Journal of Zhejiang University SCIENCE A
ISSN 1673-565X (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

# Stress analysis of anisotropic thick laminates in cylindrical bending using a semi-analytical approach*

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Received Dec. 29, 2006; revision accepted Mar. 9, 2007

**Abstract:** Semi-analytical elasticity solutions for bending of angle-ply laminates in cylindrical bending are presented using the state-space-based differential quadrature method (SSDQM). Partial differential state equation is derived from the basic equations of elasticity based on the state space concept. Then, the differential quadrature (DQ) technique is introduced to discretize the longitudinal domain of the plate so that a series of ordinary differential state equations are obtained at the discrete points. Meanwhile, the edge constrained conditions are handled directly using the stress and displacement components without the Saint-Venant principle. The thickness domain is solved analytically based on the state space formalism along with the continuity conditions at interfaces. The present method is validated by comparing the results to the exact solutions of Pagano's problem. Numerical results for fully clamped thick laminates are presented, and the influences of ply angle on stress distributions are discussed.

**Key words:** Semi-analytical elasticity solution, State-space-based differential quadrature method (SSDQM), Angle-ply laminates, Cylindrical bending
**doi:**10.1631/jzus.2007.A1740 **Document code:** A **CLC number:** O34

## INTRODUCTION

Cylindrical bending of composite laminated plates has been posing a preliminary research focus during the past decades. Pagano (1969; 1970) obtained two exact elasticity solutions for laminated plates in cylindrical bending, one for the plane strain problem of cross-ply laminates, the other for 3D problem of angle-ply laminates. Pagano's solutions provide a standard benchmark for validating a wide range of subsequent numerical approaches and 2D simplified theories (Shu and Soldatos, 2000; Messina, 2001; Messina and Soldatos, 2002). However, simplified global theories are confined to the prediction of global responses, such as gross deflection or natural frequencies, for thin or moderately thick plates to such an extent that the solution accuracy reduces increasingly as the laminates become thicker. On the other hand, if the component layer is analyzed individually using 2D theories and the global analysis is implemented by combining the algebraic equations of all layers along with the continuity conditions at interfaces, a large number of unknowns will be encountered, thus leading to cumbersome work and heavy storage for laminates with large number of layers. Meanwhile, 2D theories deliver inaccurate local response especially the stresses and strains at the ply level near geometric and materials discontinuities, which always play an important role on the failure of laminates. This is partly due to the fact that only part of the elastic constants is considered in the constitutive relations, indicating that the solutions will remain unchanged regardless of the variation of the values of elastic constants not considered.

* Project supported by the National Natural Science Foundation of China (No. 10432030), the China Postdoctoral Science Foundation (No. 20060401071) and the Program for New Century Excellent Talent in University of China (No. NCET-05-0510)

To meet the pursuit of high accuracy for composite thick laminates, 3D solution is required. Nevertheless, existing 3D analyses for plates in cylindrical bending are mainly limited to fully simply supported boundary conditions (Pagano, 1970; Chen and Lee, 2004b; Chen *et al.*, 2004b; Bian *et al.*, 2005; Yan *et al.*, 2007), since these conditions can be exactly captured by expanding all stress and displacement components into trigonometric series. Such treatment, however, is invalid for non-simple supports, such as clamped or free constraints, for which numerical techniques should be employed.

Differential quadrature method (DQM) is a highly efficient numerical technique among others for obtaining numerical solutions for boundary/initial value problems (Bellman *et al.*, 1972; Bert and Malik, 1996; Shu, 2000; Wang *et al.*, 2003; Pu and Zheng, 2006). It was recently successfully combined with the state space method (SSM), termed as the state-space-based differential quadrature method (SSDQM), by Chen *et al.*(2003b; 2004a) for laminated beams. This hybrid method allows precise treatment of edge boundary conditions point by point along the thickness direction, and, hence, the Saint-Venant principle is no longer necessary. Although SSDQM was applied by Chen and Lee (2004a) for free vibration analysis of cross-ply laminates in cylindrical bending with arbitrary boundary conditions, the problem under investigation is exactly a 2D problem. Most recently, Lü *et al.*(2007) used this method to analyze the free vibration of anisotropic laminates in cylindrical bending. In the present work, SSDQM is extended to obtain semi-analytical 3D elasticity solutions for generally laminated thick plates in cylindrical bending. A set of simultaneous partial differential equations governing cylindrical bending of plates is derived directly from the basic equations of 3D elasticity. The differential quadrature (DQ) technique is applied to approximate variables along the longitudinal direction, while the thickness domain of plates is exactly solved using SSM. Global analysis of plates is implemented using the transfer matrix method according to the continuity conditions at the interfaces. Numerical results, especially for strongly thick laminates with non-fully simply supported edges, are presented and hoped to provide a benchmark for future numerical analyses.

