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Convexity-preserving interpolation of trigonometric polynomial curves with a shape parameter*

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Abstract: In computer aided geometric design (CAGD), it is often needed to produce a convexity-preserving interpolating curve according to the given planar data points. However, most existing pertinent methods cannot generate convexity-preserving interpolating transcendental curves; even constructing convexity-preserving interpolating polynomial curves, it is required to solve a system of equations or recur to a complicated iterative process. The method developed in this paper overcomes the above drawbacks. The basic idea is: first to construct a kind of trigonometric polynomial curves with a shape parameter, and interpolating trigonometric polynomial parametric curves with C^2 (or G^1) continuity can be automatically generated without having to solve any system of equations or do any iterative computation. Then, the convexity of the constructed curves can be guaranteed by the appropriate value of the shape parameter. Performing the method is easy and fast, and the curvature distribution of the resulting interpolating curves is always well-proportioned. Several numerical examples are shown to substantiate that our algorithm is not only correct but also usable.

Key words: Computer aided geometric design (CAGD), α-trigonometric polynomial curves, Interpolation, Convexity-preserving, Shape parameter
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INTRODUCTION

In geometric shape design and reverse engineering, constructing a convexity-preserving interpolating curve according to the given planar data points is a familiar and basal problem. So far, some techniques have been invented to obtain this kind of curves [see (Tai and Wang, 2004) and the references therein], but almost all of them have to solve a system of equations or recur to a complicated iterative process; although there is no mention of the fact that, in spite of doing a lot of the above troublesome work, they could not generate convexity-preserving interpolating transcendental curves, such as cycloid, spiral and some other common curves in shape design (Tai and Wang, 2004). As for constructing convexity-preserving interpolating curve by using NURBS, a user rarely adopts them because of the intangibility of weight for curve shape and the complexity in computation of rational curve.

Just to overcome the above drawbacks, this paper constructs a kind of trigonometric polynomial curve with a shape parameter and investigates its convexity-preserving interpolation algorithm. Trigonometric spline was introduced by Schoenberg (1964), who studied the interpolation but did not take shape preservation into account. Subsequently, many people explored further properties of trigonometric splines (Lyche and Winther, 1979; Koch *et al.*, 1995; Walz, 1997a; 1997b; Peña, 1997). Later, Zhang (1996;

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1997) developed C-curves using the basis {sin *t*, cos *t*, *t*, 1}, and unified C-curves and H-curves that use basis {sinh *t*, cosh *t*, *t*, 1} as a complete curve family, named F-splines (Zhang *et al.*, 2005; Zhang and Krause, 2005). Some other properties of C-curves were studied in (Wang and Chen, 2004; Wang and Li, 2006). However, they did not process interpolation. This paper presents a novel trigonometric polynomial curve with a shape parameter. The technique is based on the Loe's idea (Loe, 1996) using singular blending, but Loe blended a cubic uniform B-spline and its control polygon to get non-interpolatory α-B-spline. Also our technique generalizes the work of (Tai and Wang, 2004), which applied the singular blending idea to actualize interpolation but cannot obtain interpolating transcendental curves.

In this paper we construct a new type of curve by blending a parametrized polygon and trigonometric polynomial splines with blending factor α which acts as shape parameter; then a family of trigonometric polynomial curves with C^2 (or G^1) continuity interpolating the given planar data points can be automatically generated without having to solve any system of equations or to go at any iterative computation; next, we research the convexity-preserving property of these curves, and find a range in which the shape parameter α takes its value in order to make the corresponding interpolating curves convexity-preserving, as well as to make its curvature distribution always well-proportioned. Lastly, several numerical examples are shown to substantiate that our algorithm is not only correct but also usable, and can generate convexity-preserving interpolating transcendental curves.

CONSTRUCTION OF α-TRIGONOMETRIC POLYNOMIAL INTERPOLATING CURVES

We will construct a kind of trigonometric polynomial curves with shape parameter to interpolate the given planar data points $\{\boldsymbol{P}_i\}_{i=1}^{n}$, $n\geq 4$. Firstly, to ensure the number of line segments formed by connecting the data points equals the number of curve segments in the trigonometric polynomial curves to be constructed, two auxiliary data points $\boldsymbol{P}_0$, $\boldsymbol{P}_{n+1}$ are introduced according to the method of (Tai and Wang, 2004); then for the C-B-splines in (Zhang, 1996), we take $\pi/2$ as the parametric step, take $u_i=i\pi/2$ (i=1,2,…,n) as the parametric nodes, so the uniform trigonometric polynomial curve can be constructed as follows:

$$\begin{aligned}\boldsymbol{C}(u)=\boldsymbol{C}_j(u)=B_{0,4}(t)\boldsymbol{P}_{j-1}+B_{1,4}(t)\boldsymbol{P}_j+\\ B_{2,4}(t)\boldsymbol{P}_{j+1}+B_{3,4}(t)\boldsymbol{P}_{j+2},\end{aligned} \tag{1}$$
$$t=u-u_j,\ u_j\le u\le u_{j+1},\ j=1,2,...,n-1,$$

where $\{\boldsymbol{P}_i\}_{i=0}^{n+1}$ represent the control points, $B_{i,4}(t)$ (i=0, 1,2,3) are the uniform trigonometric polynomial splines basis:

$$\begin{cases}B_{0,4}(t)=(\pi/2-t-\cos t)/\pi,\\ B_{1,4}(t)=(t-\sin t+2\cos t)/\pi,\\ B_{2,4}(t)=(\pi/2-t+2\sin t-\cos t)/\pi,\\ B_{3,4}(t)=(t-\sin t)/\pi.\end{cases} \tag{2}$$

Thus, the curve (1) has the following properties (Zhang, 1996):

$$\begin{cases}\boldsymbol{C}(u_j)=(1/2-1/\pi)(\boldsymbol{P}_{j-1}+\boldsymbol{P}_{j+1})+2\boldsymbol{P}_j/\pi,\\ \boldsymbol{C}(u_{j+1})=(1/2-1/\pi)(\boldsymbol{P}_j+\boldsymbol{P}_{j+2})+2\boldsymbol{P}_{j+1}/\pi,\\ \boldsymbol{C}'(u_j)=(\boldsymbol{P}_{j+1}-\boldsymbol{P}_{j-1})/\pi,\ \boldsymbol{C}'(u_{j+1})=(\boldsymbol{P}_{j+2}-\boldsymbol{P}_j)/\pi,\\ \boldsymbol{C}(u)\in C^2[u_1,u_n].\end{cases} \tag{3}$$

To avoid solving global equation systems and to introduce a shape adjustment parameter without modifying the order of continuity anywhere, we blend a parametrized polygon and the above trigonometric polynomial curve $\boldsymbol{C}(u)$ using a blending factor α. The vertices of this polygon can be directly computed from the interpolation conditions. In order to let the curve to be constructed have the basis {sin *t*, cos *t*, *t*, 1}, like the curve $\boldsymbol{C}(u)$, we construct singular blending function (Loe, 1996) $S_j(u)$ to parametrize the polygon using the basis {sin *t*, cos *t*, *t*, 1}; so we take $S_j(u)\in C^2[u_j, u_{j+1}]$ and satisfying the following conditions:

$$\begin{aligned}&S_j(u_j)=0,\ S_j(u_{j+1})=1,\ S_j'(u_j)=S_j''(u_j)=0,\\ &S_j'(u_{j+1})=S_j''(u_{j+1})=0,\ j=1,2,...,n-1.\end{aligned} \tag{4}$$

In this way, after some calculation, we get

$S_j(u)=\overline{S}(u-u_j),\ u_j \le u \le u_{j+1},\ j=1,2,\ldots,n-1,$

where

$$\overline{S}(t)=\begin{cases}6(2+\sqrt{3})(t-\sin t)/\pi, & 0\le t\le \pi/6,\\ 3(5+3\sqrt{3})(\sin t-\cos t)/\pi-6(3+2\sqrt{3})t/\pi & \\ \quad +5+3\sqrt{3}, & \pi/6\le t\le \pi/3,\\ 6(2+\sqrt{3})(t+\cos t)/\pi-5-3\sqrt{3}, & \pi/3\le t\le \pi/2.\end{cases} \tag{5}$$

