Journal of Zhejiang University SCIENCE A
ISSN 1673-565X (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Mobility and equilibrium stability analysis of pin-jointed mechanisms with equilibrium matrix SVD*

LU Jin-yu, LUO Yao-zhi†‡, LI Na

(*Department of Civil Engineering, Zhejiang University, Hangzhou 310027, China*)

†E-mail: luoyz@zju.edu.cn

Received Oct. 20, 2006; revision accepted Jan. 17, 2007

Abstract: Under certain load pattern, the geometrically indeterminate pin-jointed mechanisms will present certain shapes to keep static equalization. This paper proposes a matrix-based method to determine the mobility and equilibrium stability of mechanisms according to the effects of the external loads. The first and second variations of the potential energy function of mechanisms under conservative force field are analyzed. Based on the singular value decomposition (SVD) method, a new criterion for the mobility and equilibrium stability of mechanisms can be concluded by analyzing the equilibrium matrix. The mobility and stability of mechanisms can be classified by unified matrix formulae. A number of examples are given to demonstrate the proposed criterion. In the end, criteria are summarized in a table.

Key words: Pin-jointed mechanisms, Criteria for stability of equilibrium, Criteria for mobility, Potential energy function, Equilibrium matrix, Singular value decomposition (SVD) method

doi:10.1631/jzus.2007.A1091 **Document code:** A **CLC number:** TU31

INTRODUCTION

Researches on the stability of physical system, especially mechanical system, can be traced back to Lagrange Times, two centuries ago. Potential energy Φ can be expressed as a function of generalized coordinates Q in conservative systems. In the field of conservative force, the first variation of Φ being equal to zero is the necessary and sufficient criterion of equilibrium. There are three different equilibrium states (Belytschko *et al.*, 2000): stable equilibrium (Fig.1a), unstable equilibrium (Fig.1b) and indifferent equilibrium (Fig.1c). Acted on by a small disturbance, the ball in Fig.1a can return to the initial place and keep in equilibrium; the ball in Fig.1b cannot return to the initial place and enters the non-equilibrium state; the ball in Fig.1c cannot return to the initial place but can still keep equilibrium in another place. For a single degree-of-freedom (SDOF) system, it is under stable equilibrium if the second variation of Φ (i.e. $\delta^2\Phi$) is positive and under unstable equilibrium if $\delta^2\Phi$ is negative.

Two types of inextensional mechanisms are discussed in this paper, which can be described as infinitesimal mechanism and finite mechanism. In a finite mechanism, the joints can be moved a certain distance without causing any change in the length of the bars; while in an infinitesimal mechanism, some small changes in length of the bars will occur when the joints move.

The "Maxwell's rule" (Maxwell, 1890) is a famous criterion to judge the geometrical stability of structures using the bar number B, node number J and constraint number C (Structure is geometrically invariant if $B\geq 3J-C$ and geometrically variant if $B<3J-C$), with it being of great importance in structural design and construction. However, this rule only gives the necessary condition of geometrical stability. Geometrical stability is stability from the purely

‡ Corresponding author

* Project supported by the National Natural Science Foundation of China (Nos. 50378083 and 50638050), and the Research Foundation for the Doctoral Program of Higher Education of China (No. 20050335097)

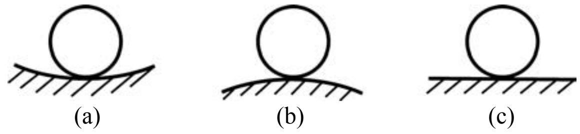

Fig.1 Equilibrium on the surface of sphere. (a) Stable equilibrium; (b) Unstable equilibrium; (c) Indifferent equilibrium

geometrical view. According to "Maxwell's rule", the structures in Fig.2a and Fig.2b are both geometrically invariant, while the structure in Fig.2b is actually an infinitesimal mechanism. After this problem was found, the geometrical stability of infinitesimal mechanism was widely studied. By analyzing the variation of strain energy function Φ, Kötter studied if infinitesimal mechanism can be stiffened by self-stress (Pellegrino, 2001). Equilibrium equations were found by calculating the first variation of Φ. Kötter pointed out that the mechanism could be stiffened if $\delta^2\Phi$ is positive definite, and the method can also be used when the number of independent states of self-stress is greater than one. Kuznetsov (1988) found that if self-stress was applied to bars, the infinitesimal mechanism would be geometrically stable. Calladine and Pellegrino classified the structural assemblies (Pellegrino, 1990; 1993). Based on singular value decomposition method, the criterion for geometrical stability of mechanism was presented (Pellegrino, 1990; 1993). If self-stress can provide first order stiffness to the mechanism displacement, it is infinitesimal mechanism. The product of geometrical force and the modes of inextensional mechanisms were used to determine a more general stability with both geometry and assembly topology considered (Pellegrino and Calladine, 1986). Fowler and Guest (2000) made a symmetry extension of Maxwell's rule. And the mobility of highly symmetric structures is also discussed (Kovács, 2004). Tarnai and Szabó (2002) analyzed the

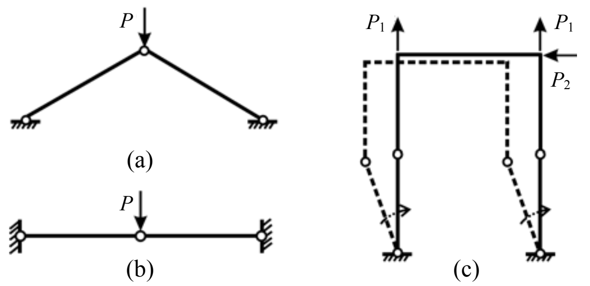

Fig.2 Structures under loads. (a) Determinate structure; (b) Infinitesimal mechanism; (c) Finite mechanism

rigidity and stability of prestressed infinitesimal mechanisms. The geometrical stability of other structures (such as cable-strut tensile structures) was studied (Luo, 2000; Luo and Dong, 2002).