## STATE EQUATION DISCRETIZED BY DQM

Consider a $p$-layered homogeneous angle-ply laminated rectangular plate having a gross thickness of $h$ and length of $l$ in $x$-direction. The fibers in an individual layer orient at an angle $\theta$ with respect to the $x$-axis. All variables and loadings applied to the plate are assumed independent of coordinate $y$ so that the plate is subjected to cylindrical bending in $y$ direction. Following the routine job of SSM (Chen *et al.*, 2003a; Yan and Chen, 2004), the first order simultaneous differential equations about coordinate $z$, called the state equation, can be derived directly from the basic equations of 3D elasticity (Chen and Lee, 2004b). If the stress components are normalized by the stiffness constant $Q_{66}^{(1)}$ and the displacement components by plate thickness $h$, the normalized state equation is then obtained as

$$\frac{\partial}{\partial\zeta}[\bar{\sigma}_z\ \bar{u}\ \bar{v}\ \bar{w}\ \bar{\tau}_{xz}\ \bar{\tau}_{yz}]^{\mathrm{T}} = \begin{bmatrix} & \mathbf{0} & & 0 & -s\partial_{,\xi} & 0 \\ & & & -s\partial_{,\xi} & c_7 & c_8 \\ & & & 0 & c_8 & c_9 \\ c_{10} & c_4 s\partial_{,\xi} & b_3 s\partial_{,\xi} & & & \\ c_4 s\partial_{,\xi} & -c_1 s^2\partial_{,\xi\xi} & -b_1 s^2\partial_{,\xi\xi} & & \mathbf{0} & \\ b_3 s\partial_{,\xi} & -b_1 s^2\partial_{,\xi\xi} & -c_6 s^2\partial_{,\xi\xi} & & & \end{bmatrix} \begin{Bmatrix} \bar{\sigma}_z \\ \bar{u} \\ \bar{v} \\ \bar{w} \\ \bar{\tau}_{xz} \\ \bar{\tau}_{yz} \end{Bmatrix}, \quad (1)$$

where $s=h/l$ is the aspect ratio, while $\partial_{,\xi}$ and $\partial_{,\xi\xi}$ respectively denotes the first- and second-derivative with respect to $\xi$. All stress and displacement variables in Eq.(1) are termed as the state variables, and the accompanying induced variables are obtained as

$$\begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\tau}_{xy} \end{Bmatrix} = \begin{bmatrix} -c_4 & c_1 s\partial_{,\xi} & b_1 s\partial_{,\xi} \\ -c_5 & c_3 s\partial_{,\xi} & b_2 s\partial_{,\xi} \\ -b_3 & b_1 s\partial_{,\xi} & c_6 s\partial_{,\xi} \end{bmatrix} \begin{Bmatrix} \bar{\sigma}_z \\ \bar{u} \\ \bar{v} \end{Bmatrix}. \quad (2)$$

In the above two equations, the coefficients $c_i$ and $b_j$ are determined by

$$c_1 = \frac{Q_{11}Q_{33} - Q_{13}^2}{Q_{66}^{(1)}Q_{33}},\ c_3 = \frac{Q_{12}Q_{33} - Q_{13}Q_{23}}{Q_{66}^{(1)}Q_{33}},\ c_4 = -\frac{Q_{13}}{Q_{33}},$$

$$c_5 = -\frac{Q_{23}}{Q_{33}},\ c_6 = \frac{Q_{66}Q_{33} - Q_{36}^2}{Q_{66}^{(1)}Q_{33}},\ c_7 = \frac{Q_{66}^{(1)}Q_{44}}{d},$$