We call the reparametrized polygon a singular polygon, whose yet-to-be-determined vertices $\boldsymbol{V}_j$ and $\boldsymbol{V}_{j+1}$ are dependent on α, and hence the singular polygon is defined as follows:

$$\boldsymbol{L}(u,\alpha)=\boldsymbol{L}_j(u,\alpha)=[1-S_j(u)]\boldsymbol{V}_j+S_j(u)\boldsymbol{V}_{j+1},\quad u_j\le u\le u_{j+1}, j=1,2,\ldots,n-1.$$

By blending the singular polygon $\boldsymbol{L}(u,\alpha)$ and the uniform trigonometric polynomial curve $\boldsymbol{C}(u)$, both defined on $[u_1, u_n]$, using α as the blending factor, we obtain a uniform trigonometric polynomial curve with an additional parameter α:

$$\boldsymbol{T}(u,\alpha)=\alpha\boldsymbol{C}(u)+(1-\alpha)\boldsymbol{L}(u,\alpha),\ u_1\le u\le u_n.$$

Each of its segments on $[u_j, u_{j+1}]$ can be more explicitly represented as follows:

$$\begin{aligned}\boldsymbol{T}(u,\alpha)&=\boldsymbol{T}_j(u,\alpha)=\alpha\boldsymbol{C}_j(u)+(1-\alpha)\boldsymbol{L}_j(u,\alpha)\\ &=\alpha\boldsymbol{C}_j(u)+(1-\alpha)\{[1-S_j(u)]\boldsymbol{V}_j+S_j(u)\boldsymbol{V}_{j+1}\}.\end{aligned} \tag{6}$$

From the condition Eq.(4) of the singular blending function $S_j(u)$ and the interpolating condition $\boldsymbol{T}(u_j,\alpha)=\boldsymbol{P}_j$ (j=1,2,…,n), we obtain

$$\begin{cases}(1-\alpha)\boldsymbol{V}_j=\boldsymbol{P}_j-\alpha\boldsymbol{C}_j(u_j), & j=1,2,...,n-1,\\ (1-\alpha)\boldsymbol{V}_n=\boldsymbol{P}_n-\alpha\boldsymbol{C}_{n-1}(u_n), & j=n.\end{cases}$$

Finally, we get the following desired interpolating uniform α-trigonometric polynomial curve

$$\begin{aligned}\boldsymbol{T}(u,\alpha)&=\boldsymbol{T}_j(u,\alpha)=\alpha\boldsymbol{C}_j(u)+[1-S_j(u)]\\ &\cdot[\boldsymbol{P}_j-\alpha\boldsymbol{C}_j(u_j)]+S_j(u)[\boldsymbol{P}_{j+1}-\alpha\boldsymbol{C}_j(u_{j+1})],\\ &u_j\le u\le u_{j+1},\ j=1,2,\ldots,n-1.\end{aligned} \tag{7}$$

The continuity of the curve $\boldsymbol{T}(u,\alpha)$ follows directly from Eq.(4) that

$$\frac{\partial}{\partial u}\boldsymbol{T}(u_j,\alpha)=\alpha\boldsymbol{C}'(u_j),\ \frac{\partial^2}{\partial u^2}\boldsymbol{T}(u_j,\alpha)=\alpha\boldsymbol{C}''(u_j). \tag{8}$$

Thus, for a fixed shape parameter α, the curve $\boldsymbol{T}(u,\alpha)$ maintains the same order of continuity as that of $\boldsymbol{C}(u)$ at every knot u_j (j=1,2,…,n). Since, everywhere away from the knots, the components of the uniform trigonometric polynomial curve $\boldsymbol{C}(u)$ are C^∞ and the singular blending functions $S_j(u)$ are C^2, the uniform α-trigonometric polynomial curve $\boldsymbol{T}(u,\alpha)\in C^2$.

For the purpose of practical shape design, we only discuss the standard α-trigonometric polynomial curve which has the restriction of $\alpha\in(0,1)$. When α=0, the α-trigonometric polynomial curve is reduced to the polygon $\boldsymbol{P}_1\boldsymbol{P}_2\ldots\boldsymbol{P}_n$ connecting the data points $\{\boldsymbol{P}_i\}_{i=1}^n$; when α=1, the α-trigonometric polynomial curve is reduced to the uniform trigonometric polynomial curve $\boldsymbol{C}(u)$, in which case the singular polygon $\boldsymbol{L}(u,\alpha)$ is non-existent.

CONVEXITY-PRESERVING INTERPOLATION RELATIVE TO GLOBAL CONVEX DATA POINTS

Definition 1 A set of planar data points denoted as $\{\boldsymbol{P}_i\}_{i=1}^n$ is said to be convex if the close polygon $\boldsymbol{P}_1\boldsymbol{P}_2\ldots\boldsymbol{P}_n\boldsymbol{P}_1$ is convex.

Definition 2 A planar curve is said to be convex if there are no inflection points or singular points (including cusps and the double points) on it.

Let $\{\boldsymbol{P}_i\}_{i=1}^n$ be convex data points in the plane with any three consecutive points $\boldsymbol{P}_{j-1}$, $\boldsymbol{P}_j$, $\boldsymbol{P}_{j+1}$ (j=2, 3,…,n−1) being not collinear and $\{\boldsymbol{P}_i\}_{i=1}^n$ being arranged counterclockwise.

Convexity of jth curve segment $\boldsymbol{T}_j(u,\alpha)$ on $[u_j, u_{j+1}]$

Let $t=u-u_j\in[0,\pi/2]$, $u\in[u_j, u_{j+1}]$, and write $\boldsymbol{T}_j(u,\alpha)$ as $\overline{\boldsymbol{T}}_j(t,\alpha)$, $\boldsymbol{C}_j(u)$ as $\overline{\boldsymbol{C}}_j(t)$ on $[u_j, u_{j+1}]$, then Eqs.(6) and (7) can be written as

$$\begin{aligned}\boldsymbol{T}_j(u,\alpha)&=\overline{\boldsymbol{T}}_j(t,\alpha)\\ &=\alpha\overline{\boldsymbol{C}}_j(t)+(1-\alpha)\{[1-\overline{S}(t)]\boldsymbol{V}_j+\overline{S}(t)\boldsymbol{V}_{j+1}\} \quad (9)\\ &=\alpha\overline{\boldsymbol{C}}_j(t)+[1-\overline{S}(t)][\boldsymbol{P}_j-\alpha\overline{\boldsymbol{C}}_j(0)]+\overline{S}(t)[\boldsymbol{P}_{j+1}-\alpha\overline{\boldsymbol{C}}_j(\pi/2)], \quad (10)\end{aligned}$$

where $0 \le t \le \pi/2$.

Denote the wedge product of two vectors $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$ as

$$[\boldsymbol{a}_i, \boldsymbol{a}_j] = \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix}, \quad \boldsymbol{a}_i=(x_i, y_i),\ \boldsymbol{a}_j=(x_j, y_j),$$

and let

$$\boldsymbol{a}_i=\boldsymbol{P}_i-\boldsymbol{P}_{i-1}, \quad i=1,2,\ldots,n+1,$$
$$\boldsymbol{c}_j=-(\boldsymbol{a}_j+\boldsymbol{a}_{j+1}+\boldsymbol{a}_{j+2}),$$

then $[\boldsymbol{c}_j, \boldsymbol{a}_j]\ge\boldsymbol{0}$, $[\boldsymbol{a}_{j+2}, \boldsymbol{c}_j]\ge\boldsymbol{0}$ hold, so if we write

$$w_j^{01}=[\boldsymbol{a}_j, \boldsymbol{a}_{j+1}],\ w_j^{12}=[\boldsymbol{a}_{j+1}, \boldsymbol{a}_{j+2}],\ w_j^{20}=[\boldsymbol{a}_{j+2}, \boldsymbol{a}_j],$$

and because $\{\boldsymbol{P}_{j-1}\boldsymbol{P}_j\boldsymbol{P}_{j+1}\boldsymbol{P}_{j+2}\}$ are convex data points arranged counterclockwise, we have

$$w_j^{01} \ge w_j^{20},\ w_j^{12} \ge w_j^{20},\ w_j^{01} > 0,\ w_j^{12} > 0. \tag{11}$$