The mechanisms discussed above are mostly infinitesimal mechanisms with the geometrical stability of finite mechanism being rarely further discussed. The researches on finite mechanism mainly concentrate on the bifurcation of motion (Tarnai and Szabó, 2000; Lengyel and You, 2004), because finite mechanism is considered as geometrically unstable by both "Maxwell's rule" and Pellegrino's criterion (Pellegrino and Calladine, 1986). Few articles deal with the more general stability of mechanisms.

Apparently, the finite mechanism in Fig.2c is in stable equilibrium under load P_1, which means it can resist perturbations. Taking the mechanism shown in Fig.3 as example, two types of external loads P are applied, and we can analyze the mechanism by calculating the first and second variation of the potential energy function.

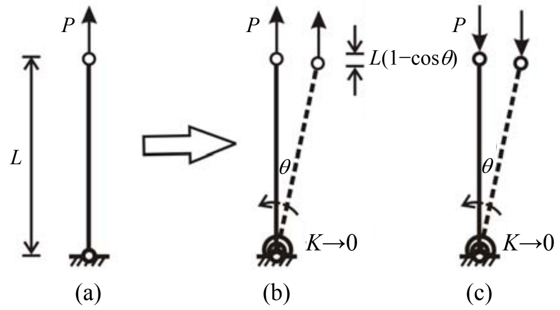

Fig.3 One-bar mechanism. (a) Topology; (b), (c) Load patterns

Supposing a spring with a rotated stiffness K is added to the constrained node, when small disturbing displacement θ is acting on the mechanism, the potential energy function of the mechanism can be found by (taking the initial state as zero potential energy state)

$$\Phi=K\theta-PL(1-\cos\theta).$$

Then the first and second partial derivatives of the potential energy are

$$\partial\Phi/\partial\theta=K-PL\sin\theta,\ \partial^2\Phi/\partial\theta^2=-PL\cos\theta.$$

Let $\partial\Phi/\partial\theta=0$ and $K\to 0$, we can get $\theta=0$. Substituting θ=0 into $\partial^2\Phi/\partial\theta^2$, we get $\partial^2\Phi/\partial\theta^2=-PL$. If P<0, as shown in Fig.3b, the mechanism is under stable equilibrium ($\partial^2\Phi/\partial\theta^2$>0); if P>0, as shown in Fig.3c, the mechanism is under unstable equilibrium ($\partial^2\Phi/\partial\theta^2$<0).

Therefore, a mechanism that is geometrically unstable cannot only bear certain external loads but also resist perturbations (Fig.2b, Fig.3b), so the stability of a mechanism is relating to several aspects. Stability of mechanism should include geometrical stability and equilibrium stability. Geometrical stability only depends on its topology with nothing to do with the external loads, while topology and load effects are considered together in equilibrium stability. Though Pellegrino introduced the concept of geometrical loads (Pellegrino and Calladine, 1986), it is still purely geometrical just as "Maxwell's rule". This paper proposes an efficient matrix-based criterion for mobility and stability of equilibrium of mechanisms instead of calculating potential energy function directly. It presents a generalized numerical algorithm and improves the stability theory of mechanism.

This paper is constructed as follows: Section 2 introduces the potential energy function of mechanism in the conservative force field, and the first and second variations of potential energy function are analyzed; Section 3 introduces the singular value decomposition (SVD) method of equilibrium matrix; in Section 4, a matrix-based criterion for the mobility of mechanism and the corresponding formulae are presented; in Section 5, the details of the criterion for the stability of equilibrium mechanism are discussed, and by calculating the states of inextensional mechanism and Hessian matrix, the formulae are derived and three different equilibrium states existing in mechanisms are defined; in Section 6, a few typical examples are given to demonstrate the criterion formulae derived in this paper; in Section 7, future work is suggested and criteria are summarized in a table.

POTENTIAL ENERGY FUNCTION

Supposing load P_i acts on generalized coordinates Q_i, the potential energy of the mechanism under kinematic constraint F_k=0 is (Pellegrino, 2001)

$$\boldsymbol{\Pi}_R = -\sum_{i=1}^{n} P_i(Q_i - Q_i^0) + \sum_{k=1}^{c} \Lambda_k F_k, \tag{1}$$

where P_i are external loads applied on nodes, Q_i^0 and Q_i are the reference and current generalized coordinates respectively, Λ_k are Lagrange multipliers (which can be internal force of bars), F_k are kinematic constraint functions: $F_k(Q_1, \ldots, Q_n)$=0, n is the number of generalized coordinates, and c is the number of generalized constraints.

The equilibrium state of this system can be obtained by taking the first variation of $\boldsymbol{\Pi}_R$ as zero:

$$\delta\boldsymbol{\Pi}_R=\mathbf{0}. \tag{2}$$

Then we have

$$\delta\boldsymbol{\Pi}_R = \begin{bmatrix} \left(\dfrac{\partial \boldsymbol{\Pi}_R}{\partial Q_i}\right)^{\mathrm{T}} & \left(\dfrac{\partial \boldsymbol{\Pi}_R}{\partial \Lambda_k}\right)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} (\delta Q_i) \\ (\delta \Lambda_k) \end{bmatrix} = \mathbf{0}.$$

So we can obtain the equilibrium equation

$$\left(\frac{\partial \boldsymbol{\Pi}_R}{\partial Q_i}\right) = -P_i + \sum_{k=1}^{c} \Lambda_k \frac{\partial F_k}{\partial Q_i} = 0,\ i = 1, \ldots, n, \tag{3}$$

and the compatibility equation

$$\left(\frac{\partial \boldsymbol{\Pi}_R}{\partial \Lambda_k}\right) = F_k = 0, k=1, \ldots, c. \tag{4}$$

Substituting the external load P_i from Eq.(3) into the second variation $\delta^2\boldsymbol{\Pi}_R$, the criterion for stability of the mechanism can be established as

$$\begin{cases} \delta^2 \boldsymbol{\Pi}_R > 0, & \text{stable;} \\ \delta^2 \boldsymbol{\Pi}_R < 0, & \text{unstable.} \end{cases} \tag{5}$$