$$c_8=-\frac{Q_{66}^{(1)}Q_{45}}{d},\ c_9=\frac{Q_{66}^{(1)}Q_{55}}{d},\ c_{10}=\frac{Q_{66}^{(1)}}{Q_{33}},$$
$$b_1=\frac{Q_{16}Q_{33}-Q_{13}Q_{36}}{Q_{66}^{(1)}Q_{33}},\ b_2=\frac{Q_{26}Q_{33}-Q_{23}Q_{36}}{Q_{66}^{(1)}Q_{33}},\ b_3=-\frac{Q_{36}}{Q_{33}},$$

where $d=Q_{44}Q_{55}-Q_{45}^2$, and $Q_{ij}$ are the elastic constants defined by the stiffness constants of constitutive equations combined with the ply angle $\theta$ (Vinson and Sierakowski, 2002), with the superscript '(1)' denoting the first layer.

Eq.(1) is susceptible to an exact solution, provided that both edges ($x$=0 and $x$=$l$) are subjected to the following form of simple supports:

$$\bar{\sigma}_x=\bar{\tau}_{xy}=\bar{w}=0\ . \tag{3}$$

This set of boundary conditions can be satisfied by expanding all state variables into the trigonometric series about coordinate $x$ (Chen and Lee, 2004b). However, if one of the edges is constrained by non-simple support, say clamped or free boundary conditions, it is very difficult to seek an exact even analytical solution to Eq.(1). In this circumstance, DQM is adopted to transfer the partial differential state equation into ordinary differential equation about $z$, which is similar to that done to laminated beams (Chen *et al.*, 2003b; 2004a). Applying the DQ rule (Shu and Richards, 1992) to the partial derivatives about $\xi$ in Eq.(1) leads to the following state equation at an arbitrary discrete point $\xi_i$ ($i$=1, 2, …, $N$):

$$\begin{cases}\dfrac{\mathrm{d}\bar{\sigma}_{z,i}}{\mathrm{d}\zeta}=-s\sum_{k=1}^{N}g_{ik}^{(1)}\bar{\tau}_{xz,k},\\ \dfrac{\mathrm{d}\bar{u}_i}{\mathrm{d}\zeta}=-s\sum_{k=1}^{N}g_{ik}^{(1)}\bar{w}_k+c_7\bar{\tau}_{xz,i}+c_8\bar{\tau}_{yz,i},\\ \dfrac{\mathrm{d}\bar{v}_i}{\mathrm{d}\zeta}=c_8\bar{\tau}_{xz,i}+c_9\bar{\tau}_{yz,i},\\ \dfrac{\mathrm{d}\bar{w}_i}{\mathrm{d}\zeta}=c_{10}\bar{\sigma}_{z,i}+c_4s\sum_{k=1}^{N}g_{ik}^{(1)}\bar{u}_k+b_3s\sum_{k=1}^{N}g_{ik}^{(1)}\bar{v}_k,\\ \dfrac{\mathrm{d}\bar{\tau}_{xz,i}}{\mathrm{d}\zeta}=c_4s\sum_{k=1}^{N}g_{ik}^{(1)}\bar{\sigma}_{z,k}-c_1s^2\sum_{k=1}^{N}g_{ik}^{(2)}\bar{u}_k-b_1s^2\sum_{k=1}^{N}g_{ik}^{(2)}\bar{v}_k,\\ \dfrac{\mathrm{d}\bar{\tau}_{yz,i}}{\mathrm{d}\zeta}=b_3s\sum_{k=1}^{N}g_{ik}^{(1)}\bar{\sigma}_{z,k}-b_1s^2\sum_{k=1}^{N}g_{ik}^{(2)}\bar{u}_k-c_6s^2\sum_{k=1}^{N}g_{ik}^{(2)}\bar{v}_k,\end{cases} \tag{4}$$

where $N$ is the discrete point number, and $g_{ik}^{(n)}$ are the weighting coefficients for the $n$th order derivative (Shu and Richards, 1992). The induced variables at the discrete point $\xi_i$, in a similar manner, can be obtained similarly as

$$\begin{cases}\bar{\sigma}_{x,i}=-c_4\bar{\sigma}_{z,i}+c_1s\sum_{k=1}^{N}g_{ik}^{(1)}\bar{u}_k+b_1s\sum_{k=1}^{N}g_{ik}^{(1)}\bar{v}_k,\\ \bar{\sigma}_{y,i}=-c_5\bar{\sigma}_{z,i}+c_3s\sum_{k=1}^{N}g_{ik}^{(1)}\bar{u}_k+b_2s\sum_{k=1}^{N}g_{ik}^{(1)}\bar{v}_k,\\ \bar{\tau}_{xy,i}=-b_3\bar{\sigma}_{z,i}+b_1s\sum_{k=1}^{N}g_{ik}^{(1)}\bar{u}_k+c_6s\sum_{k=1}^{N}g_{ik}^{(1)}\bar{v}_k.\end{cases} \tag{5}$$