Moreover, it is easy to get

$$\bar{\boldsymbol{C}}_j'(t)=[(1-\sin t)\boldsymbol{a}_j+(\cos t+\sin t)\boldsymbol{a}_{j+1}+(1-\cos t)\boldsymbol{a}_{j+2}]/\pi,$$

thus

$$\begin{aligned}
\frac{\partial}{\partial t}\bar{\boldsymbol{T}}_j(t,\alpha)
&=\alpha\bar{\boldsymbol{C}}_j'(t)+\bar{S}'(t)\{(\boldsymbol{P}_{j+1}-\boldsymbol{P}_j)-\alpha[\bar{\boldsymbol{C}}_j(\pi/2)-\bar{\boldsymbol{C}}_j(0)]\}\\
&=\alpha[(1-\sin t)/\pi-\bar{S}'(t)(1/2-1/\pi)]\boldsymbol{a}_j\\
&\quad+[\alpha(\cos t+\sin t)/\pi+\bar{S}'(t)-2\alpha\bar{S}'(t)/\pi]\boldsymbol{a}_{j+1}\\
&\quad+\alpha[(1-\cos t)/\pi-\bar{S}'(t)(1/2-1/\pi)]\boldsymbol{a}_{j+2},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}\bar{\boldsymbol{T}}_j(t,\alpha)&=\alpha[-\cos t/\pi-\bar{S}''(t)(1/2-1/\pi)]\boldsymbol{a}_j+\\
&\quad[\alpha(\cos t-\sin t)/\pi+\bar{S}''(t)-2\alpha\bar{S}''(t)/\pi]\boldsymbol{a}_{j+1}+\\
&\quad\alpha[\sin t/\pi-\bar{S}''(t)(1/2-1/\pi)]\boldsymbol{a}_{j+2}.
\end{aligned}$$

Let

$$\bar{M}_j(t,\alpha)=\frac{\pi^2}{\alpha}\left[\frac{\partial}{\partial t}\bar{\boldsymbol{T}}_j(t,\alpha),\frac{\partial^2}{\partial t^2}\bar{\boldsymbol{T}}_j(t,\alpha)\right],$$
$$h_1=6(2+\sqrt{3})/\pi,\ h_2=6(5+3\sqrt{3})/\pi,$$

after some algebraic calculation in subinterval $[0, \pi/6]$, $[\pi/6,\pi/3]$ and $[\pi/3,\pi/2]$, $\bar{M}_j(t,\alpha)$ can be expressed as follows respectively:

$$\begin{aligned}
\bar{M}_j(t,\alpha)&\triangleq F_1^{(j)}(t,\alpha)=\{6(2+\sqrt{3})(-1+\sin t+\cos t)\\
&+\alpha[7+3\sqrt{3}+h_1+(5+3\sqrt{3}-3h_1)\sin t\\
&-(5+3\sqrt{3}+h_1)\cos t]\}w_j^{01}+\alpha(5+3\sqrt{3}-h_1)[(-1\\
&+\sin t+\cos t)w_j^{20}+(-1-\sin t+\cos t)w_j^{12}],\ t\in[0,\pi/6];
\end{aligned}$$

$$\begin{aligned}
\bar{M}_j(t,\alpha)&\triangleq F_2^{(j)}(t,\alpha)=\{3(5+3\sqrt{3})(1-\sin t)-\\
&3(1+\sqrt{3})\cos t+\alpha[1-h_2-(10+6\sqrt{3}+h_1-2h_2)\sin t+\\
&(10+6\sqrt{3}-h_1)\cos t]\}w_j^{01}+\alpha[16+9\sqrt{3}-h_2-(10+6\sqrt{3}\\
&+h_1-h_2)(\sin t+\cos t)]w_j^{20}+\{3(5+3\sqrt{3})(1-\cos t)\\
&-3(1+\sqrt{3})\sin t+\alpha[1-h_2+(10+6\sqrt{3}-h_1)\sin t\\
&-(10+6\sqrt{3}+h_1-2h_2)\cos t]\}w_j^{12},\ t\in[\pi/6,\pi/3];
\end{aligned}$$

$$\begin{aligned}
\bar{M}_j(t,\alpha)&\triangleq F_3^{(j)}(t,\alpha)=\alpha(5+3\sqrt{3}-h_1)[(-1+\sin t\\
&-\cos t)w_j^{01}+(-1+\sin t+\cos t)w_j^{20}]+\{6(2+\sqrt{3})\cdot\\
&(-1+\sin t+\cos t)+\alpha[7+3\sqrt{3}+h_1-(5+3\sqrt{3}+h_1)\sin t\\
&+(5+3\sqrt{3}-3h_1)\cos t]\}w_j^{12},\ t\in[\pi/3,\pi/2].
\end{aligned}$$

Noting that the relative curvature of the curve $\bar{\boldsymbol{T}}_j(t,\alpha)$ $(0\le t\le\pi/2)$ is (Farin, 2005)

$$k_j(t,\alpha)=\left[\frac{\partial}{\partial t}\bar{\boldsymbol{T}}_j(t,\alpha),\frac{\partial^2}{\partial t^2}\bar{\boldsymbol{T}}_j(t,\alpha)\right]\bigg/\left|\frac{\partial}{\partial t}\bar{\boldsymbol{T}}_j(t,\alpha)\right|^3,$$

so, there are no inflection points on $\bar{\boldsymbol{T}}_j(t,\alpha)$ $(0\le t\le\pi/2)$ if and only if $\bar{M}_j(t,\alpha)\ge 0$, $\forall t\in[0,\pi/2]$. In order to obtain the range of the shape parameter α in which $\bar{M}_j(t,\alpha)\ge 0$ holds $\forall t\in[0,\pi/2]$ in the three subintervals of $[0, \pi/2]$, the following three lemmas are needed. Here we just prove Lemma 1, proofs of the other two lemmas are similar.

Lemma 1 Let $f(x)=f_0+f_1\sin x+f_2\cos x$, then $f(x)\ge 0$, $\forall x\in[0,\pi/6]$ if and only if

$$R_0^{(1)}\ge 0,\ R_2^{(1)}\ge 0,\ R_1^{(1)}+\sqrt{2R_0^{(1)}R_2^{(1)}}\ge 0,$$

where

$$\begin{cases}
R_0^{(1)}=f_0+f_2,\\
R_1^{(1)}=f_0+(2-\sqrt{3})f_1+f_2,\\
R_2^{(1)}=(4-2\sqrt{3})f_0+(2-\sqrt{3})f_1-(3-2\sqrt{3})f_2.
\end{cases}$$

Proof Do trigonometric substitution to the functions $\sin x$ and $\cos x$, and let $\tan(x/2)=t$, we can write $f(x)$ as

$$f(x)=\frac{f_0+f_2+2f_1t+(f_0-f_2)t^2}{1+t^2},\ t\in[0,\ 2-\sqrt{3}].$$

So

$$f(x)\ge 0,\ x\in[0,\pi/6]\Leftrightarrow g(t)\ge 0,\ t\in[0,2-\sqrt{3}],$$

where

$$g(t)=f_0+f_2+2f_1t+(f_0-f_2)t^2,\ t\in[0,2-\sqrt{3}].$$

Then let $v=t/(2-\sqrt{3}),\ t\in[0,2-\sqrt{3}]$, and using the conversion formula

$$[1,v,v^2]=[B_0^2(v),B_1^2(v),B_2^2(v)]\begin{bmatrix}1 & 0 & 0\\ 1 & 1/2 & 0\\ 1 & 1 & 1\end{bmatrix}$$

between the monomial basis and Bernstein basis, we can convert $g(t)$ into a Bernstein polynomial:

$$g(t)=\bar{g}(v)=[B_0^2(v),B_1^2(v),B_2^2(v)]\cdot\begin{pmatrix}f_0+f_2\\ f_0+(2-\sqrt{3})f_1+f_2\\ 2[(4-2\sqrt{3})f_0+(2-\sqrt{3})f_1-(3-2\sqrt{3})f_2]\end{pmatrix}.$$

Hence, the positive conditions of quadratic Bernstein polynomial (Chang, 1995) gives Lemma 1.

