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Weak and strong convergence of an explicit iteration scheme with perturbed mapping for nonexpansive mappings

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Abstract: In this paper, we consider an explicit iteration scheme with perturbed mapping for nonexpansive mappings in real q-uniformly smooth Banach spaces. Some weak and strong convergence theorems for this explicit iteration scheme are established. In particular, necessary and sufficient conditions for strong convergence of this explicit iteration scheme are obtained. At last, some useful corollaries for strong convergence of this explicit iteration scheme are given.

Key words: Nonexpansive mapping, Iteration scheme with perturbed mapping, Opial condition, Completely continuous
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INTRODUCTION AND PRELIMINARIES

Let E be a real Banach space with dual space E^*, $\langle\cdot,\cdot\rangle$ be the duality pairing between elements of E and those of E^*, and 2^{E^*} denote the family of all the nonempty subsets of E^*. The generalized duality mapping J_q:$E\to 2^{E^*}$ is defined by $J_q(x)=\{f\in E^*:\langle x,f\rangle=\|x\|^q, \|f\|=\|x\|^{q-1}\},\forall x\in E$, where $q>1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is well known that, in general, $J_q(x)=\|x\|^{q-2}J_2(x)$ $\forall x\neq 0$. A mapping T with domain $D(T)$ and range $R(T)$ in E is called κ-Lipschitz continuous if there exists a constant $\kappa>0$ such that $\|Tx-Ty\|\leq\kappa\|x-y\|$, for all $x,y\in D(T)$, and if $\kappa=1$, then T is called nonexpansive. The mapping F:$D(F)\subset E\to E$ is called η-strongly accretive if for all $x,y\in D(F)$ there exists a constant $\eta>0$ and $j_q(x-y)\in J_q(x-y)$ such that $\langle F(x)-F(y), j_q(x-y)\rangle\geq\eta\|x-y\|^q$. Note that if $E\equiv H$, a real Hilbert space, then J_2 becomes the identity mapping on H, and on this condition strongly accretive mappings reduce to strongly monotone mappings.

Recently, Wang (2007) studied an explicit iterative scheme with perturbed mapping F in Hilbert spaces and proved the following theorem:

Let H be a real Hilbert space, T:$H\to H$ a nonexpansive mapping with $F(T)=\{x\in H:Tx=x\}\neq\varnothing$, and F:$H\to H$ an η-strongly monotone and κ-Lipschitzian mapping. For any given $x_0\in H$, $\{x_n\}$ is defined by

$$x_{n+1}=\alpha_n x_n+(1-\alpha_n)T^{\lambda_{n+1}}x_n,\ n\geq 0, \tag{1}$$

where $T^{\lambda_{n+1}}x_n=Tx_n-\lambda_{n+1}\mu F(Tx_n)$, $\{\alpha_n\}$ and $\{\lambda_n\}\subset[0,1)$ satisfy the following conditions: (i) $\alpha<\alpha_n<\beta$ for some $\alpha,\beta\in(0,1)$; (ii) $\sum_{n=1}^{\infty}\lambda_n<\infty$; (iii) $0<\mu<2\eta/\kappa^2$. Then

(1) $\{x_n\}$ converges weakly to a fixed point of T;

(2) $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty}d(x_n,F(T))=0$.

Motivated and inspired by the above research work of (Wang, 2007), in this paper, we will consider the explicit iteration scheme with perturbed mapping for nonexpansive mappings $\{x_n\}$ defined by Eq.(1) in a real q-uniformly smooth Banach space E. We will establish some weak and strong convergence theorems for this explicit iteration scheme.

Recall that E is said to satisfy Opial condition, if for each sequence $\{x_n\}$ in E, the condition that the sequence $x_n \to x$ weakly implies that $\limsup_{n\to\infty} \| x_n - x \| < \limsup_{n\to\infty} \| x_n - y \|$ $\forall y \in E$ with $y \neq x$.

A Banach space E is called smooth if for every $x \in E$ with $\|x\|=1$, there exists a unique $f \in E^*$ such that $\|f\|=f(x)=1$. The modulus of smoothness of E is the function $\rho_E:[0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) = \sup\left\{ \frac{\| x+y \| + \| x-y \|}{2} - 1 : x, y \in E,\ \| x \| = 1,\ \| y \| = \tau \right\}.$$

Definition 1 The Banach space E is said to be (i) uniformly smooth, if $\lim_{\tau\to 0} \rho_E(\tau)/\tau = 0$; (ii) q-uniformly smooth, for $q>1$, if there exists a constant $c>0$ such that $\rho_E(\tau) \le c\tau^q$, $\tau \in [0,\infty)$.