## SOLUTIONS FOR STATIC BENDING PROBLEM

In order to obtain a unique solution for a practical problem, the edge constrained conditions should be incorporated into the state equation, Eq.(4). Assemblage of the state equations at all discrete points leads to the global state equation for the $k$th layer as follows:

$$\frac{\mathrm{d}}{\mathrm{d}z}\boldsymbol{\delta}^{(k)}(\zeta)=\boldsymbol{A}_k\boldsymbol{\delta}^{(k)}(\zeta)\ , \tag{6}$$

in which $\boldsymbol{\delta}^{\mathrm{T}}=\left[\bar{\boldsymbol{\sigma}}_z^{\mathrm{T}}\quad \bar{\boldsymbol{u}}^{\mathrm{T}}\quad \bar{\boldsymbol{v}}^{\mathrm{T}}\quad \bar{\boldsymbol{w}}^{\mathrm{T}}\quad \bar{\boldsymbol{\tau}}_{xz}^{\mathrm{T}}\quad \bar{\boldsymbol{\tau}}_{yz}^{\mathrm{T}}\right]$ is the global state vector composed of the unknown state variables about coordinate $z$, and coefficient matrix $\boldsymbol{A}_k$ is obtained from Eq.(4) by considering all the constrained conditions.

According to the matrix theory, the general solution of Eq.(6) is

$$\boldsymbol{\delta}^{(k)}(\zeta)=\boldsymbol{T}_k\boldsymbol{\delta}^{(k)}(\zeta_{k-1}),\quad \zeta_{k-1}\le\zeta\le\zeta_k\ , \tag{7}$$

for $k$=1, 2, …, $p$, where $\boldsymbol{T}_k=\exp[(\zeta-\zeta_{k-1})\boldsymbol{A}_k]$ is the transfer matrix, by which the state vector at lower surface of the $k$th layer $\boldsymbol{\delta}^{(k)}(\zeta_{k-1})$ is transferred to that at an arbitrary coordinate $z$ in the $k$th layer. In this sense, the state vector at upper surface of the layer can be obtained using Eq.(7). Considering the continuity conditions at the interfaces of any adjacent layers, one gets

$$\boldsymbol{\delta}^{(p)}(1)=\boldsymbol{T}\boldsymbol{\delta}^{(1)}(0)\ , \tag{8}$$

which is known as the global transfer relation, and $\boldsymbol{T}=\boldsymbol{T}_p\boldsymbol{T}_{p-1}\ldots\boldsymbol{T}_2\boldsymbol{T}_1$ is the global transfer matrix. Considering the conditions on the lateral surfaces of the plate in Eq.(8), the following governing equation is extracted:

$$\begin{Bmatrix} \bar{\boldsymbol{\sigma}}_z \\ \bar{\boldsymbol{\tau}}_{xz} \\ \bar{\boldsymbol{\tau}}_{yz} \end{Bmatrix}_1^{(p)} = \begin{bmatrix} \boldsymbol{t}_{12} & \boldsymbol{t}_{13} & \boldsymbol{t}_{14} \\ \boldsymbol{t}_{52} & \boldsymbol{t}_{53} & \boldsymbol{t}_{54} \\ \boldsymbol{t}_{62} & \boldsymbol{t}_{63} & \boldsymbol{t}_{64} \end{bmatrix} \begin{Bmatrix} \bar{\boldsymbol{u}} \\ \bar{\boldsymbol{v}} \\ \bar{\boldsymbol{w}} \end{Bmatrix}_0^{(1)} + \begin{bmatrix} \boldsymbol{t}_{11} & \boldsymbol{t}_{15} & \boldsymbol{t}_{16} \\ \boldsymbol{t}_{51} & \boldsymbol{t}_{55} & \boldsymbol{t}_{56} \\ \boldsymbol{t}_{61} & \boldsymbol{t}_{65} & \boldsymbol{t}_{66} \end{bmatrix} \begin{Bmatrix} \bar{\boldsymbol{\sigma}}_z \\ \bar{\boldsymbol{\tau}}_{xz} \\ \bar{\boldsymbol{\tau}}_{yz} \end{Bmatrix}_0^{(1)}, \tag{9}$$

