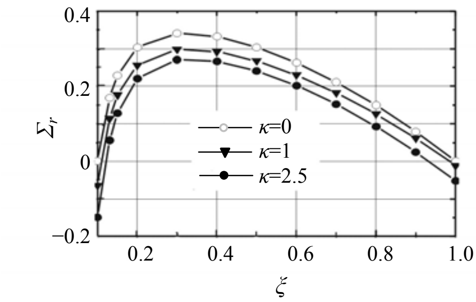

Fig.4 Dimensionless radial stress at z=0

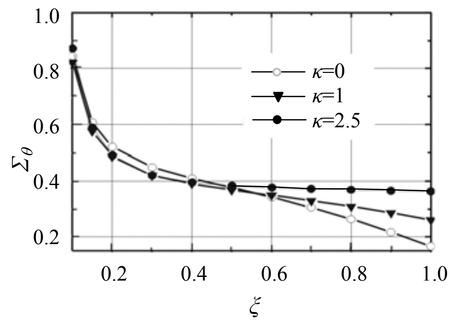

Fig.5 Dimensionless hoop stress at z=0

Table 1 Parameters employed in the numerical example

Parameters	Values
a (m)	1
b (m)	0.1
h (m)	0.1
ω (rad/s)	62.8
ρ_0 (kg/m^3)	7.6×10^3
c_{11}^0 (GPa)	286
c_{12}^0 (GPa)	173
c_{13}^0 (GPa)	170
c_{33}^0 (GPa)	269.5
c_{44}^0 (GPa)	45.3

tion at the internal boundary of the annular plate (near ξ=0.1) is clearly shown in Fig.5, and the hoop stress decreases with ξ. A more uniform distribution of hoop stress versus the radial coordinate is obtained when κ=2.5. The result for κ=0 coincides with those obtained by Chen *et al.*(2003), where only the homogeneous case was considered.

CONCLUSION

The 3D analytical solution for a rotating transversely isotropic FGM annular plate is derived. There is no restriction imposed on graded properties along the thickness. The solution can be readily degenerated into that for a solid disc obtained by Chen *et al.*(2007).

Numerical results show that the elastic parameters varying in the z-direction can affect the distribution of hoop stress in the r-direction. Hence the stress level in the rotating annular plate may be optimally designed by selecting a proper distribution of material properties.

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