Remark 1 J_q is single-valued if E is uniformly smooth and all Hilbert spaces and L_q (or l_q) ($2 \le q < \infty$) spaces are 2-uniformly smooth, while for $1<q<2$, L_q (or l_q) spaces are q-uniformly smooth. And a uniformly smooth Banach space is reflexive and smooth.

Definition 2 Let D be a closed subset of E and $T:D\to D$ be a mapping: (1) T is said to be demi-closed at a point $x \in D$, if for each sequence $\{x_n\}$ in D, the conditions $x_n \to x_n \in D$ weakly and $Tx_n \to y$ imply $Tx=y$; (2) T is said to be semi-compact, if for any bounded sequence $\{x_n\} \subset D$ such that $\| x_n - Tx_n \| \to 0$ $(n \to \infty)$, then there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \to x^* \in D$.

In order to prove our main results, we shall make use of the following lemmas in the sequel.

Lemma 1 (Xu, 1991) Let $q>1$ be a real number and let E be a real uniformly smooth Banach space. Then E is q-uniformly smooth if and only if there exists a constant $c_q>0$ such that $\forall x,y \in E$, $\| x+y \|^q \le \| x \|^q + q\langle y, J_q(x) \rangle + c_q \| y \|^q$.

Lemma 2 (Jung, 2005) Let E be a reflexive Banach space which satisfies Opial condition, let C be a nonempty closed convex subset of E and suppose $T:C\to E$ is nonexpansive. Then the mapping $I-T$ is demi-closed on C, where I is the identity mapping.

Lemma 3 (Osilike *et al*., 2002) Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality $a_{n+1} \le (1+\delta_n)a_n + b_n$, $n \ge 1$. If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If in addition $\{a_n\}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 4 (Tan and Xu, 1993) Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers satisfying the inequality $a_{n+1} \le a_n + b_n$, $\forall n \ge 0$. If $\sum_{n=0}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.

Lemma 5 (Suzuki, 2005) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ $\forall n \in \mathbb{N}$ and $\limsup_{n\to\infty} (\| y_{n+1} - y_n \| - \| x_{n+1} - x_n \|) \le 0$. Then $\lim_{n\to\infty} \| y_n - x_n \| = 0$.

Lemma 6 Let E be a real q-uniformly smooth Banach space, $T:E\to E$ be nonexpansive, and $F:E\to E$ be κ-Lipschitz continuous and η-strongly accretive. If $\lambda \in [0,1)$ and $\mu^{q-1} \in (0, q\eta/(c_q\kappa^q))$, a mapping $T^\lambda:E\to E$ is defined by $T^\lambda x := Tx - \lambda\mu F(Tx)$, $\forall x \in E$. Then

$$\| T^\lambda x - T^\lambda y \| \le \sqrt[q]{1 - q\lambda\mu\eta + c_q\lambda^q\mu^q\kappa^q}\, \| x-y \| \le \| x-y \|.$$

Proof By Lemma 1, we obtain

$$\begin{aligned}
\| T^\lambda x - T^\lambda y \|^q &= \| Tx - Ty - \lambda\mu(F(Tx) - F(Ty)) \|^q \\
&\le \| Tx - Ty \|^q - q\lambda\mu \left\langle F(Tx) - F(Ty), J_q(Tx - Ty) \right\rangle \\
&\quad + c_q\lambda^q\mu^q \| F(Tx) - F(Ty) \|^q \\
&\le (1 - q\lambda\mu\eta + c_q\lambda^q\mu^q\kappa^q) \| Tx - Ty \|^q \\
&\le (1 - q\lambda\mu\eta + c_q\lambda^q\mu^q\kappa^q) \| x - y \|^q .
\end{aligned}$$

Since $\mu^{q-1} \in (0, q\eta/(c_q\kappa^q))$, it is easy to see that $1 - q\lambda\mu\eta + c_q\lambda^q\mu^q\kappa^q \le 1$. The proof is completed.

