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Analytical solution for functionally graded anisotropic cantilever beam under thermal and uniformly distributed load*

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Abstract: The bending problem of a functionally graded anisotropic cantilever beam subjected to thermal and uniformly distributed load is investigated, with material parameters being arbitrary functions of the thickness coordinate. The heat conduction problem is treated as a 1D problem through the thickness. Based on the elementary formulations for plane stress problem, the stress function is assumed to be in the form of polynomial of the longitudinal coordinate variable, from which the stresses can be derived. The stress function is then determined completely with the compatibility equation and boundary conditions. A practical example is presented to show the application of the method.

Key words: Functionally graded material (FGM), Anisotropic, Thermal stress, Analytical solution, Cantilever beam
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INTRODUCTION

Functionally graded materials (FGMs) are a kind of material possessing properties that vary gradually with respect to the spatial coordinates. The material properties can be designed so as to improve its strength, toughness, high temperature withstanding ability, etc. Compared with traditional laminated composite structures, structures made of FGM have no obvious interfaces of material property, so that the phenomena of stress concentration can be eliminated or weakened greatly. Thus, FGMs have broad potential applications in aeronautics/astronautics manufacturing industry, nuclear power plant, etc.

Within the last two decades, the classic, first-order and higher-order shear deformation theories (Wetherhold *et al*., 1996; Almajid *et al*., 2001; Wu *et al*., 2002) had been developed for FGM structures. While a series of 3D elasticity solutions have been obtained for FGM plates and shells with special geometries (Ootao and Tanigawa, 2000; Chen and Ding, 2002; Chen *et al*., 2001; 2004; Zhong and Shang, 2005), 2D analytical solutions for plates and beams have also been presented (Sankar, 2001; Zhu and Sankar, 2004; Ding *et al*., 2007; Huang *et al*., 2007). Sankar and Tzeng (2002) investigated the thermal stress problem of a simply supported orthotropic beam with thermoelastic constants varying exponentially through the thickness. However, associated studies for FGM anisotropic beams with material parameters varying arbitrarily through the thickness have not been reported yet.

In this paper, we present an analytical solution for FGM anisotropic cantilever beam, whose thermoelastic parameters change arbitrarily along the thickness direction, and the heat conduction problem is treated as a 1D problem along the thickness coordinate. A numerical example is considered with numerical results presented.

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BASIC EQUATIONS AND BOUNDARY CONDITIONS

Let us consider an FGM anisotropic cantilever beam, as shown in Fig.1. We assume that the beam is in the state of plane stress, and is subjected to a uniformly distributed load q on its upper surface, with the initial temperature of the beam being T_0. Because of the change of the environment temperature, the temperatures on the upper and lower surfaces are changed to and kept at T_a and T_b. It is assumed that the temperature T is independent of the coordinate x and time t. The thermal condition is adiabatic at the two ends as well as two surfaces of the beam. Thus the heat conduction problem can be treated as a 1D problem. The material properties vary only in the thickness direction, i.e. we have the elastic compliance parameters as $s_{ij}=s_{ij}(y)$ $(i,j=1,2,6)$, the thermal conductivity coefficient as $k_2=k_2(y)$, and the thermal expansion coefficients as $\alpha_i=\alpha_i(y)$.

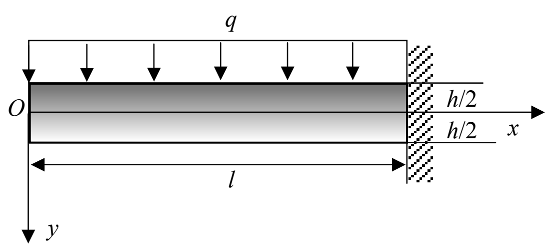

Fig.1 A cantilever beam under a uniform load and the coordinates

The heat conduction equation in the steady state (Ding *et al.*, 2006) is

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[k_2(y)\frac{\mathrm{d}T(y)}{\mathrm{d}y}\right]=0. \tag{1}$$

The boundary conditions for the temperature field are expressed as:

$$T(x,-h/2)=T_a,\ T(x,h/2)=T_b,\ \partial T/\partial x=\partial T/\partial z=0. \tag{2}$$

The basic equations of elasticity (Ding *et al.*, 2006) are

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0,\quad \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0, \tag{3}$$