Expanding $\delta^2\boldsymbol{\Pi}_R$ to matrix form, then we get

$$\delta^2 \boldsymbol{\Pi}_R = [(\delta Q_i)^{\mathrm{T}}\ \ (\delta \Lambda_k)^{\mathrm{T}}] \begin{bmatrix} \left(\displaystyle\sum_{h=1}^{c} \Lambda_h \frac{\partial^2 F_h}{\partial Q_i \partial Q_j}\right) & \left(\dfrac{\partial F_i}{\partial Q_k}\right) \\ \left(\dfrac{\partial F_k}{\partial Q_i}\right) & (\boldsymbol{F}) \end{bmatrix} \times \begin{bmatrix} (\delta Q_j) \\ (\delta \Lambda_k) \end{bmatrix}. \tag{6}$$

Applying the following notations

$$\boldsymbol{H}=\left[\sum_{h=1}^{c}\Lambda_h\frac{\partial^2 F_h}{\partial Q_i\partial Q_j}\right]_{n\times n},\quad \boldsymbol{J}=\left[\frac{\partial F_k}{\partial Q_i}\right]_{c\times n},$$

$\mathrm{d}\boldsymbol{Q}=[\delta Q_i]_{n\times 1}$ and $\mathrm{d}\boldsymbol{P}=[\delta P_j]_{n\times 1}$,

we have

$$\delta^2\boldsymbol{\Pi}_R=[\mathrm{d}\boldsymbol{Q}^{\mathrm{T}}\quad \mathrm{d}\boldsymbol{\Lambda}^{\mathrm{T}}]\begin{bmatrix}\mathrm{d}\boldsymbol{P}\\ \mathrm{d}\boldsymbol{p}\end{bmatrix}, \tag{7}$$

$$\begin{bmatrix}\mathrm{d}\boldsymbol{P}\\ \mathrm{d}\boldsymbol{p}\end{bmatrix}=\begin{bmatrix}\boldsymbol{H} & \boldsymbol{J}^{\mathrm{T}}\\ \boldsymbol{J} & \boldsymbol{F}\end{bmatrix}\begin{bmatrix}\mathrm{d}\boldsymbol{Q}\\ \mathrm{d}\boldsymbol{\Lambda}\end{bmatrix}, \tag{8}$$

where $\boldsymbol{H}$ is the Hessian matrix of $\sum_{h=1}^{c}\Lambda_h F_h$, $\boldsymbol{J}$ is the Jacobian matrix of F_k, $\boldsymbol{F}$ is the flexibility matrix, $\mathrm{d}\boldsymbol{Q}$ is the increment of nodal displacement, $\mathrm{d}\boldsymbol{\Lambda}$ is the increment of bar internal force, $\mathrm{d}\boldsymbol{P}$ is the increment of nodal load, $\mathrm{d}\boldsymbol{p}$ is the increment of kinematic load.

For mechanism consisting of rigid bars and pin-joints, F_k in Eq.(1) can be expressed as

$$F_k=\sqrt{(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2}-L_k=0. \tag{9}$$

We also have $c<n$.

As the elastic deformation of the bars can be neglected, the increment of bar internal force equals zero and the bar flexibility is infinitesimal, i.e.

$$\boldsymbol{F}=\boldsymbol{0}, \tag{10}$$
$$\mathrm{d}\boldsymbol{\Lambda}=\boldsymbol{0}. \tag{11}$$

Substituting Eq.(10) and Eq.(11) into Eq.(7) and Eq.(8), we have

$$\delta^2\boldsymbol{\Pi}_R=\mathrm{d}\boldsymbol{Q}^{\mathrm{T}}\cdot\mathrm{d}\boldsymbol{P}, \tag{12}$$
$$\mathrm{d}\boldsymbol{P}=\boldsymbol{H}\cdot\mathrm{d}\boldsymbol{Q}. \tag{13}$$

$\delta^2\boldsymbol{\Pi}_R$ can be simplified as

$$\delta^2\boldsymbol{\Pi}_R=\mathrm{d}\boldsymbol{Q}^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\mathrm{d}\boldsymbol{Q}.$$

So the necessary and sufficient criterion for stable mechanism is

$$\mathrm{d}\boldsymbol{Q}^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\mathrm{d}\boldsymbol{Q}>0. \tag{14}$$

MATRIX ANALYSIS BASED ON SINGULAR VALUE DECOMPOSITION METHOD

To set up the equilibrium equation at node i, the sub-structure including all the bars connected to node i, as shown in Fig.4, is considered. The Cartesian coordinates of node i at its current position are taken as x_i, y_i and z_i, and coordinates of nodes j and k are similarly denoted. The lengths of elements k and l are taken as L_k and L_l. Assuming $\boldsymbol{P}_i$ is the external force vector at node i, $\boldsymbol{P}_i=(P_{ix},P_{iy},P_{iz})^{\mathrm{T}}$, t_k and t_l are the internal forces of elements k and l. According to Eq.(3), the equilibrium equation at node i can be expressed as

$$\begin{cases}\dfrac{x_i-x_h}{L_l}t_l+\dfrac{x_i-x_j}{L_k}t_k=P_{ix},\\ \dfrac{y_i-y_h}{L_l}t_l+\dfrac{y_i-y_j}{L_k}t_k=P_{iy},\\ \dfrac{z_i-z_h}{L_l}t_l+\dfrac{z_i-z_j}{L_k}t_k=P_{iz}.\end{cases} \tag{15}$$

Rewrite Eq.(15) in matrix form as

$$\begin{pmatrix}\dfrac{x_i-x_h}{L_l} & \dfrac{x_i-x_j}{L_k}\\ \dfrac{y_i-y_h}{L_l} & \dfrac{y_i-y_j}{L_k}\\ \dfrac{z_i-z_h}{L_l} & \dfrac{z_i-z_j}{L_k}\end{pmatrix}\begin{Bmatrix}t_l\\ t_k\end{Bmatrix}=\begin{Bmatrix}P_{ix}\\ P_{iy}\\ P_{iz}\end{Bmatrix}. \tag{16}$$