MAIN RESULTS

Theorem 1 Let E be a real q-uniformly smooth Banach space satisfying Opial condition, $T:E\to E$ be a nonexpansive mapping with $F(T)=\{x \in E: Tx=x\} \neq \varnothing$, and $F:E\to E$ be κ-Lipschitz continuous and η-strongly accretive. Suppose that $\{\lambda_n\}_{n=1}^{\infty} \subset [0,1)$, $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$

satisfy the following conditions: (i) $\alpha\le\alpha_n\le\beta$ for some $\alpha,\beta\in(0,1)$; (ii) $\sum_{n=1}^{\infty}\lambda_n<\infty$; (iii) $\mu^{q-1}\in(0, q\eta/(c_q\kappa^q))$. Then for any given $x_0\in E$, the sequence $\{x_n\}$ defined by Eq.(1) converges weakly to a fixed point of T.

Proof Let p be an arbitrary element of $F(T)$, it follows from Eq.(1) and Lemma 6 that

$$\begin{aligned}\|x_{n+1}-p\|&\le\alpha_n\|x_n-p\|+(1-\alpha_n)\|T^{\lambda_{n+1}}x_n-T^{\lambda_{n+1}}p\|\\&\quad+(1-\alpha_n)\|T^{\lambda_{n+1}}p-p\|\\&\le\alpha_n\|x_n-p\|+(1-\alpha_n)\|x_n-p\|\\&\quad+(1-\alpha_n)\lambda_{n+1}\mu\|F(p)\|\\&\le\|x_n-p\|+\lambda_{n+1}\mu\|F(p)\|.\end{aligned}\tag{2}$$

Since $\sum_{n=1}^{\infty}\lambda_n<\infty$, from Lemma 4 we deduce that $\lim_{n\to\infty}\|x_n-p\|$ exists. Hence $\{x_n\}$ is bounded. We can also obtain that $\{T(x_n)\}$, $\{F(x_n)\}$ and $\{F(Tx_n)\}$ are all bounded. Put $M=\max\{\sup_{n\ge0}\|F(x_n)\|,\ \sup_{n\ge0}\|F(Tx_n)\|\}$. We note that

$$\begin{aligned}&\|T^{\lambda_{n+2}}x_{n+1}-T^{\lambda_{n+1}}x_n\|\\&\le\|T^{\lambda_{n+2}}x_{n+1}-T^{\lambda_{n+2}}x_n\|+\|T^{\lambda_{n+2}}x_n-T^{\lambda_{n+1}}x_n\|\\&\le\|x_{n+1}-x_n\|+\mu|\lambda_{n+1}-\lambda_{n+2}|\cdot\|F(Tx_n)\|.\end{aligned}$$

Therefore

$$\begin{aligned}&\limsup_{n\to\infty}(\|T^{\lambda_{n+2}}x_{n+1}-T^{\lambda_{n+1}}x_n\|-\|x_{n+1}-x_n\|)\\&\quad\le\limsup_{n\to\infty}(|\lambda_{n+1}-\lambda_{n+2}|\mu M)=0.\end{aligned}$$

From Lemma 5 and the condition (i), we have $\lim_{n\to\infty}\|x_n-T^{\lambda_{n+1}}x_n\|=0$. It follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\alpha_n)\|T^{\lambda_{n+1}}x_n-x_n\|=0.\tag{3}$$

Observe that

$$\begin{aligned}&\|x_n-Tx_n\|\le\|x_{n+1}-x_n\|+\|x_{n+1}-Tx_n\|\\&=\|x_{n+1}-x_n\|+\|\alpha_n(x_n-Tx_n)-(1-\alpha_n)\lambda_{n+1}\mu F(Tx_n)\|\\&\le\|x_{n+1}-x_n\|+\alpha_n\|x_n-Tx_n\|+(1-\alpha_n)\lambda_{n+1}\mu M.\end{aligned}$$

This together with the condition (i) yields

$$\|x_n-Tx_n\|\le\|x_{n+1}-x_n\|/(1-\beta)+\lambda_{n+1}\mu M\to0,\ (n\to\infty)$$

that is

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0.\tag{4}$$

From Remark 1 we know that E is reflexive and again since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ which converges weakly to some $x^*\in E$ and we have $\lim_{j\to\infty}\|x_{n_j}-Tx_{n_j}\|=0$. By Lemma 2, we have $(I-T)x^*=0$, i.e., x^* is a fixed point of T.