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \gamma_{xy}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}, \tag{4}$$

$$\begin{aligned}\varepsilon_x&=s_{11}\sigma_x+s_{12}\sigma_y+s_{16}\tau_{xy}+\alpha_1\Delta T,\\ \varepsilon_y&=s_{12}\sigma_x+s_{22}\sigma_y+s_{26}\tau_{xy}+\alpha_2\Delta T,\\ \gamma_{xy}&=s_{16}\sigma_x+s_{26}\sigma_y+s_{66}\tau_{xy}+\alpha_6\Delta T,\end{aligned} \tag{5}$$

where σ_x, σ_y and τ_{xy} denote the stress components, ε_x, ε_y and γ_{xy} are the strain components, u and v denote the displacement components, and

$$\Delta T=T(y)-T_0. \tag{6}$$

The strain compatibility equation can be derived from Eq.(4)

$$\frac{\partial^2 \varepsilon_x}{\partial y^2}+\frac{\partial^2 \varepsilon_y}{\partial x^2}-\frac{\partial^2 \gamma_{xy}}{\partial x \partial y}=0. \tag{7}$$

The boundary conditions of elasticity at the upper and lower surfaces are

$$\sigma_y(x,-h/2)=-q,\ \sigma_y(x,h/2)=0,\ \tau_{xy}(x,\pm h/2)=0. \tag{8}$$

The boundary conditions at the left end (free) of the beam are

$$N_0=0,\quad M_0=0,\quad Q_0=0, \tag{9}$$

where N_0, M_0 and Q_0 denote the axial force, moment and shear force at x=0. The boundary conditions for the fixed end at the right end (fixed) of the beam are taken as:

$$u(l,0)=0,\ v(l,0)=0,\ \partial v(l,0)/\partial x=0. \tag{10}$$

DETERMINATION OF THE TEMPERATURE FIELD AND STRESS FUNCTION METHOD

Firstly, the temperature field of the beam can be solved from Eq.(1) with the boundary conditions in Eq.(2) as:

$$T(y)=T_a+(T_b-T_a)\int_{-h/2}^{y}\frac{\mathrm{d}\xi}{k_2(\xi)}\Big/\int_{-h/2}^{h/2}\frac{\mathrm{d}\xi}{k_2(\xi)}. \tag{11}$$

Secondly, a stress function ϕ is introduced such that

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2\phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}. \tag{12}$$

So that Eq.(3) is satisfied automatically. We then assume that

$$\phi=\phi_0(y) + x\phi_1(y) + x^2\phi_2(y), \tag{13}$$

where $\phi_0(y)$, $\phi_1(y)$ and $\phi_2(y)$ are unknown functions to be determined. Substitution of Eq.(13) into Eq.(12) gives

$$\sigma_x=\phi_0''+ x\phi_1''+ x^2\phi_2'',\ \sigma_y=2\phi_2,\ \tau_{xy}=-(\phi_1'+ 2x\phi_2'). \tag{14}$$

Substitution of Eq.(14) into Eq.(5), and then into Eq.(7), gives rise to

$$\begin{aligned}&(s_{11}\phi_2'')'' = 0, \quad (s_{11}\phi_1''-2s_{16}\phi_2')'' - 2(s_{16}\phi_2'')' = 0,\\ &(s_{11}\phi_0''-s_{16}\phi_1'+2s_{12}\phi_2+\alpha_1\Delta T)''\\ &\qquad +(2s_{66}\phi_2'-s_{16}\phi_1'')'+2s_{12}\phi_2''=0.\end{aligned} \tag{15}$$

Integration of Eq.$(15)_1$ yields

$$\begin{aligned}&\phi_2'' = a_1A_1 + a_2A_2, \quad \phi_2' = a_1A_1^0 + a_2A_2^0 + a_3,\\ &\phi_2 = a_1A_1^1 + a_2A_2^1 + a_3y + a_9,\end{aligned} \tag{16}$$

where and hereafter a_1~a_9 are integral constants, and

$$A_1 = \frac{y}{s_{11}}, \quad A_2 = \frac{1}{s_{11}},$$

$$A_i^n(y) = \frac{1}{n!}\int_{-h/2}^{y}(y-\xi)^n A_i\,\mathrm{d}\xi \quad (i=1,2; n=0,1). \tag{17}$$