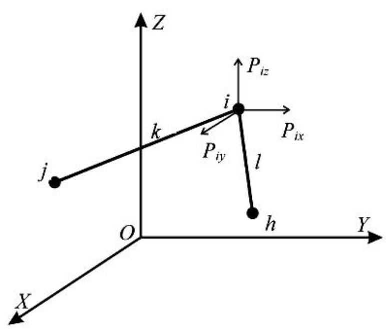

Fig.4 Geometrical relationships

Assembling all the node equilibrium equations belonging to the same structure, we can get the global equilibrium equation (Pellegrino, 1990; 1993)

$$\boldsymbol{At}=\boldsymbol{P}, \quad (17)$$

where $\boldsymbol{A}$ is equilibrium matrix, $\boldsymbol{t}$ is internal force vector, and $\boldsymbol{P}$ is external load vector. Here $\boldsymbol{A}$ and $\boldsymbol{t}$ are corresponding to $\boldsymbol{J}^{\mathrm{T}}$ and d$\boldsymbol{\Lambda}$ in the last section, respectively.

Take the dimension of $\boldsymbol{A}$ to be $n_r \times n_c$, and the rank of $\boldsymbol{A}$ to be r, according to the Linear Algebra Theory, the SVD expression of $\boldsymbol{A}$ is

$$\exists \boldsymbol{U} \in \mathbb{C}^{n_r \times n_r},\ \boldsymbol{V} \in \mathbb{C}^{n_c \times n_c}, \text{ and } \boldsymbol{S} \in \mathbb{C}^{r \times r},$$

$$\text{subject to} \quad \boldsymbol{A} = \boldsymbol{U} \begin{bmatrix} \boldsymbol{S} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \boldsymbol{V}^{\mathrm{T}}, \quad (18)$$

where $n_r=3J-C$, $n_c=B$, and J, B and C mean the numbers of nodes, elements and generalized constraints. $\boldsymbol{S}=\mathrm{diag}(s_{11},s_{22},\ldots,s_{rr})$, which are non-zero singular values and $s_{11}\geq s_{22}\geq\ldots\geq s_{rr}\geq 0$. $\boldsymbol{U}$ and $\boldsymbol{V}$ are orthogonal matrices, and can be decomposed as $\boldsymbol{U}=[\boldsymbol{U}_r\ \boldsymbol{U}_m]$, $\boldsymbol{V}=[\boldsymbol{V}_r\ \boldsymbol{V}_s]$.

Two variables m and s are introduced to represent the number of inextensional mechanism and the number of independent states of self-stress, with them being defined as $m=n_r-r$ and $s=n_c-r$. The orthogonalities of SVD method are presented as

$$\boldsymbol{A}\boldsymbol{V}_s=\boldsymbol{0}, \quad (19)$$

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{U}_m=\boldsymbol{0}, \quad (20)$$

where $\boldsymbol{U}_m$ are the states of inextensional mechanisms, and $\boldsymbol{V}_s$ are independent states of self-stress. The states of inextensional mechanisms that represent the mode of inextensional deformations are a group of orthogonal basis for space of nodal displacements, and the independent states of self-stress that represent the mode of internal force in bars are orthogonal basis for space of internal forces.

The classification of structural assemblies is given in Table 1 (Pellegrino, 1990; 1993). The mechanisms that are discussed in this paper belong to type II and type IV. Type II is a finite mechanism and Eq.(17) has a unique solution $\boldsymbol{t}$ for some particular load $\boldsymbol{P}$, which can be defined as the equilibrium load of the corresponding mechanism. Type IV is mostly infinitesimal mechanisms and Eq.(17) has infinite s-dimensional solution $\boldsymbol{t}=\boldsymbol{t}'+\boldsymbol{V}_s\alpha$.

Table 1 Classification of structural assemblies

Assembly type	Static and kinematic properties
I	s=0: statically determinate; m=0: kinematically determinate
II	s=0: statically determinate; m>0: kinematically indeterminate
III	s>0: statically indeterminate; m=0: kinematically determinate
IV	s>0: statically indeterminate; m>0: kinematically indeterminate

MATRIX-BASED CRITERION FOR THE MOBILITY OF MECHANISM

In the reference configuration, without external loads ($\boldsymbol{P}=\boldsymbol{0}$), $\boldsymbol{t}$ is called self-stress vector. It satisfies

$$\boldsymbol{At}=\boldsymbol{0}. \quad (21)$$

Coefficient matrix $\boldsymbol{A}$ is full-column-rank when $m>0$ and $s=0$. It corresponds to type II in Table 1. According to Linear Algebra Theory, homogenous equation Eq.(21) does not have non-zero solution ($\boldsymbol{t}\equiv\boldsymbol{0}$), which means that finite mechanism cannot keep equilibrium state depending on the transfer of internal force. However, when applied with certain external load $\boldsymbol{P}$, the mechanism could maintain force equilibrium with Eq.(17) having a unique non-zero solution $\boldsymbol{t}$.

The criterion for mechanism mobility is discussed (on whether a mechanism is in equilibrium state). We concluded that when external load vector $\boldsymbol{P}$ orthogonalizes to the states of inextensional mechanisms $\boldsymbol{U}_m$ ($\boldsymbol{U}_m^{\mathrm{T}}\boldsymbol{P}=\boldsymbol{0}$), the mechanism would be stiffened by load $\boldsymbol{P}$ and cannot move. It can be proved as follows.

The equivalent question of whether the mechanism is in equilibrium state is whether Eq.(17) has non-zero solution, so non-zero solution $\boldsymbol{t}$ for Eq.(17) is the necessary and sufficient condition for immobility of the mechanism.

The necessary and sufficient condition for Eq.(17) having non-zero solution is

$$\mathrm{rank}(\boldsymbol{A})=\mathrm{rank}(\boldsymbol{A}|\boldsymbol{P}), \quad (22)$$

where $\mathrm{rank}(\boldsymbol{A})=r=n_c$.