Next we show that $\{x_n\}$ converges weakly to x^*. Suppose the contrary, then there exists some subsequence $\{x_{n_k}\}\subset\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $\hat{x}$ and $\hat{x}\ne x^*$. Then by the same method as given above, we can also prove that $\hat{x}\in F(T)$. By virtue of Opial condition of E, we conclude that

$$\begin{aligned}&\lim_{n\to\infty}\|x_n-x^*\|=\limsup_{j\to\infty}\|x_{n_j}-x^*\|<\limsup_{j\to\infty}\|x_{n_j}-\hat{x}\|\\&=\lim_{n\to\infty}\|x_n-\hat{x}\|=\limsup_{k\to\infty}\|x_{n_k}-\hat{x}\|<\limsup_{k\to\infty}\|x_{n_k}-x^*\|\\&=\lim_{n\to\infty}\|x_n-x^*\|.\end{aligned}$$

This is a contradiction. Hence $x^*=\hat{x}$. This implies that $\{x_n\}$ converges weakly to x^*. The proof is completed.

Lemma 7 Let E be a real q-uniformly smooth Banach space, $T:E\to E$ be a nonexpansive mapping with $F(T)\ne\varnothing$, and $F:E\to E$ be κ-Lipschitz continuous and η-strongly accretive. Suppose that $\{\lambda_n\}_{n=1}^{\infty}\subset[0,1)$, $\{\alpha_n\}_{n=1}^{\infty}\subset(0,1)$ satisfy the following conditions: (i) $\alpha\le\alpha_n\le\beta$ for some $\alpha,\beta\in(0,1)$; (ii) $\sum_{n=1}^{\infty}\lambda_n<\infty$; (iii) $\mu^{q-1}\in(0, q\eta/(c_q\kappa^q))$. Let $x_0\in E$ and $\{x_n\}$ be defined by Eq.(1). Then

(1) $\lim_{n\to\infty}\|x_n-x^*\|$ exists for each $x^*\in F(T)$;

(2) $\lim_{n\to\infty}d(x_n,F(T))$ exists, where $d(x_n,F(T))=\inf_{p\in F(T)}\|x_n-p\|$;

(3) $\liminf_{n\to\infty}\|x_n-Tx_n\|=0$.

Proof From Eqs.(2) and (4), the conclusions (1) and (3) hold.

Next we shall prove the conclusion (2). Indeed, for each $p\in F(T)$,

$$\|F(p)\|\le\|F(p)-F(x_n)\|+\|F(x_n)\|\le\kappa\|x_n-p\|+M,$$

where M is as appeared in Theorem 1. This together with Eq.(2) yields

$$\begin{aligned}\|x_{n+1}-p\| &\le \|x_n-p\|+\lambda_{n+1}\mu\|F(p)\| \\ &\le \|x_n-p\|+\lambda_{n+1}\mu\kappa\|x_n-p\|+\lambda_{n+1}\mu M \\ &=(1+\delta_n)\|x_n-p\|+b_n, \qquad (5)\end{aligned}$$

and so

$$d(x_{n+1}, F(T))\le(1+\delta_n)d(x_n, F(T))+b_n, \qquad (6)$$

where $\delta_n=\lambda_{n+1}\mu\kappa$, $b_n=\lambda_{n+1}\mu M$, $n\ge0$. By virtue of $\sum_{n=1}^{\infty}\lambda_n<\infty$, we have $\sum_{n=0}^{\infty}\delta_n<\infty$, $\sum_{n=0}^{\infty}b_n<\infty$. By Eq.(6) and Lemma 3, we deduce $\lim_{n\to\infty}d(x_n,F(T))$ exists. This completes the proof.

Theorem 2 Let E be a real q-uniformly smooth Banach space, $T:E\to E$ be a nonexpansive mapping with $F(T)\ne\varnothing$, and $F:E\to E$ be κ-Lipschitz continuous and η-strongly accretive. Suppose that $\{\lambda_n\}_{n=1}^{\infty}\subset[0,1)$, $\{\alpha_n\}_{n=1}^{\infty}\subset(0,1)$ satisfy the following conditions: (i) $\alpha\le\alpha_n\le\beta$ for some $\alpha,\beta\in(0,1)$; (ii) $\sum_{n=1}^{\infty}\lambda_n<\infty$; (iii) $\mu^{q-1}\in(0, q\eta/(c_q\kappa^q))$. Then for any given $x_0\in E$, the sequence $\{x_n\}$ defined by Eq.(1) converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty}d(x_n,F(T))=0$.