where $\boldsymbol{t}_{ij}$ ($i, j$=1, 2, …, 6) denote the blocked matrices of matrix $\boldsymbol{T}$, and subscripts '1' and '0' denote the top surface of the $p$th layer and bottom surface of the first layer, respectively. The displacement vector at the bottom surface of the plate is solved from Eq.(9) as

$$\begin{Bmatrix} \bar{\boldsymbol{u}} \\ \bar{\boldsymbol{v}} \\ \bar{\boldsymbol{w}} \end{Bmatrix}_0^{(1)} = \begin{bmatrix} \boldsymbol{t}_{12} & \boldsymbol{t}_{13} & \boldsymbol{t}_{14} \\ \boldsymbol{t}_{52} & \boldsymbol{t}_{53} & \boldsymbol{t}_{54} \\ \boldsymbol{t}_{62} & \boldsymbol{t}_{63} & \boldsymbol{t}_{64} \end{bmatrix}^{-1} \cdot \left( \begin{Bmatrix} \bar{\boldsymbol{\sigma}}_z \\ \bar{\boldsymbol{\tau}}_{xz} \\ \bar{\boldsymbol{\tau}}_{yz} \end{Bmatrix}_1^{(p)} - \begin{bmatrix} \boldsymbol{t}_{11} & \boldsymbol{t}_{15} & \boldsymbol{t}_{16} \\ \boldsymbol{t}_{51} & \boldsymbol{t}_{55} & \boldsymbol{t}_{56} \\ \boldsymbol{t}_{61} & \boldsymbol{t}_{65} & \boldsymbol{t}_{66} \end{bmatrix} \begin{Bmatrix} \bar{\boldsymbol{\sigma}}_z \\ \bar{\boldsymbol{\tau}}_{xz} \\ \bar{\boldsymbol{\tau}}_{yz} \end{Bmatrix}_0^{(1)} \right). \tag{10}$$

Combining Eq.(10) with the traction conditions at the bottom surface, and after repeated use of the general solution Eq.(7), the state vector at an arbitrary coordinate $z$ is easily obtainable.

## NUMERICAL EXAMPLES

Numerical examples are implemented to validate the convergence and accuracy of the present method for 3D analysis on cylindrical bending of anisotropic thick laminates ($s$=0.25) with the top surface, unless mentioned otherwise, applied by upward normal sinusoidal load $\sigma_z|_{z=h}=q_0\sin(\pi x)$ or uniform load $\sigma_z|_{z=h}=q_0$. The plates in consideration are assumed subjected to fully simply supported (SS) or clamped (CC) edge constraints. The corresponding boundary conditions, in the DQ framework, are expressed as:

$$\begin{aligned} &\text{SS: } \bar{\sigma}_{x,1} = \bar{\sigma}_{x,N} = 0,\ \bar{\tau}_{xy,1} = \bar{\tau}_{xy,N} = 0,\ \bar{w}_1 = \bar{w}_N = 0; \\ &\text{CC: } \bar{u}_1 = \bar{u}_N = 0,\ \bar{v}_1 = \bar{v}_N = 0,\ \bar{w}_1 = \bar{w}_N = 0. \end{aligned} \tag{11}$$

The laminated plate is obtained by stacking up $p$ layers of orthotropic composite fibers having elastic properties of $E_1$=25$E_2$, $E_3$=$E_2$, $G_{12}$=$G_{13}$=0.5$E_2$, $G_{23}$=0.2$E_2$, $\nu_{12}$=$\nu_{13}$=$\nu_{23}$=0.25.