Applying SVD to equilibrium matrix $\boldsymbol{A}$, and according to Eq.(20), $\boldsymbol{U}_m$ satisfies

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{U}_m=\boldsymbol{0}.$$

Pre-multiply both sides of Eq.(20) by non-zero vector $\boldsymbol{t}^{\mathrm{T}}$ and transform it, and then we can get

$$\begin{gathered}\boldsymbol{t}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{U}_m=\boldsymbol{t}^{\mathrm{T}}\cdot\boldsymbol{0};\\ (\boldsymbol{t}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{U}_m)^{\mathrm{T}}=\boldsymbol{0};\\ \boldsymbol{U}_m^{\mathrm{T}}\boldsymbol{A}\boldsymbol{t}=\boldsymbol{0}.\end{gathered} \tag{23}$$

Substituting Eq.(17) into Eq.(23),

$$\boldsymbol{U}_m^{\mathrm{T}}\boldsymbol{P}=\boldsymbol{0}. \tag{24}$$

It can be used as a criterion for the mobility of the mechanism. A mechanism is not mobile if its external load satisfies Eq.(24). Otherwise, if

$$\boldsymbol{U}_m^{\mathrm{T}}\boldsymbol{P}\neq\boldsymbol{0}, \tag{25}$$

the mechanism is mobile. The external load $\boldsymbol{P}$ that satisfies Eq.(24) can be defined as equilibrium load.

One more thing should be pointed out is that when using Eq.(24) to determine the mechanism mobility with $m>1$, all the vectors $\boldsymbol{U}_m$ ($\boldsymbol{U}_m^{(r)}$, $r=1, \ldots, m$) should be checked.

MATRIX-BASED CRITERION FOR THE EQUILIBRIUM STABILITY OF MECHANISM

A mechanism is in the equilibrium state $\boldsymbol{\Psi}^0$ when equilibrium load $\boldsymbol{P}$ is applied. Under perturbations $\delta\boldsymbol{d}$, a mechanism in equilibrium will behave in three different states: returning to $\boldsymbol{\Psi}^0$, moving to another equilibrium state $\boldsymbol{\Psi}'$ and being equilibrium in state $\boldsymbol{\Psi}^{\delta\boldsymbol{d}}$, which is corresponding to Figs.1a, 1b and 1c (Belytschko *et al.*, 2000). Indeed, stability of equilibrium can imply resistance to perturbations.

The internal force vector of bars is given in Eq.(17). Since the increment of joint displacement $\mathrm{d}\boldsymbol{Q}$ in Eq.(14) equals inextensional mechanisms

$$\mathrm{d}\boldsymbol{Q}=\boldsymbol{U}_m. \tag{26}$$

The second variation of potential energy can be rewritten as

$$\delta^2\boldsymbol{\Pi}_R=\boldsymbol{U}_m{}^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{U}_m. \tag{27}$$

We can discuss the procedure of calculation of Hessian matrix $\boldsymbol{H}$. The sub-matrix of $\boldsymbol{H}$ at node i corresponding to Fig.4 is

$$\boldsymbol{H}^{(i)}=\begin{bmatrix}\frac{t_l}{L_l}+\frac{t_k}{L_k} & 0 & 0 & -\frac{t_k}{L_k} & 0 & 0 & -\frac{t_l}{L_l} & 0 & 0\\ 0 & \frac{t_l}{L_l}+\frac{t_k}{L_k} & 0 & 0 & -\frac{t_k}{L_k} & 0 & 0 & -\frac{t_l}{L_l} & 0\\ 0 & 0 & \frac{t_l}{L_l}+\frac{t_k}{L_k} & 0 & 0 & -\frac{t_k}{L_k} & 0 & 0 & -\frac{t_l}{L_l}\end{bmatrix}\begin{matrix}\cdots x_i\\ \cdots y_i\\ \cdots z_i\end{matrix}, \tag{28}$$

where t_l and t_k are internal forces of bar l and k. They are both components of vector $\boldsymbol{t}$. The number of columns for $\boldsymbol{H}^{(i)}$ denoted as ζ_i equals the number of degree-of-freedom (DOF) that connects to node i including the DOF belonging to node i (e.g. $\zeta_i=9$ in Fig.4). t_l/L_l and t_k/L_k are force density coefficients of bars l and k. $\boldsymbol{H}_{n\times n}$ is constituted by $\boldsymbol{H}^{(i)}$.

Since our aim is to set up an easy computational framework for all types of mechanisms, it is important that the dimensions of all $\boldsymbol{H}^{(i)}$ are equal. However, ζ_i varies from node to node in the same mechanism, which is quite difficult to program. So we re-assemble $\boldsymbol{H}_{\kappa_i}^{(k)}$ based on bar k and new variable κ_i:

$$\begin{aligned}\boldsymbol{H}_{\kappa_i=1}^{(k)}&=\begin{bmatrix}\frac{t_k}{L_k} & 0 & 0 & -\frac{t_k}{L_k} & 0 & 0\end{bmatrix},\\ \boldsymbol{H}_{\kappa_i=2}^{(k)}&=\begin{bmatrix}0 & \frac{t_k}{L_k} & 0 & 0 & -\frac{t_k}{L_k} & 0\end{bmatrix},\\ \boldsymbol{H}_{\kappa_i=3}^{(k)}&=\begin{bmatrix}0 & 0 & \frac{t_k}{L_k} & 0 & 0 & -\frac{t_k}{L_k}\end{bmatrix},\end{aligned} \tag{29}$$

where κ_i (=1, 2, 3) denotes the DOF number of the beginning node that attaches to bar k. Considering boundary conditions of the mechanism, we can get $\boldsymbol{H}_{n\times n}$ by assembling $\boldsymbol{H}_{\kappa_i}^{(k)}$. This method is suitable for computer coding and analysis of different mechanisms.