Proof For $\{\delta_n\}$ in Eq.(5), since $\sum_{n=0}^{\infty}\delta_n<\infty$, let $L=\prod_{n=0}^{\infty}(1+\delta_n)$, then $1\le L<\infty$.

Suppose that $\{x_n\}$ converges strongly to a fixed point p of T, then $\lim_{n\to\infty}\|x_n-p\|=0$. Since $0\le d(x_n,F(T))\le\|x_n-p\|\to0$ $(n\to\infty)$, we have $\liminf_{n\to\infty}d(x_n,F(T))=0$.

Conversely suppose $\liminf_{n\to\infty}d(x_n,F(T))=0$, then the conclusion (2) of Lemma 7 implies that $\lim_{n\to\infty}d(x_n,F(T))=0$. Thus for arbitrary $\varepsilon>0$, there exists a positive integer N_0 such that $d(x_n,F(T))<\varepsilon/(4L)$, $\forall n\ge N_0$. And for $\{b_n\}$ in Eq.(5), the condition $\sum_{n=0}^{\infty}b_n<\infty$ implies that there exists a positive integer N_1 such that $\sum_{j=n}^{\infty}b_j<\varepsilon/(4L)$, $\forall n\ge N_1$. Let $N_*=\max\{N_0, N_1\}$. From Eq.(5), $\forall n$, $m>N_*$ and for any $p\in F(T)$, we derive

$$\begin{aligned}&\|x_n-x_m\|\le\|x_n-p\|+\|x_m-p\| \\ &\le(1+\delta_{n-1})\|x_{n-1}-p\|+b_{n-1}+(1+\delta_{m-1})\|x_{m-1}-p\|+b_{m-1} \\ &\le\prod_{i=N_*}^{n-1}(1+\delta_i)\|x_{N_*}-p\|+b_{n-1}+\sum_{j=N_*}^{n-2}b_j\prod_{i=j+1}^{n-1}(1+\delta_i) \\ &\quad+\prod_{i=N_*}^{m-1}(1+\delta_i)\|x_{N_*}-p\|+b_{m-1}+\sum_{j=N_*}^{m-2}b_j\prod_{i=j+1}^{m-1}(1+\delta_i) \\ &\le2L\|x_{N_*}-p\|+L\sum_{j=N_*}^{n-1}b_j+L\sum_{j=N_*}^{m-1}b_j \\ &\le2L\|x_{N_*}-p\|+2L\sum_{j=N_*}^{\infty}b_j<2L\|x_{N_*}-p\|+\varepsilon/2.\end{aligned}$$

Taking the infimum over all $p\in F(T)$, we have $\|x_n-x_m\|\le2Ld(x_{N_*},F(T))+\varepsilon/2<\varepsilon$. This implies that $\{x_n\}$ is a Cauchy sequence. Let $x_n\to\tilde{x}\in E$. Then we derive from Eq.(4) that $\tilde{x}\in F(T)$. The proof is completed.

Similar to the proof of Corollaries 3.3, 3.4 and Theorem 3.5 in (Wang, 2007), we can obtain the following conclusions:

Corollary 1 Under the conditions of Lemma 7, if T is completely continuous, then $\{x_n\}$ converges strongly to a fixed point of T.

Corollary 2 Under the conditions of Lemma 7, if T is semi-compact, then $\{x_n\}$ converges strongly to a fixed point of T.

Senter and Dotson (1974) introduced Condition (A). Later on, Maiti and Ghosh (1989) studied Condition (A) and pointed out that Condition (A) is weaker than the requirement of semi-compactness for nonexpansive mappings. A mapping $T:K\to K$ with $F(T)=\{x\in K:Tx=x\}\ne\varnothing$ is said to satisfy Condition (A) if there exists a nondecreasing function $f:[0,\infty)\to[0,\infty)$ with $f(0)=0$ and $f(t)>0$ $\forall t\in(0,\infty)$ such that $\|x-Tx\|\ge f(d(x,F(T)))$ $\forall x\in K$.

Theorem 3 Under the conditions of Lemma 7, if T satisfies Condition (A), then $\{x_n\}$ converges strongly to a fixed point of T.

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