Lemma 2 Let $f(x)=f_0+f_1\sin x+f_2\cos x$, then $f(x)\ge 0$, $\forall x\in[\pi/6,\pi/3]$ if and only if

$$R_0^{(2)}\ge 0,\ R_2^{(2)}\ge 0,\ R_1^{(2)}+2\sqrt{R_0^{(2)}R_2^{(2)}/3}\ge 0,$$

where

$$\begin{cases}R_0^{(2)}=(4-2\sqrt{3})f_0+(2-\sqrt{3})f_1-(3-2\sqrt{3})f_2,\\ R_1^{(2)}=2(f_1+f_2)+2\sqrt{3}(f_0-f_1-f_2)/3,\\ R_2^{(2)}=2f_0+\sqrt{3}f_1+f_2.\end{cases}$$

Lemma 3 Let $f(x)=f_0+f_1\sin x+f_2\cos x$, then $f(x)\ge 0$, $\forall x\in[\pi/3,\pi/2]$ if and only if

$$R_0^{(3)}\ge 0,\ R_2^{(3)}\ge 0,\ R_1^{(3)}+2\sqrt{R_0^{(3)}R_2^{(3)}/3}\ge 0,$$

where

$$\begin{cases}R_0^{(3)}=2f_0+\sqrt{3}f_1+f_2,\\ R_1^{(3)}=f_0+f_1+f_2+\sqrt{3}(f_0+f_1-f_2)/3,\\ R_2^{(3)}=f_0+f_1.\end{cases}$$

Next, we will use Lemmas 1~3 to obtain the range of the shape parameter α in which $\bar{M}_j(t,\alpha)\ge 0$ holds $\forall t\in[0,\pi/2]$.

Let

$$\begin{cases}A_1^{(j)}=[8-2\sqrt{3}-6(1+\sqrt{3})/\pi]w_j^{01}+[2-6(\sqrt{3}-1)/\pi]w_j^{20}\\ \qquad +[-2\sqrt{3}+6(3-\sqrt{3})/\pi]w_j^{12},\\ A_2^{(j)}=(3+\sqrt{3}-18/\pi)w_j^{01}+(1+\sqrt{3}-6/\pi)(w_j^{20}-w_j^{12}),\\ A_3^{(j)}=-[6+4\sqrt{3}-6(3+\sqrt{3})/\pi]w_j^{01}+[4+2\sqrt{3}\\ \qquad -6(1+\sqrt{3})/\pi]w_j^{20}+[10+4\sqrt{3}-6(5+3\sqrt{3})/\pi]w_j^{12},\\ A_4^{(j)}=[\sqrt{3}-6(3-\sqrt{3})/\pi](w_j^{01}+w_j^{12})\\ \qquad +[3-6(3-\sqrt{3})/\pi]w_j^{20},\\ A_5^{(j)}=[2+4\sqrt{3}/3-2(3+\sqrt{3})/\pi](w_j^{20}-w_j^{01})\\ \qquad +[4+2\sqrt{3}-6(3+\sqrt{3})/\pi]w_j^{12},\end{cases}$$

$$\begin{cases}k_1^{(j)}=(A_2^{(j)})^2-4A_1^{(j)}w_j^{01},\\ k_2^{(j)}=12w_j^{01}A_2^{(j)}-24(\sqrt{3}-1)(w_j^{01})^2,\\ k_3^{(j)}=36(w_j^{01})^2,\ k_4^{(j)}=(A_4^{(j)})^2-3A_1^{(j)}A_3^{(j)},\\ k_5^{(j)}=6(3-\sqrt{3})(w_j^{01}+w_j^{12})A_4^{(j)}-18(\sqrt{3}+1)w_j^{12}A_1^{(j)}\\ \qquad -18(\sqrt{3}-1)w_j^{01}A_3^{(j)},\\ k_6^{(j)}=(9-3\sqrt{3})^2(w_j^{01}+w_j^{12})^2-216w_j^{01}w_j^{12},\\ k_7^{(j)}=(A_5^{(j)})^2-8A_3^{(j)}w_j^{12}/3,\\ k_8^{(j)}=4(3+\sqrt{3})A_5^{(j)}w_j^{12}-16(\sqrt{3}+1)(w_j^{12})^2,\\ k_9^{(j)}=4(3+\sqrt{3})^2(w_j^{12})^2,\end{cases}$$

$$\Delta_1^{(j)}=(k_2^{(j)})^2-4k_1^{(j)}k_3^{(j)},\ \Delta_2^{(j)}=(k_5^{(j)})^2-4k_4^{(j)}k_6^{(j)},$$
$$\Delta_3^{(j)}=(k_8^{(j)})^2-4k_7^{(j)}k_9^{(j)},$$

$$\begin{cases}\alpha_1^{(j)}=6(1-\sqrt{3})w_j^{01}/A_1^{(j)},\ \alpha_2^{(j)}=-6w_j^{01}/A_2^{(j)},\\ \alpha_3^{(j)}=-6(1+\sqrt{3})w_j^{12}/A_3^{(j)},\\ \alpha_4^{(j)}=3(\sqrt{3}-3)(w_j^{01}+w_j^{12})/A_4^{(j)},\\ \alpha_5^{(j)}=-2(3+\sqrt{3})w_j^{12}/A_5^{(j)},\\ \alpha_1^{*(j)},\alpha_2^{*(j)}=\left(-k_2^{(j)}\pm\sqrt{\Delta_1^{(j)}}\right)/2k_1^{(j)},\ (\alpha_1^{*(j)}\le\alpha_2^{*(j)}),\\ \alpha_3^{*(j)},\alpha_4^{*(j)}=\left(-k_5^{(j)}\pm\sqrt{\Delta_2^{(j)}}\right)/2k_4^{(j)},\ (\alpha_3^{*(j)}\le\alpha_4^{*(j)}),\\ \alpha_5^{*(j)},\alpha_6^{*(j)}=\left(-k_8^{(j)}\pm\sqrt{\Delta_3^{(j)}}\right)/2k_7^{(j)},\ (\alpha_5^{*(j)}\le\alpha_6^{*(j)}).\end{cases}\qquad(12)$$

Then it is easy to see that $A_i^{(j)}<0$ (i=1,2,3,4,5), $k_1^{(j)}>0$, $k_4^{(j)}<0$, $k_7^{(j)}>0$.

In the subinterval [0,π/6], $F_1^{(j)}(t,\alpha)$ can be written as

$$F_1^{(j)}(t,\alpha)=f_0^{(j)}+f_1^{(j)}\sin t+f_2^{(j)}\cos t,$$

where

$$\begin{aligned}f_0^{(j)}&=-6(2+\sqrt{3})w_j^{01}+\alpha[(7+3\sqrt{3}+h_1)w_j^{01}\\&\quad-(5+3\sqrt{3}-h_1)(w_j^{20}+w_j^{12})],\\f_1^{(j)}&=6(2+\sqrt{3})w_j^{01}+\alpha[(5+3\sqrt{3}-3h_1)w_j^{01}\\&\quad+(5+3\sqrt{3}-h_1)(w_j^{20}-w_j^{12})],\\f_2^{(j)}&=6(2+\sqrt{3})w_j^{01}+\alpha[-(5+3\sqrt{3}+h_1)w_j^{01}\\&\quad+(5+3\sqrt{3}-h_1)(w_j^{20}+w_j^{12})].\end{aligned}$$

Then we can see from Lemma 1 that $F_1^{(j)}(t,\alpha)\geq 0$, $t\in$[0,π/6] if and only if the following three formulae hold synchronously:

(1) $R_0^{(1)}=2\alpha w_j^{01}\geq 0$;

(2) $R_2^{(1)}=6(\sqrt{3}-1)w_j^{01}+\alpha A_1^{(j)}\geq 0 \Leftrightarrow \alpha\leq\alpha_1^{(j)}$;

(3) $R_1^{(1)}+\sqrt{2R_0^{(1)}R_2^{(1)}}\geq 0$

$\Leftrightarrow R_1^{(1)}\geq 0$, or $R_1^{(1)}<0$ and $2R_0^{(1)}R_2^{(1)}\geq (R_1^{(1)})^2$

$\Leftrightarrow 6w_j^{01}+\alpha A_2^{(j)}\geq 0$, or $6w_j^{01}+\alpha A_2^{(j)}<0$ and

$H_1^{(j)}(\alpha)=k_1^{(j)}\alpha^2+k_2^{(j)}\alpha+k_3^{(j)}\leq 0.$

When the root discriminant $\Delta_1^{(j)}\geq 0$ of the quadratic polynomial $H_1^{(j)}(\alpha)$, $H_1^{(j)}(\alpha)$ has the zeroes $\alpha_1^{*(j)}$ and $\alpha_2^{*(j)}$. So combining (1) and (2), we get

$F_1^{(j)}(t,\alpha)\geq 0$, $t\in$[0,π/6]

$\Leftrightarrow 0<\alpha\leq\min(\alpha_1^{(j)},\alpha_2^{(j)},1).$ $(\Delta_1^{(j)}<0)$

$F_1^{(j)}(t,\alpha)\geq 0$, $t\in$[0,π/6]