Thus, we get a criterion for stability of equilibrium:

Case 1 For single DOF mechanism, $m=1$,

$$\boldsymbol{U}_m^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{U}_m>0\rightarrow\text{stable equilibrium},$$
$$\boldsymbol{U}_m^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{U}_m=0\rightarrow\text{indifferent equilibrium},$$
$$\boldsymbol{U}_m^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{U}_m<0\rightarrow\text{unstable equilibrium}. \quad (30)$$

Case 2 For multi-DOF mechanism, $m>1$,

$$\boldsymbol{U}_m^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{U}_m \text{ is positive definite}\rightarrow\text{stable equilibrium},$$
$$\boldsymbol{U}_m^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{U}_m=\boldsymbol{0}\rightarrow\text{indifferent equilibrium},$$
$$\boldsymbol{U}_m^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{U}_m \text{ is negative definite}\rightarrow\text{unstable equilibrium}. \quad (31)$$

If $\boldsymbol{U}_m^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{U}_m$ is positive semidefinite, higher order of potential energy $\delta^t\boldsymbol{\Pi}_R$ ($t>2$) should be considered.

According to Linear Algebra Theory, we can get

$$\boldsymbol{H}_{n\times n} \text{ is positive definite}\leftrightarrow\forall\boldsymbol{X}_{n\times 1}\ (=\{x_1,x_2,\ldots,x_n\}^{\mathrm{T}}) \text{ subject to } \boldsymbol{X}^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{X}>0. \quad (32)$$

Obviously, linear space formed by $\boldsymbol{X}$ includes that formed by vector $\boldsymbol{U}_m$, $\boldsymbol{R}^n(\boldsymbol{U}_m)\subseteq\boldsymbol{R}^n(\boldsymbol{X})$. So $\boldsymbol{H}_{n\times n}$ can be defined as the sufficient criterion for stability of equilibrium of mechanisms. We have

$$\boldsymbol{H}_{n\times n} \text{ is positive definite}\rightarrow\text{stable equilibrium}, \quad (33)$$
$$\boldsymbol{H}_{n\times n} \text{ is negative definite}\rightarrow\text{unstable equilibrium}. \quad (34)$$

EXAMPLES

Equilibrium stability evaluation for a single-bar mechanism

Fig.3 shows a simple finite mechanism that $m=1$ and $s=0$, and it is applied with two different external forces.

For the case shown in Fig.3b, there are

$$\boldsymbol{H}=\begin{bmatrix}1&0\\0&1\end{bmatrix},\ \boldsymbol{U}_m=\begin{bmatrix}1\\0\end{bmatrix},\ \boldsymbol{P}=\begin{bmatrix}0\\1\end{bmatrix}.$$

Hence $\boldsymbol{U}_m^{\mathrm{T}}\boldsymbol{P}=0$, and the mechanism is not mobile. $\boldsymbol{H}$ is positive definite, and the equilibrium of the mechanism is stable.

For the case shown in Fig.3c, there are

$$\boldsymbol{H}=\begin{bmatrix}-1&0\\0&-1\end{bmatrix},\ \boldsymbol{U}_m=\begin{bmatrix}1\\0\end{bmatrix},\ \boldsymbol{P}=\begin{bmatrix}0\\-1\end{bmatrix}.$$

Hence $\boldsymbol{U}_m^{\mathrm{T}}\boldsymbol{P}=0$, and the mechanism is not mobile. $\boldsymbol{H}$ is negative definite, and the equilibrium of the mechanism is unstable.

Equilibrium stability evaluation for a four-bar mechanism

In this example, a four-bar mechanism, consisting of rigid bars and pin joints, subjected to several different load patterns is considered. The main data of the mechanism are illustrated in Fig.5a. It is a finite mechanism, $m=1$, $s=0$, which can be indicated by analyzing the equilibrium matrix

$$\boldsymbol{A}=\begin{bmatrix}0&-1&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

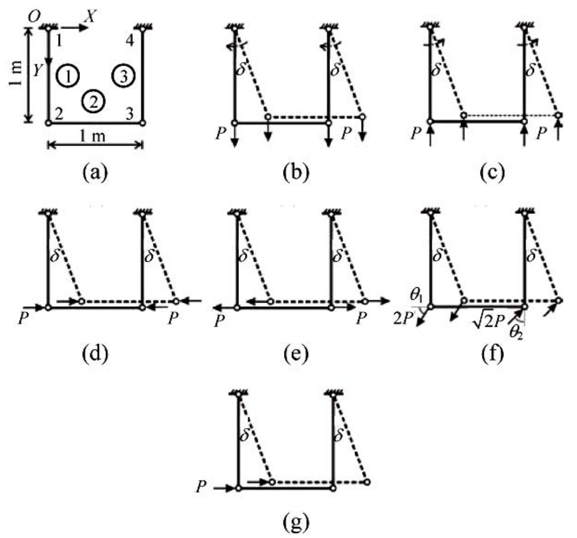

Fig.5 A four-bar mechanism. (a) Geometrical parameters; (b)~(g) Applied with different load patterns

When certain concentrated loads are applied, the internal forces of elements and inextensional mechanism can be calculated. The Hessian matrix can also be obtained by Eq.(29). Mobility and stability of equilibrium of mechanism in Figs.5b~5g can be determined by checking criterion formulae Eq.(24) and Eq.(30). The detailed numerical calculation is shown as follows:

For the case shown in Fig.5b,

$$\boldsymbol{H}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\ \boldsymbol{U}_m=\begin{bmatrix}-0.707\\0\\-0.707\\0\end{bmatrix},$$

$$\boldsymbol{t}=\begin{bmatrix}1\\0\\1\end{bmatrix},\ \boldsymbol{P}=\begin{bmatrix}0\\1\\0\\1\end{bmatrix}.$$

Hence $\boldsymbol{U}_m^{\mathrm{T}}\boldsymbol{P}$=0, the mechanism is not mobile. $\boldsymbol{U}_m^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{U}_m$=1. The equilibrium of the mechanism is stable.

For the case shown in Fig.5c,

$$\boldsymbol{H}=\begin{bmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix},\ \boldsymbol{U}_m=\begin{bmatrix}-0.707\\0\\-0.707\\0\end{bmatrix},$$

$$\boldsymbol{t}=\begin{bmatrix}-1\\0\\-1\end{bmatrix},\ \boldsymbol{P}=\begin{bmatrix}0\\-1\\0\\-1\end{bmatrix}.$$

Hence $\boldsymbol{U}_m^{\mathrm{T}}\boldsymbol{P}$=0, and the mechanism is not mobile. $\boldsymbol{U}_m^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{U}_m$=−1. The equilibrium of the mechanism is unstable.