For comparison and validation, numerical results given in what follows are transferred into the following normalized parameters:

$$\begin{cases} W = 100\bar{w}E_2 s^4 / q_0, \\ (\sigma_\xi, \sigma_\eta, \tau_{\xi\eta}) = (\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}_{xy})Q_{66}^{(1)} s^2 / q_0, \\ (\tau_{\xi\zeta}, \tau_{\eta\zeta}) = (\bar{\tau}_{xz}, \bar{\tau}_{yz})Q_{66}^{(1)} s / q_0, \\ \sigma_\zeta = \bar{\sigma}_z Q_{66}^{(1)} / q_0. \end{cases} \tag{12}$$

Note that uniformly distributed sampling points are adopted for the implementation of DQ procedure.

Firstly, stress and deflection parameters of a two layered SS plate under sinusoidal loading are calculated using different values of $N$. Numerical results for plates with stacking sequences of [−30°/30°] and [−45°/45°] are presented in Table 1. It is seen that the results converge rapidly to the exact solutions (Chen and Lee, 2004b) as $N$ increases.

Next, numerical values of stress and deflection parameters of a single-layered CC plate are listed in Table 2, as well as the normalized elongation $\Delta W = 10E_2[\bar{w}(l/2,h) - \bar{w}(l/2,0)]/q_0$ of the normal to the mid-surface of the plate. Results for CC plates are again observed to converge fast. For the orthotropic plates [0°], the present results are also validated by the analytical results given by Vel and Batra (2000) by solving the generalized plane strain deformations based on Eshelby-Stroh formalism. It is noted from Table 2 that if the orienting angle of fibers changes from 0° to 45°, the deflection parameter is nearly doubled and the longitudinal normal stress parameter $\sigma_\xi$ is reduced by over 25%, but the transverse normal stress $\sigma_\zeta$ and the elongation $\Delta W$ remain almost unchanged.

Finally, effects of ply angle $\theta$ on the stress field of a CC plate with symmetric lamina scheme of [$\theta$/90°/$\theta$] are investigated. The plate is assumed subjected to uniform pressure on the top surface, and the discrete point number is taken as $N$=9. Figs.1a and 1b exhibit the along-thickness distributions of axial normal and shear stress parameters $\sigma_\xi(l/2,z)$ and $\tau_{\xi\zeta}(l/4,z)$, respectively, for different values of $\theta$. It is

**Table 1 Stresses and deflection parameters of two-layered SS plate subjected to sinusoidal loading**

| Stacking sequences | $N$ | $W(l/2,h/2)$ | $\sigma_\xi(l/2,0)$ | $\sigma_\xi(l/2,h)$ | $\tau_{\xi\eta}(l/2,0)$ | $\tau_{\xi\zeta}(0,h/2)$ | $\tau_{\eta\zeta}(0,h/2)$ | $\sigma_\zeta(l/2,3h/4)$ |
|---|---|---|---|---|---|---|---|---|
| [−30°/30°] | 5 | 4.23649 | −1.15004 | 1.18288 | 0.52633 | 0.20134 | −0.0127032 | 0.78600 |
| | 7 | 3.90327 | −1.03864 | 1.07325 | 0.47556 | 0.21786 | −0.0054575 | 0.78631 |
| | 9 | 3.92018 | −1.04457 | 1.07909 | 0.47826 | 0.21736 | −0.0056568 | 0.78633 |
| | 11 | 3.91973 | −1.04441 | 1.07894 | 0.47819 | 0.21737 | −0.0056536 | 0.78633 |
| | Exact | 3.91974 | −1.04442 | 1.07894 | 0.478195 | 0.217369 | −0.00565371 | 0.786324 |
| | 2D | 3.95800 | −1.06610 | 1.06610 | − | 0.2143 | 0.0000 | − |
| [−45°/45°] | 5 | 6.88752 | −1.06126 | 1.07899 | 0.73730 | 0.23697 | −0.0182137 | 0.79474 |
| | 7 | 6.25542 | −0.95464 | 0.97431 | 0.66294 | 0.25169 | −0.0082586 | 0.79543 |
| | 9 | 6.28735 | −0.96029 | 0.97987 | 0.66689 | 0.25128 | −0.0085177 | 0.79545 |
| | 11 | 6.28652 | −0.96015 | 0.97973 | 0.66678 | 0.25129 | −0.0085139 | 0.79545 |
| | Exact | 6.28654 | −0.960194 | 0.979728 | 0.666786 | 0.251289 | −0.00851361 | 0.795447 |
| | 2D | 6.34600 | −0.97580 | 0.97580 | − | 0.24840 | 0.0000 | − |