$\Leftrightarrow 0<\alpha\leq\min(\alpha_1^{(j)},\alpha_2^{(j)},1)$, or

$\max(\alpha_2^{(j)},\alpha_1^{*(j)},0)\leq\alpha\leq\min(\alpha_1^{(j)},\alpha_2^{*(j)},1).$ $(\Delta_1^{(j)}\geq 0)$

Similarly, using Lemmas 2 and 3 on $F_2^{(j)}(t,\alpha)$ and $F_3^{(j)}(t,\alpha)$ respectively gives

$F_2^{(j)}(t,\alpha)\geq 0$, $t\in$[π/6, π/3]

$\Leftrightarrow 0<\alpha\leq\min(\alpha_1^{(j)},\alpha_3^{(j)}\alpha_4^{(j)},1).$ $(\Delta_2^{(j)}<0)$

$F_2^{(j)}(t,\alpha)\geq 0$, $t\in$[π/6, π/3]

$\Leftrightarrow 0<\alpha\leq\min(\alpha_1^{(j)},\alpha_3^{(j)}\alpha_4^{(j)},1)$, or

$\max(\alpha_4^{(j)},0)\leq\alpha\leq\min(\alpha_1^{(j)},\alpha_3^{(j)},\alpha_3^{*(j)},1)$, or

$\max(\alpha_4^{(j)},\alpha_4^{*(j)},0)\leq\alpha\leq\min(\alpha_1^{(j)},\alpha_3^{(j)},1).$ $(\Delta_2^{(j)}\geq 0)$

$F_3^{(j)}(t,\alpha)\geq 0$, $t\in$[π/3,π/2]

$\Leftrightarrow 0<\alpha\leq\min(\alpha_3^{(j)},\alpha_5^{(j)},1)$. $(\Delta_3^{(j)}<0)$

$F_3^{(j)}(t,\alpha)\geq 0$, $t\in$[π/3,π/2]

$\Leftrightarrow 0<\alpha\leq\min(\alpha_3^{(j)},\alpha_5^{(j)},1)$, or

$\max(\alpha_5^{(j)},\alpha_5^{*(j)},0)\leq\alpha\leq\min(\alpha_3^{(j)},\alpha_6^{*(j)},1)$. $(\Delta_3^{(j)}\geq 0)$

Summarizing the results on the above three subintervals, if we denote

$$\begin{cases}\alpha_{l1}^{(j)}=0,\ \alpha_{l2}^{(j)}=\max(\alpha_2^{(j)},\alpha_1^{*(j)},0),\\\alpha_{l3}^{(j)}=\max(\alpha_4^{(j)},0),\ \alpha_{l4}^{(j)}=\max(\alpha_4^{(j)},\alpha_4^{*(j)},0),\\\alpha_{l5}^{(j)}=\max(\alpha_5^{(j)},\alpha_5^{*(j)},0),\ \alpha_{l6}^{(j)}=\max(\alpha_{l2}^{(j)},\alpha_{l3}^{(j)}),\\\alpha_{l7}^{(j)}=\max(\alpha_{l2}^{(j)},\alpha_{l4}^{(j)}),\ \alpha_{l8}^{(j)}=\max(\alpha_{l2}^{(j)},\alpha_{l5}^{(j)}),\\\alpha_{l9}^{(j)}=\max(\alpha_{l3}^{(j)},\alpha_{l5}^{(j)}),\ \alpha_{l10}^{(j)}=\max(\alpha_{l4}^{(j)},\alpha_{l5}^{(j)}),\\\alpha_{l11}^{(j)}=\max(\alpha_{l2}^{(j)},\alpha_{l3}^{(j)},\alpha_{l5}^{(j)}),\ \alpha_{l12}^{(j)}=\max(\alpha_{l2}^{(j)},\alpha_{l4}^{(j)},\alpha_{l5}^{(j)}),\\\alpha_{r1}^{(j)}=\min(\alpha_1^{(j)},\alpha_2^{(j)},\alpha_3^{(j)},\alpha_4^{(j)},\alpha_5^{(j)},1),\\\alpha_{r2}^{(j)}=\min(\alpha_1^{(j)},\alpha_2^{*(j)},\alpha_3^{(j)},\alpha_4^{(j)},\alpha_5^{(j)},1),\\\alpha_{r3}^{(j)}=\min(\alpha_1^{(j)},\alpha_2^{(j)},\alpha_3^{(j)},\alpha_3^{*(j)},\alpha_5^{(j)},1),\\\alpha_{r4}^{(j)}=\min(\alpha_1^{(j)},\alpha_2^{(j)},\alpha_3^{(j)},\alpha_5^{(j)},1),\\\alpha_{r5}^{(j)}=\min(\alpha_1^{(j)},\alpha_2^{(j)},\alpha_3^{(j)},\alpha_4^{(j)},\alpha_6^{*(j)},1),\\\alpha_{r6}^{(j)}=\min(\alpha_1^{(j)},\alpha_2^{*(j)},\alpha_3^{(j)},\alpha_3^{*(j)},\alpha_5^{(j)},1),\\\alpha_{r7}^{(j)}=\min(\alpha_1^{(j)},\alpha_2^{*(j)},\alpha_3^{(j)},\alpha_5^{(j)},1),\\\alpha_{r8}^{(j)}=\min(\alpha_1^{(j)},\alpha_2^{*(j)},\alpha_3^{(j)},\alpha_4^{(j)},\alpha_6^{*(j)},1),\\\alpha_{r9}^{(j)}=\min(\alpha_1^{(j)},\alpha_2^{(j)},\alpha_3^{(j)},\alpha_3^{*(j)},\alpha_6^{*(j)},1),\\\alpha_{r10}^{(j)}=\min(\alpha_1^{(j)},\alpha_2^{(j)},\alpha_3^{(j)},\alpha_6^{*(j)},1),\\\alpha_{r11}^{(j)}=\min(\alpha_1^{(j)},\alpha_2^{*(j)},\alpha_3^{(j)},\alpha_3^{*(j)},\alpha_6^{*(j)},1),\\\alpha_{r12}^{(j)}=\min(\alpha_1^{(j)},\alpha_2^{*(j)},\alpha_3^{(j)},\alpha_6^{*(j)},1),\end{cases}\tag{13}$$

and set

$$\begin{cases}B_1^{*(j)}=\{\alpha\,|\,0<\alpha\leq\alpha_{r1}^{(j)}\},\\B_i^{*(j)}=\{\alpha\,|\,\alpha_{li}^{(j)}\leq\alpha\leq\alpha_{ri}^{(j)}\},\ i=2,3,...,12,\end{cases}\tag{14}$$

we can state the following

Theorem 1 Let $\{\boldsymbol{P}_i\}_{i=j-1}^{j+2}$ be convex data points in the plane and any three consecutive points are not collinear, let $\overline{\boldsymbol{T}}_j(t,\alpha)$ ($0\le t\le\pi/2$) be the uniform α-trigonometric polynomial interpolating curve segment defined by Eq.(10), then there are no inflection points on $\overline{\boldsymbol{T}}_j(t,\alpha)$ ($0\le t\le\pi/2$) if and only if $\alpha\in U_j^*$, where

(1) When $\Delta_1^{(j)}, \Delta_2^{(j)}, \Delta_3^{(j)}<0$, $U_j^*=B_1^{*(j)}$;

(2) When $\Delta_1^{(j)}\ge 0$ and $\Delta_2^{(j)}, \Delta_3^{(j)}<0$, $U_j^*=B_1^{*(j)}\bigcup B_2^{*(j)}$;

(3) When $\Delta_2^{(j)}\ge 0$ and $\Delta_1^{(j)}, \Delta_3^{(j)}<0$, $U_j^*=B_1^{*(j)}\bigcup B_3^{*(j)}\bigcup B_4^{*(j)}$;

(4) When $\Delta_3^{(j)}\ge 0$ and $\Delta_1^{(j)}, \Delta_2^{(j)}<0$, $U_j^*=B_1^{*(j)}\bigcup B_5^{*(j)}$;

(5) When $\Delta_1^{(j)}, \Delta_2^{(j)}\ge 0$ and $\Delta_3^{(j)}<0$, $U_j^*=B_1^{*(j)}\bigcup B_2^{*(j)}\bigcup B_3^{*(j)}\bigcup B_4^{*(j)}\bigcup B_6^{*(j)}\bigcup B_7^{*(j)}$;

(6) When $\Delta_1^{(j)}, \Delta_3^{(j)}\ge 0$ and $\Delta_2^{(j)}<0$, $U_j^*=B_1^{*(j)}\bigcup B_2^{*(j)}\bigcup B_5^{*(j)}\bigcup B_8^{*(j)}$;

(7) When $\Delta_2^{(j)},\Delta_3^{(j)}\ge 0$ and $\Delta_1^{(j)}<0$, , $U_j^*=B_1^{*(j)}\bigcup B_3^{*(j)}\bigcup B_4^{*(j)}\bigcup B_5^{*(j)}\bigcup B_9^{*(j)}\bigcup B_{10}^{*(j)}$;

(8) When $\Delta_1^{(j)}, \Delta_2^{(j)}, \Delta_3^{(j)}\ge 0$, $U_j^*=\bigcup_{i=1}^{12}B_i^{*(j)}$.