For the case shown in Fig.5d,

$$\boldsymbol{H}=\begin{bmatrix}1&0&1&0\\0&-1&0&1\\-1&0&-1&0\\0&1&0&-1\end{bmatrix},\ \boldsymbol{U}_m=\begin{bmatrix}-0.707\\0\\-0.707\\0\end{bmatrix},$$

$$\boldsymbol{t}=\begin{bmatrix}0\\-1\\0\end{bmatrix},\ \boldsymbol{P}=\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}.$$

Hence $\boldsymbol{U}_m^{\mathrm{T}}\boldsymbol{P}$=0, and the mechanism is not mobile. $\boldsymbol{U}_m^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{U}_m$=0. The mechanism is under indifferent equilibrium.

For the case shown in Fig.5e,

$$\boldsymbol{H}=\begin{bmatrix}1&0&-1&0\\0&1&0&-1\\-1&0&1&0\\0&-1&0&1\end{bmatrix},\ \boldsymbol{U}_m=\begin{bmatrix}-0.707\\0\\-0.707\\0\end{bmatrix},$$

$$\boldsymbol{t}=\begin{bmatrix}0\\1\\0\end{bmatrix},\ \boldsymbol{P}=\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}.$$

Hence $\boldsymbol{U}_m^{\mathrm{T}}\boldsymbol{P}$=0, and the mechanism is not mobile. $\boldsymbol{U}_m^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{U}_m$=0. The mechanism is under indifferent equilibrium.

For the case shown in Fig.5f (θ_1=60°, θ_2=45°),

$$\boldsymbol{H}=\begin{bmatrix}2.732&0&-1&0\\0&2.732&0&-1\\-1&0&0&0\\0&-1&0&0\end{bmatrix},\ \boldsymbol{U}_m=\begin{bmatrix}-0.707\\0\\-0.707\\0\end{bmatrix},$$

$$\boldsymbol{t}=\begin{bmatrix}1.732\\1\\-1\end{bmatrix},\ \boldsymbol{P}=\begin{bmatrix}-1\\1.732\\1\\-1\end{bmatrix}.$$

Hence $\boldsymbol{U}_m^{\mathrm{T}}\boldsymbol{P}$=0, and the mechanism is not mobile. $\boldsymbol{U}_m^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{U}_m$=0.366. The equilibrium of the mechanism is stable.

For the case shown in Fig.5g,

$$\boldsymbol{U}_m=\begin{bmatrix}-0.707\\0\\-0.707\\0\end{bmatrix},\ \boldsymbol{P}=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}.$$

Hence $\boldsymbol{U}_m^{\mathrm{T}}\boldsymbol{P}$=−0.707≠0, and the mechanism is mobile.

The displacement path of node 2 is shown in Fig.6 when perturbation δ is acting. Node 2 in Figs.5b and 5f returns to the initial equilibrium state and that in Fig.5c is away from the initial state. Particularly, Figs.5d and 5e show a neutral equilibrium state (indifferent equilibrium state).

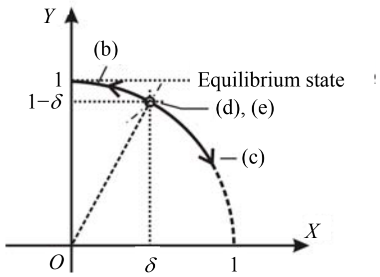

Fig.6 Trace of node 2

Equilibrium stability evaluation for a multi-mode mechanism

Fig.7 shows a multi-mode mechanism consisting of two equal Watt type mechanisms, which are connected by a horizontal bar. It has been used as an example in (Tarnai and Szabó, 2000) to analyze the motion bifurcation. The paper analyzes the mobility and stability of this mechanism under different load patterns. The rank of the corresponding equilibrium matrix $\boldsymbol{A}$ is equal to the column of $\boldsymbol{A}$ and less than the row of $\boldsymbol{A}$ by 2, i.e. m=2, s=0. Based on the SVD method, the states of inextensional mechanism can be calculated as follows:

$$\boldsymbol{U}_m=\begin{bmatrix} 0 & 0.486 & 0 & 0.486 & 0 & 0.486 & 0 & 0.486 & 0 & 0.116 & 0 & 0.116 & 0 & 0.116 & 0 & 0.116 \\ 0 & -0.116 & 0 & -0.116 & 0 & -0.116 & 0 & -0.116 & 0 & 0.486 & 0 & 0.486 & 0 & 0.486 & 0 & 0.486 \end{bmatrix}^{\mathrm{T}}.$$

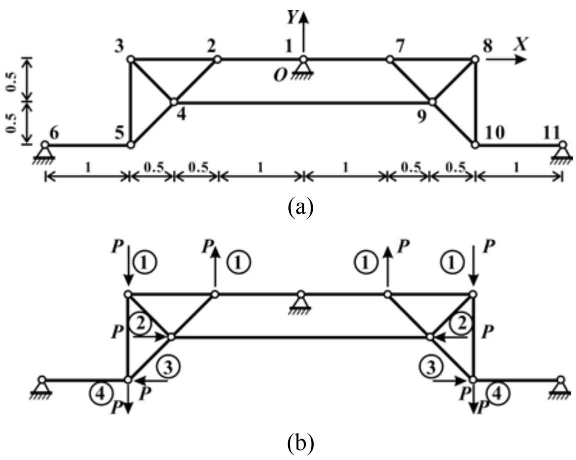

Fig.7 Multi-modes mechanism. (a) Geometrical parameters; (b) Under different load patterns

Under load (1): for $\boldsymbol{U}_m^{\mathrm{T}}\boldsymbol{P}=\boldsymbol{0}$, the mechanism is not mobile; for $\boldsymbol{U}_m^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{U}_m=\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$, the equilibrium of the mechanism is stable.

Under load (2): for $\boldsymbol{U}_m^{\mathrm{T}}\boldsymbol{P}=\boldsymbol{0}$, the mechanism is not mobile; for $\boldsymbol{U}_m^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{U}_m=\boldsymbol{0}$, the mechanism is under indifferent equilibrium.