Note: '2D' denotes results obtained based on global 2D plate theory with 5 DOF (Shu and Soldatos, 2000), and '−' denotes the result not available

**Table 2 Stress and deflection parameters of single-layered CC plate subjected to sinusoidal loading**

| Orienting angle | $N$ | $W(l/2,h/2)$ | $\sigma_\xi(l/2,h)$ | $\sigma_\zeta(l/2,h/2)$ | $\Delta W$ |
|---|---|---|---|---|---|
| [0°] | 5 | −1.4753 | −0.5109 | −0.4903 | −4.6263 |
| | 7 | −1.4875 | −0.4840 | −0.4901 | −4.6234 |
| | 9 | −1.4914 | −0.4893 | −0.4901 | −4.6239 |
| | 11 | −1.4928 | −0.4884 | −0.4901 | −4.6238 |
| | 13 | −1.4935 | −0.4886 | −0.4901 | −4.6238 |
| | Anal. | −1.4946 | −0.4887 | −0.4900 | −4.6238 |
| [45°] | 5 | −2.5431 | −0.3745 | −0.4961 | −4.7320 |
| | 7 | −2.5586 | −0.3570 | −0.4959 | −4.7272 |
| | 9 | −2.5693 | −0.3605 | −0.4959 | −4.7278 |
| | 11 | −2.5734 | −0.3600 | −0.4959 | −4.7277 |
| | 13 | −2.5755 | −0.3601 | −0.4959 | −4.7277 |
| | Anal. | − | − | − | − |

Note: 'Anal.' denotes the analytical results from (Vel and Batra, 2000), and '−' denotes the result not available

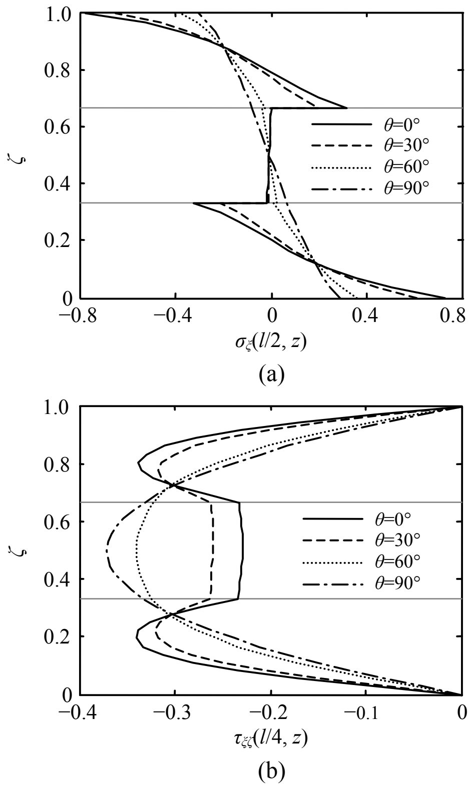


**Fig.1 Along-thickness distributions of axial normal stress parameter $\sigma_\xi(l/2, z)$ (a) and stress parameter $\tau_{\xi\zeta}(l/4, z)$ (b) of a three-layered ($\theta$/90°/$\theta$) CC plate subjected to uniform pressure on the top surface ($N$=9)**

established from Fig.1a that the discontinuity of the interfacial normal stress reduces monotonically as the ply angle of the face lamina increases from 0° to 90°, and that the stress suffered by the upper surface fiber of the central layer shifted from tensile to pressure. Fig.1b indicates that variation of ply angle of the surface layer poses significant effects on the distribution of the shear stress. For example, when $\theta$ increases from 30° to 60°, the maximal shear stress of the central layer increases by more than 25%; this value exceeds 50% when $\theta$ shifts from 0° to 90°.

## CONCLUSION

Bending analysis of angle-ply laminates in cylindrical bending is conducted via the newly developed SSDQM. No displacement and stress approximations are adopted along plate thickness direction, enabling the current method suitable for laminates with arbitrary thickness. On the other hand, introduction of the DQ technique makes it feasible to model the supporting edges directly using stress and displacement components, and hence the Saint-Venant principle becomes unnecessary. Numerical comparisons indicate that the present semi-analytical results are reliable, especially for fully clamped thick laminates, and can serve as benchmark solutions for future numerical analyses.

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