Substituting "<" for "≤" in $B_i^{*(j)}$ (i=1,2,…,12), and denoting the corresponding sets as $B_i^{(j)}$ (i=1,2,…,12), U_j^* in Lemma 1 as U_j, then when $\alpha\in U_j$, $\overline{M}_j(t,\alpha)>0$, that is $\partial\overline{\boldsymbol{T}}_j(t,\alpha)/\partial t\ne\boldsymbol{0}$, which indicates that there are no cusp points on the curve segment $\overline{\boldsymbol{T}}_j(t,\alpha)$ ($0\le t\le\pi/2$).

Next, let us consider the double points on this curve segment. Suppose that there exist t_1, t_2, $0\le t_1<t_2\le\pi/2$ such that $\overline{\boldsymbol{T}}_j(t_1,\alpha)=\overline{\boldsymbol{T}}_j(t_2,\alpha)$. From Eq.(10), after a simple calculation, we have

$$\alpha\boldsymbol{a}_j(t_2+\cos t_2-t_1-\cos t_1)/\pi+\alpha\boldsymbol{a}_{j+1}(\sin t_2-\cos t_2-\sin t_1+\cos t_1)/\pi+\alpha\boldsymbol{a}_{j+2}(t_2-\sin t_2-t_1+\sin t_1)/\pi+[\overline{S}(t_2)-\overline{S}(t_1)][(1-2\alpha/\pi)\boldsymbol{a}_{j+1}-\alpha(1/2-1/\pi)(\boldsymbol{a}_j+\boldsymbol{a}_{j+2})]=\boldsymbol{0}.$$

Then calculating the wedge product of both sides of the above equation with $\boldsymbol{a}_{j+2}$ gives

$$-\alpha w_j^{20}(t_2+\cos t_2-t_1-\cos t_1)/\pi+\alpha w_j^{12}(\sin t_2-\cos t_2-\sin t_1+\cos t_1)/\pi+[\overline{S}(t_2)-\overline{S}(t_1)]\cdot[(1-2\alpha/\pi)w_j^{12}+\alpha(1/2-1/\pi)w_j^{20}]=0. \quad (15)$$

But, when $w_j^{20}\ge 0$, Eq.(15) can be written as

$$\alpha(w_j^{12}-w_j^{20})(\sin t_2-\cos t_2-\sin t_1+\cos t_1)/\pi+\alpha(t_1-t_2-2\cos t_2+2\cos t_1+\sin t_2-\sin t_1)w_j^{20}/\pi+[\overline{S}(t_2)-\overline{S}(t_1)]\cdot[(1-2\alpha/\pi)w_j^{12}+\alpha(1/2-1/\pi)w_j^{20}]=0. \quad (16)$$

The left side of Eq.(16) is larger than zero because of the monotonicity of the function $\overline{S}(t)$ on [0,π/2] and Eq.(11). This contradiction shows that there are no double points on the curve segment $\overline{\boldsymbol{T}}_j(t,\alpha)$ ($0\le t\le\pi/2$); and, when $w_j^{20}<0$, if we choose α satisfying

$$0<\alpha\le\alpha_0^{(j)}=\min\left\{\pi w_j^{12}\big/[2w_j^{12}+(1-\pi/2)w_j^{20}],1\right\},$$

then the left side of Eq.(15) is larger than zero. So the same reasoning shows the inexistence of double points on the curve segment $\overline{\boldsymbol{T}}_j(t,\alpha)$ ($0\le t\le\pi/2$). Combining the above discussion and Theorem 1, if we let

$$V_j=\{\alpha\,|\,0<\alpha\le\alpha_0^{(j)}\}, \quad (17)$$

we can state the following

Theorem 2 Let $\{\boldsymbol{P}_i\}_{i=j-1}^{j+2}$ and $\overline{\boldsymbol{T}}_j(t,\alpha)$ ($0\le t\le\pi/2$) be the same as in Theorem 1, if according to the different cases in Theorem 1, we choose D_j as follows:

$$D_j=U_j, \quad (w_j^{20}\ge 0). \quad (18)$$

$$D_j=U_j\bigcap V_j, \quad (w_j^{20}<0). \quad (19)$$

Then, when $\alpha\in D_j$, the corresponding curve segment $\overline{\boldsymbol{T}}_j(t,\alpha)$ ($0\le t\le\pi/2$) is convex and C^2-continuous.

Convexity of *α*-trigonometric polynomial interpolating curve *T*(*u*, *α*) ($u_1\le u\le u_n$)

1. Taking uniform shape parameter for each curve segment

From Section 3.1, we can get

Theorem 3 Let $\{\boldsymbol{P}_i\}_{i=1}^n$ be convex data points in the plane, with any three consecutive points being not collinear; suppose that $\boldsymbol{P}_0$ and $\boldsymbol{P}_{n+1}$ are two auxiliary data points introduced according to the method of (Tai and Wang, 2004) and $\boldsymbol{T}(u, \alpha)$ ($u_1 \le u \le u_n$) is the uniform α-trigonometric polynomial curve defined by Eq.(7). If the shape parameter α satisfies

$$\alpha \in \bigcap_{j=1}^{n-1} D_j, \tag{20}$$

then the whole curve $\boldsymbol{T}(u, \alpha)$ ($u_1 \le u \le u_n$) is convex and C^2-continuous.

Proof From Theorem 2 we can see each segment $\boldsymbol{T}_j(u,\alpha)$ ($u_j \le u \le u_{j+1}$), j=1,2,…,n−1 of the curve $\boldsymbol{T}(u,\alpha)$ ($u_1 \le u \le u_n$) with the shape parameter $\alpha \in \bigcap_{j=1}^{n-1} D_j$ is convex. Also, from Eq.(8) and the fact that trigonometric polynomial curve is convexity-preserving about its control polynomial (Zhang, 1997), we can conclude there are no cusp points on the curve $\boldsymbol{T}(u,\alpha)$ ($u_1 \le u \le u_n$).

When the variable $u \in [u_j, u_{j+1}]$, substituting t=π/4 into Eq.(10) gives

$$\overline{\boldsymbol{T}}_j(\pi/4,\alpha) = (\boldsymbol{P}_j + \boldsymbol{P}_{j+1})/2 + \alpha(\sqrt{2}-1)/\pi \cdot [(\boldsymbol{P}_j + \boldsymbol{P}_{j+1})/2 - (\boldsymbol{P}_{j-1} + \boldsymbol{P}_{j+2})/2].$$

Since $0<\alpha<1$, it is easy to prove that each data point $\boldsymbol{P}_j$ (j=1,2,…,n) is not the inflection point of $\boldsymbol{T}(u,\alpha)$ ($u_1 \le u \le u_n$). Again, the definition of $\boldsymbol{T}(u,\alpha)$ and the fact of $\boldsymbol{T}(u,\alpha) \in C^2[u_1,u_n]$ reveal that the joint point between any two consecutive curve segments $\boldsymbol{T}_j(u,\alpha)$ and $\boldsymbol{T}_{j+1}(u,\alpha)$ (j=1,2,…,n−2) do not comprise a double point. So, curve $\boldsymbol{T}(u,\alpha)$ ($u_1 \le u \le u_n$) is convex.