Under load (3): for $\boldsymbol{U}_m^{\mathrm{T}}\boldsymbol{P}=\boldsymbol{0}$, the mechanism is not mobile; for $\boldsymbol{U}_m^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{U}_m=\begin{bmatrix} -0.25 & 0 \\ 0 & -0.25 \end{bmatrix}$, the equilibrium of the mechanism is unstable.

Under load (4): for $\boldsymbol{U}_m^{\mathrm{T}}\boldsymbol{P}=\begin{bmatrix} -0.602 \\ -0.371 \end{bmatrix}\neq\boldsymbol{0}$, the mechanism is mobile.

CONCLUSION

This paper presents a matrix-based criterion for mobility and stability of mechanism. As external loads are considered, stability of equilibrium is more generalized than geometrical stability. The idea of stability theory in structures has been introduced into mechanism. It shows that three types of equilibrium states exist in the field of mechanism: stable equilibrium, unstable equilibrium and indifferent equilibrium. The first and second variations of potential energy function of mechanism are analyzed under the field of conservative force. The Linear Algebra Theory is used throughout the calculations. Several efficient criterion formulae are listed and classified in Table 2.

Table 2 Criterion formulae for mobility and equilibrium stability of mechanisms

	Criterion for mobility	Criterion for stability of equilibrium
m=1	Immobility: $\boldsymbol{U}_m^{\mathrm{T}}\boldsymbol{P}$=0	Stable equilibrium: $\boldsymbol{U}_m^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{U}_m>0$
		Indifferent equilibrium: $\boldsymbol{U}_m^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{U}_m=0$
	Mobile: $\boldsymbol{U}_m^{\mathrm{T}}\boldsymbol{P}\neq0$	Unstable equilibrium: $\boldsymbol{U}_m^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{U}_m<0$
m>1	Immobility: $\boldsymbol{U}_m^{\mathrm{T}}\boldsymbol{P}=\boldsymbol{0}$	Stable equilibrium: $\boldsymbol{U}_m^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{U}_m$ (positive definite)
		Indifferent equilibrium: $\boldsymbol{U}_m^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{U}_m=\boldsymbol{0}$
	Mobile: $\boldsymbol{U}_m^{\mathrm{T}}\boldsymbol{P}\neq\boldsymbol{0}$	Unstable equilibrium: $\boldsymbol{U}_m^{\mathrm{T}}\cdot\boldsymbol{H}\cdot\boldsymbol{U}_m$ (negative definite)

The criterion for mobility is quite helpful for determining whether the finite mechanisms applied with external load are in equilibrium state. And the criterion for stability of equilibrium classifies the equilibrium into three states as mentioned above. Compared to other criterion for the stability of mechanisms, the criterion proposed in this paper is better in theory and much easier in practice.

Finally, it should be noted that further work on the criterion for stability of equilibrium of higher order mechanisms still need to be done.

We believe that the criterion formulae shown in Table 2 will be useful to other people working on mechanisms.

ACKNOWLEDGEMENT

The authors are grateful to Dr Zhong You of Oxford University for his suggestions and advice. Also we appreciate the efforts of Professor Tibor Tarnai of Budapest University of Technology and Economics in reading the draft of this paper.

References

Belytschko, T., Liu, W.K., Moran, B., 2000. Nonlinear Finite Elements for Continua and Structures. John Wiley & Sons, New York, US.

Fowler, P.W., Guest, S.D., 2000. A symmetry extension of Maxwell's rule for rigidity of frames. *Int. J. Solids Struct.*, **37**(12):1793-1804. [doi:10.1016/S0020-7683(98)00326-6]

Kovács, F., 2004. Mobility and stress analysis of highly symmetric generalized bar-and-joint structures. *Journal of Computational and Applied Mechanics*, **5**:65-78.

Kuznetsov, E.N., 1988. Underconstrained structural systems. *Int. J. Solids Struct.*, **24**(2):153-163. [doi:10.1016/0020-7683(88)90026-1]

Lengyel, A., You, Z., 2004. Bifurcations of SDOF mechanisms using catastrophe theory. *Int. J. Mech. Sci.*, **41**:559-568. [doi:10.1016/j.ijsolstr.2003.09.024]

Luo, Y.Z., 2000. Geometrical stability analysis of cable-strut tensile structures. *J. Zhejiang University (Science Edition)*, **27**:608-611 (in Chinese).

Luo, Y.Z., Dong, S.L., 2002. Nonlinear force method analysis for space truss with mobile mechanisms. *Acta Mechanica Solida Sinica*, **23**:288-294 (in Chinese).

Maxwell, J.C., 1890. On the Calculation of the Equilibrium and Stiffness of Frames. Cambridge University Press, Cambridge, UK.

Pellegrino, S., 1990. Analysis of pre-stressed mechanisms. *Int. J. Solids Struct.*, **26**(12):1329-1350. [doi:10.1016/0020-7683(90)90082-7]

Pellegrino, S., 1993. Structure computations with the singular value decomposition of the equilibrium matrix. *Int. J. Solids Struct.*, **30**(21):3025-3035. [doi:10.1016/0020-7683(93)90210-X]

Pellegrino, S. (Ed.), 2001. Deployable Structures. CISM Course and Lectures No. 412. Springer, Wien, New York.

Pellegrino, S., Calladine, C.R., 1986. Matrix analysis of statically and kinematically indeterminate frameworks. *Int. J. Solids Struct.*, **22**(4):409-428. [doi:10.1016/0020-7683(86)90014-4]

Tarnai, T., Szabó, J., 2000. On the exact equation of inextensional, kinematically indeterminate assemblies. *Comp. Struct.*, **75**(2):145-155. [doi:10.1016/S0045-7949(99)00090-5]

Tarnai, T., Szabó, J., 2002. Rigidity and Stability of Prestressed Infinitesimal Mechanisms. *In*: Drew, H.R., Pellegrino, S. (Eds.), New Approaches to Structural Mechanics, Shells and Biological Structures. Kluwer, Dordrecht, p.245-256.