Substituting Eqs.$(16)_1$ and $(16)_2$ into Eq.$(15)_2$, and then integrating, we get

$$\phi_1''=\sum_{i=1}^{5}a_iB_i, \quad \phi_1' = \sum_{i=1}^{5}a_iB_i^0 + a_6, \tag{18}$$

where

$$B_1 = \frac{2}{s_{11}}\left(s_{16}A_1^0 + \int_{-h/2}^{y}s_{16}A_1\,\mathrm{d}\xi\right),$$

$$B_2 = \frac{2}{s_{11}}\left(s_{16}A_2^0 + \int_{-h/2}^{y}s_{16}A_2\,\mathrm{d}\xi\right),$$

$$B_3 = \frac{2s_{16}}{s_{11}}, \quad B_4 = A_1, \quad B_5 = A_2,$$

$$B_i^n(y) = \frac{1}{n!}\int_{-h/2}^{y}(y-\xi)^n B_i\,\mathrm{d}\xi \ \ (i=1,2,...,5; n=0,1). \tag{19}$$

Substituting Eqs.(16), (18) into Eq.$(15)_3$, and performing integration twice, we obtain

$$\phi_0''=\sum_{i=1}^{10}a_iC_i, \tag{20}$$

where

$$\begin{aligned}&a_{10}=1,\\ &C_1 = \frac{1}{s_{11}}\left[\int_{-h/2}^{y}s_{16}B_1\,\mathrm{d}\xi + s_{16}B_1^0 - 2\int_{-h/2}^{y}(y-\xi)s_{12}A_1\,\mathrm{d}\xi\right.\\ &\qquad\left.-2\int_{-h/2}^{y}s_{66}A_1^0\,\mathrm{d}\xi - 2s_{12}A_1^1\right],\\ &C_2 = \frac{1}{s_{11}}\left[\int_{-h/2}^{y}s_{16}B_2\,\mathrm{d}\xi + s_{16}B_2^0 - 2\int_{-h/2}^{y}(y-\xi)s_{12}A_2\,\mathrm{d}\xi\right.\\ &\qquad\left.-2\int_{-h/2}^{y}s_{66}A_2^0\,\mathrm{d}\xi - 2s_{12}A_2^1\right],\\ &C_3 = \frac{1}{s_{11}}\left[\int_{-h/2}^{y}s_{16}B_3\,\mathrm{d}\xi + s_{16}B_3^0 - 2s_{12}y - 2\int_{-h/2}^{y}s_{66}\,\mathrm{d}\xi\right],\\ &C_4 = \frac{1}{s_{11}}\left[s_{16}B_4^0 + \int_{-h/2}^{y}s_{16}B_4\,\mathrm{d}\xi\right],\\ &C_5 = \frac{1}{s_{11}}\left[s_{16}B_5^0 + \int_{-h/2}^{y}s_{16}B_5\,\mathrm{d}\xi\right],\\ &C_6 = \frac{s_{16}}{s_{11}}, \quad C_7 = A_1, \quad C_8 = A_2,\\ &C_9 = -\frac{2s_{12}}{s_{11}}, C_{10} = -\frac{\alpha_1\Delta T}{s_{11}},\end{aligned} \tag{21}$$

$$C_i^n(y) = \frac{1}{n!}\int_{-h/2}^{y}(y-\xi)^n C_i\,\mathrm{d}\xi \ (i=1,2,...,10; n=0,1). \tag{22}$$

AXIAL FORCE, BENDING MOMENT, SHEAR FORCE AND DISPLACEMENTS

The axial force N_0, bending moment M_0 and shearing force Q_0 at the left end of the beam can be obtained by integrating Eqs.$(14)_1$ and $(14)_3$ according to

$$N_0 = \int_{-h/2}^{h/2} \sigma_x(0,y)\,dy = \sum_{i=1}^{10} a_i C_i^0(h/2),$$

$$M_0 = \int_{-h/2}^{h/2} \sigma_x(0,y)\,y\,dy = \frac{h}{2}\sum_{i=1}^{10} a_i C_i^0(h/2) - \sum_{i=1}^{10} a_i C_i^1(h/2),$$

$$Q_0 = \int_{-h/2}^{h/2} \tau_{xy}(0,y)\,dy = -\sum_{i=1}^{5} a_i B_i^1(h/2) - a_6 h \quad (23)$$