Remark In Theorems 1~3, we suppose that any three consecutive points among $\{\boldsymbol{P}_i\}_{i=1}^n$ are not collinear. This supposition is necessary to avoid $\bigcap_{j=1}^{n-1} D_j = \varnothing$ or $\bigcap_{j=1}^{n-1} D_j = \{\alpha|\alpha=0\}$. If there are three points $\boldsymbol{P}_{j-1}$, $\boldsymbol{P}_j$ and $\boldsymbol{P}_{j+1}$ ($j \ge 2$) collinear (Fig.1), the case is trivial: as the curve segments $\boldsymbol{T}_{j-1}(u,\alpha)$ ($u_{j-1} \le u \le u_j$) and $\boldsymbol{T}_j(u,\alpha)$ ($u_j \le u \le u_{j+1}$), $w_{j-1}^{12} = w_j^{01} = [\boldsymbol{a}_j, \boldsymbol{a}_{j+1}] = 0$ hold, so $\alpha_3^{(j-1)} = \alpha_1^{(j)} = 0$ from Eq.(12), and then $\alpha_{ri}^{(j-1)} = \alpha_{ri}^{(j)} = 0$ in Eq.(13) (i=1,2,…,12). That is, $D_{j-1}=D_j=\{\alpha|\alpha=0\}$. So, $\boldsymbol{T}_{j-1}(u,\alpha)$ and $\boldsymbol{T}_j(u,\alpha)$ are reduced to the straight line segments $\boldsymbol{P}_{j-1}\boldsymbol{P}_j$ and $\boldsymbol{P}_j\boldsymbol{P}_{j+1}$ respectively. If we take the uniform shape parameter for each curve segment according to Eq.(20), then α=0, and $\boldsymbol{T}(u,0)$ ($u_1 \le u \le u_n$) is reduced to the polygon $\boldsymbol{P}_1\boldsymbol{P}_2\ldots\boldsymbol{P}_n$ connecting the data points $\{\boldsymbol{P}_i\}_{i=1}^n$.

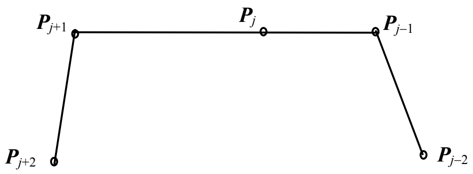

Fig.1 Three collinear consecutive points $\boldsymbol{P}_{j-1}$, $\boldsymbol{P}_j$, $\boldsymbol{P}_{j+1}$

2. Taking respective shape parameters for each curve segment

In practical applications, when there is a large number of data points, and some parameter value ranges D_j calculated according to Eq.(18) or Eq.(19) are very small, α obtained by Eq.(20) will be small (close to zero). Then, the curve $\boldsymbol{T}(u,\alpha)$ ($u_1 \le u \le u_n$) cannot reveal the information of some curve segments, and lacks flexibility. In this subsection, we will improve the algorithm in Section 3.2.1 and obtain a more useful convexity-preserving algorithm. That is, for each curve segment $\boldsymbol{T}_j(u,\alpha)$ ($u_j \le u \le u_{j+1}$), j=1,2,…, n−1, we may select the respective shape parameter α_j^* such that $\alpha_j^* \in D_j$. Accordingly, write each curve segment as $\boldsymbol{T}_j(u,\alpha_j^*)$ ($u_j \le u \le u_{j+1}$), the whole curve as $\boldsymbol{T}(u)$ ($u_1 \le u \le u_n$), and the singular polygon as $\boldsymbol{L}(u)$ ($u_1 \le u \le u_n$), each of which is $\boldsymbol{L}_j(u,\alpha_j^*)$. It is worthwhile to note that at this time the singular polygon here is discontinuous, however, $\boldsymbol{T}(u)$ is a convexity-preserving interpolating curve with G^1 continuity and each curve segment $\boldsymbol{T}_j(u,\alpha_j^*) \in C^2[u_j, u_{j+1}]$ [see Section 5 of (Tai and Wang, 2004)]. It is more suitable for application than the curve $\boldsymbol{T}(u,\alpha)$ ($u_1 \le u \le u_n$) defined in Section 3.2.1, with the uniform shape parameter α (see the examples in Section 5).

CONVEXITY-PRESERVING INTERPOLATION RELATIVE TO GENERAL DATA POINTS

In this section, we will discuss the convexity-preserving interpolation relative to general data points.

The detail is as follows. Search starting from $\boldsymbol{P}_1$, we find the first convex subset $I_1=\{\boldsymbol{P}_1\boldsymbol{P}_2...\boldsymbol{P}_{i_1}\}$ of the data points according to Definition 1; as $I_1=\{\boldsymbol{P}_1\boldsymbol{P}_2...\boldsymbol{P}_{i_1}\}$ seems to be the same as $\{\boldsymbol{P}_i\}_{i=1}^n$ in Section 2, we can obtain the auxiliary point $\boldsymbol{P}_0$; then search starting from $\boldsymbol{P}_n$ reversely, we find the last convex subset $I_m=\{\boldsymbol{P}_{i_{m-1}}\boldsymbol{P}_{i_{m-1}+1}...\boldsymbol{P}_n\}$ of the data points according to Definition 1, similarly, we can obtain the auxiliary point $\boldsymbol{P}_{n+1}$. Next, using $\{\boldsymbol{P}_i\}_{i=0}^{n+1}$ as the control vertices, we construct a uniform trigonometric polynomial curve $\boldsymbol{C}(u)$ by method similar to the one mentioned in Section 2.

Finally, we construct uniform α-trigonometric polynomial convexity-preserving interpolating curve relative to the general data points.

Noting that construction of the jth curve segment, interpolation of the data points $\boldsymbol{P}_j$ and $\boldsymbol{P}_{j+1}$, of the α-trigonometric polynomial interpolating curve, must use the points $\boldsymbol{P}_{j-1}$, $\boldsymbol{P}_j$, $\boldsymbol{P}_{j+1}$, $\boldsymbol{P}_{j+2}$ (j=1,2,…,n−1). If the point set $\{\boldsymbol{P}_{j-1}\boldsymbol{P}_j\boldsymbol{P}_{j+1}\boldsymbol{P}_{j+2}\}$ (j=1,2,…,n−1) is convex, the range of the shape parameter α: D_j, which makes the jth curve segment convex, can be obtained according to the algorithms in Section 3.1; else, we take $D_j=\{\alpha|0<\alpha<1\}$. Thus, when taking $\alpha\in\bigcap_{j=1}^{n-1}D_j$, the corresponding curve $\boldsymbol{T}(u,\alpha)$ ($u_1\le u\le u_n$) defined by Eq.(7) is convexity-preserving and interpolates $\{\boldsymbol{P}_i\}_{i=1}^n$ with C^2 continuity.

On the other hand, similar to Section 3.2.2, for each curve segment $\boldsymbol{T}_j(u,\alpha)$ ($u_j\le u\le u_{j+1}$), j=1,2,…,n−1, we may select the respective shape parameter α_j^* such that $\alpha_j^*\in D_j$. Then $\boldsymbol{T}(u)$ ($u_1\le u\le u_n$) is a convexity-preserving interpolating curve with G^1 continuity and each curve segment $\boldsymbol{T}_j(u,\alpha_j^*)\in C^2[u_j,u_{j+1}]$ (j=1,2,…, n−1).

NUMERICAL EXAMPLES

In this section, we will give some numerical examples; in each example, the data points are marked with small black points, we depict α-trigonometric polynomial interpolating curve. In Examples 3 and 4, we also give the corresponding curvature plot.

Example 1 The data points are shown in Table 1, and the corresponding interpolating curves with different values of shape parameter α are plotted in Fig.2.

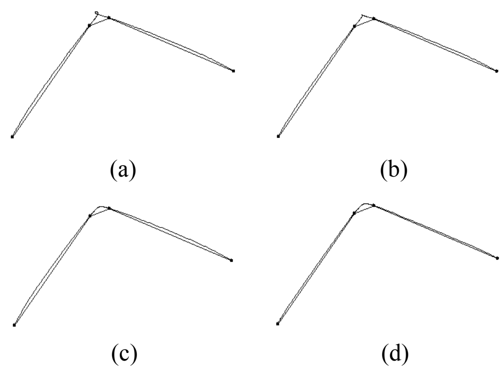

Fig.2 α-trigonometric polynomial interpolating curve with given α. (a) α=0.9; (b) α=0.74; (c) Each segment having respective shape parameter α_j^* (j=1,2,3); (d) α=0.44301

Table 1 The data points in Fig.2

x_n	12	23	25.7	43.5
y_n	28.7	13	12	19.4

Fig.2a shows the result with the uniform shape parameter α=0.9. There is a double point on the α-trigonometric polynomial interpolating curve. Fig.2b shows the result with the uniform shape parameter α=0.74, there is a cusp point on the α-trigonometric polynomial interpolating curve. Fig.2c shows the result with respective shape parameter for each curve segment, where $D_1=D_3=\{\alpha|0<\alpha<1\}$, $D_2=\{\alpha|0<\alpha<0.44302\}$ according to Theorems 1 and Theorem 2; and we take $\alpha_1^*=0.99$, $\alpha_2^*=0.44301$, $\alpha_3^*=0.99$, which avoids cusp and double points, and the α-trigonometric polynomial interpolating curve is convexity-preserving. Fig.2d shows the result with the uniform shape parameter α= 0.44302. The α-trigonometric polynomial interpolating curve is convexity-preserving. Apparently, Fig.2c has better convexity-preserving effect than Fig.2d.