Substituting Eqs.(16), (18) and (20) into Eq.(14), then into Eq.(5), and finally performing the integration, we could obtain the displacement components as follows:

$$u = x(s_{11}\sigma_0 + s_{16}\tau_1 + 2s_{12}\sigma_2 + \alpha_1 T) + \frac{x^2}{2}(s_{11}\sigma_1' + 2s_{16}\sigma_2') + \frac{x^3}{3}s_{11}\sigma_2' + \int_{-h/2}^{y}(s_{16}\sigma_0 + s_{66}\tau_1 + 2s_{26}\sigma_2 + \alpha_6 T)\,d\eta + \int_{-h/2}^{y}(y-\eta)(s_{12}\sigma_1'' + 2s_{26}\sigma_2')\,d\eta - \omega y + u_0,$$

$$v = \int_{-h/2}^{y}(s_{12}\sigma_0 + s_{26}\tau_1 + 2s_{22}\sigma_2 + \alpha_2 T)\,d\eta - \frac{a_1}{12}x^4 - \frac{a_4}{6}x^3 - \int_{-h/2}^{y} s_{12}\sigma_2''\,d\eta\, x^2 - \frac{a_7}{2}x^2 - \int_{-h/2}^{y}(s_{12}\sigma_1'' + 2s_{26}\sigma_2')\,d\eta\, x + \omega x + v_0, \quad (24)$$

where u_0, v_0 and ω are integral constants.

DETERMINATION OF STRESSES AND DISPLACEMENTS

Substitution of Eqs.(16)$_2$, (16)$_3$ and (18)$_2$ into Eqs.(14)$_2$ and (14)$_3$, and then into Eq.(8), yields

$$a_3=0,\ a_6=0,\ a_9=-q/2,\ \sum_{i=1}^{2} a_i A_i^0(h/2) = 0,$$

$$2\sum_{i=1}^{2} a_i A_i^1(h/2) - q = 0,\ \sum_{i=1}^{5} a_i B_i^0(h/2) = 0. \quad (25)$$

Substitution of Eq.(23) into Eq.(9) gives

$$\sum_{i=1}^{10} a_i C_i^0(h/2) = 0,\ \sum_{i=1}^{10} a_i C_i^1(h/2) = 0,$$

$$\sum_{i=1}^{5} a_i B_i^1(h/2) = 0. \quad (26)$$

The unknown constants a_1 and a_2 can be obtained from Eqs.(25)$_4$ and (25)$_5$, then a_4 and a_5 from Eqs.(25)$_6$ and (26)$_3$, and finally a_7 and a_8 from Eqs.(26)$_1$ and (26)$_2$. Thus, all the undetermined constants are fixed, and the stress function ϕ is determined completely. The stress components are then obtained from Eq.(14).

Substitution of Eq.(24) into Eq.(10) yields three equations, from which u_0, v_0 and ω can be obtained. Thus, the displacement components u and v are determined.

EXAMPLE

In the numerical example, we take the span of the beam (with a unit width) l=1 m, height h=0.1 m, the uniform load q=10^6 N/m^2, and the initial temperature T_0=0 °C. The temperature on the upper surface is T_a=300 °C, and that on the lower surface is T_b=0 °C. The material properties vary according to the following gradient model:

$$M = M_{\rm I}(0.5 - y/h)^{\lambda} + M_{\rm II}[1 - (0.5 - y/h)^{\lambda}],$$

where M stands for the property at the location y, $M_{\rm I}$ and $M_{\rm II}$ stand for the properties on the upper and lower surfaces, and λ is the gradient index. The material properties of $M_{\rm I}$ and $M_{\rm II}$ are listed in Table 1.

Some results of stress and displacement are shown in Figs.2~4. Fig.2 shows that the distribution of σ_x/q at x=l/3 along the thickness is linear when the material is homogeneous (i.e. λ=0), but becomes nonlinear when the material is inhomogeneous (i.e. λ≠0). We also can find that the maximum tensile stress of the inhomogeneous beam is much larger than that of the homogeneous beam. From Fig.3, we find that the distribution curve for τ_{xy}/q is a parabola when λ=0 with the extreme value locating at y=0. However, the one for λ≠0 is no longer a parabola, and the extreme value deviates from the position y=0 and approaches to that of the upper surface. Also, small difference exists between the extreme values of τ_{xy}/q for different λ. From Figs.2 and 3, it can be concluded that the