Example 2 The data points are shown in Table 2, and the corresponding interpolating curves with different α are plotted in Fig.3.

Table 2 The data points in Fig.3

x_n	7.2	10.6	17.6	28.6	32.2
y_n	24.6	7.8	7.5	8.4	21.0

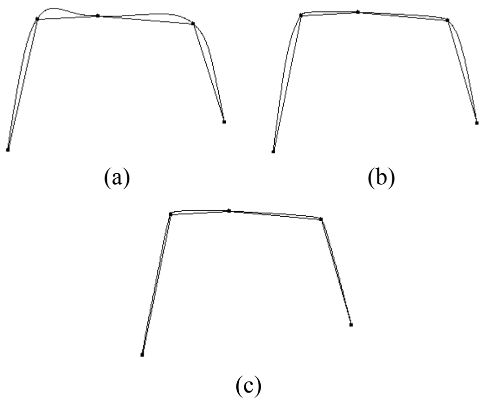

Fig.3 α-trigonometric polynomial interpolating curve with given α. (a) α=0.8; (b) Each segment having respective shape parameter α_j^* (j=1,2,3,4); (c) α=0.204647

Fig.3a shows the result with the uniform shape parameter α=0.8. There are inflection points on the α-trigonometric polynomial interpolating curve. Fig.3b shows the result with respective shape parameter for each curve segment, where $D_1=\{\alpha|0<\alpha<1\}$, $D_2=\{\alpha|0<\alpha<0.204648\}$, $D_3=\{\alpha|0<\alpha<0.244841\}$, $D_4=\{\alpha|0<\alpha<0.725503\}$ according to Theorems 1 and 2; and we take $\alpha_1^*=0.99$, $\alpha_2^*=0.204647$, $\alpha_3^*=0.24484$, $\alpha_4^*=0.725502$; the α-trigonometric polynomial interpolating curve is convexity-preserving. Fig.3c shows the result with the uniform shape parameter α= 0.204647, where the α-trigonometric polynomial interpolating curve is convexity-preserving. Apparently, Fig.3b has better convexity-preserving effect than Fig.3c.

Example 3 The data points are taken from (Fletcher and McAllister, 1990) and shown in Table 3, and the corresponding interpolating curves with different α are plotted in Fig.4.

Table 3 The data points in Fig.4

x_n	0	6	13	15	16	20	24
y_n	0	−6	−6	−2	14	16	14

Fig.4 shows the convexity-preserving α-trigonometric polynomial interpolating curves and their curvature plots. According to Theorems 1 and 2, we can get $D_j=\{\alpha|0<\alpha<1\}$, j=1,2,6; $D_3=\{\alpha|0<\alpha<0.733471\}$, $D_5=\{\alpha|0<\alpha<0.603574\}$, and the 4th segment is an intergradation between two convex subsets of the data points, and we take $D_4=\{\alpha|0<\alpha<1\}$. Fig.4a shows the result with the respective shape parameter for each curve segment, here, $\alpha_j^*=0.99$, j=1,2,6,

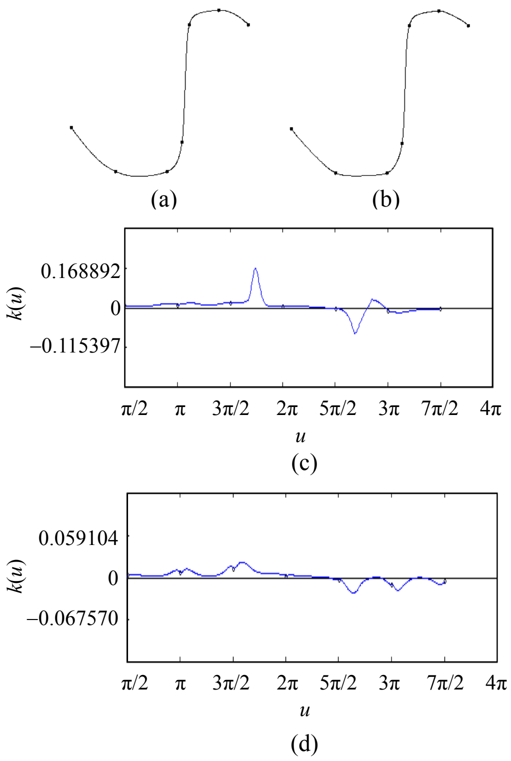

Fig.4 α-trigonometric polynomial interpolating curve and their curvature plots $k(u)$. (a) Each segment having respective shape parameter; (b) The whole curve having the uniform shape parameter; (c) The curvature plots of curve in (a) with k_{max}=0.168892, k_{min}=−0.115397; (d) The curvature plots of curve in (b) with k_{max}=0.059104, k_{min}=−0.067570

$\alpha_3^*=0.733470$, $\alpha_4^*=0.5$, $\alpha_5^*=0.6$. We must point out that the value of α_4^* does not affect the convexity-preserving property of α-trigonometric polynomial interpolating curve, considering that α-trigonometric polynomial curve is constructed by blending a parametrized polygon and the trigonometric polynomial curve, we might as well take α_4^*=0.5. Fig.4b shows the result with the uniform shape parameter α=0.6. Fig.4c shows the curvature plots of Fig.4a with k_{max}=0.168892 and k_{min}=−0.115397. Fig.4d shows the curvature plots of Fig.4b with k_{max}=0.059104 and k_{min}=−0.067570.

Example 4 The data points are taken from the cycloid

$$\begin{cases} x = a(\theta - \sin\theta), \\ y = a(1-\cos\theta), \end{cases} \quad a = 100, \ 0 \le \theta \le 2\pi,$$

where $\theta = i\pi/5$, $i = 0, 1, 2, \ldots, 10$.

Fig.5 shows the convexity-preserving α-trigonometric polynomial interpolating curve with the respective shape parameter for each curve segment (Fig.5a) and its curvature plot (Fig.5b). According to Theorems 1 and 2, here, $D_j=\{\alpha|0<\alpha<1\}$ (j=1,3,4,5,6,7, 8,10), $D_2=D_9=\{\alpha|0<\alpha<0.543712\}$, and we take $\alpha_j^*=0.99$ (j=1,3,4,5, 6,7,8,10), $\alpha_2^*=\alpha_9^*=0.543711$. It is easy to see that for transcendental curves, the effect of our method is very good.

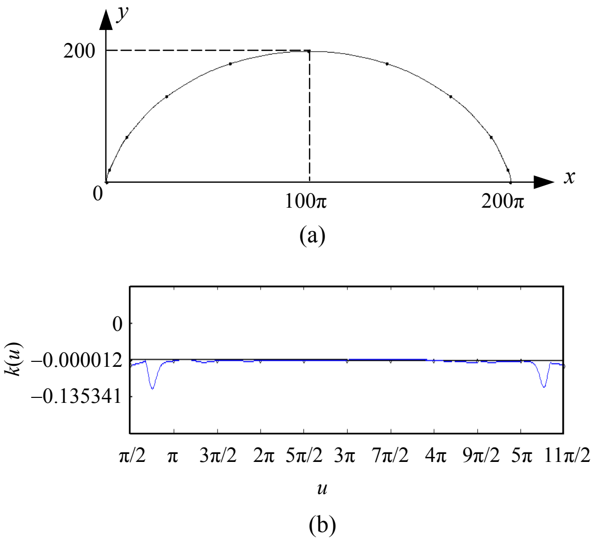

Fig.5 (a) α-trigonometric polynomial interpolating curve with the respective shape parameter for each curve segment; (b) The curvature plot $k(u)$ of curve in (a) with k_{max}=−0.000012, k_{min}=−0.135341

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