Journal of Zhejiang University SCIENCE A
ISSN 1673-565X (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

A note on the Marcinkiewicz integral operators on $F_p^{\alpha,q}$*

ZHANG Chun-jie†1, QIAN Rui-rui[2]
([1]Department of Mathematics, Zhejiang University, Hangzhou 310027, China)
([2]Sci-Tech Section, Zhejiang Sci-Tech University, Hangzhou 210018, China)
†E-mail: purezhang@zju.edu.cn

Received Mar. 2, 2007; revision accepted Apr. 11, 2007

Abstract: In this paper, we shall prove that the Marcinkiewicz integral operator μ_Ω, when its kernel Ω satisfies the L^1-Dini condition, is bounded on the Triebel-Lizorkin spaces. It is well known that the Triebel-Lizorkin spaces are generalizations of many familiar spaces such as the Lebesgue spaces and the Sobolev spaces. Therefore, our result extends many known theorems on the Marcinkiewicz integral operator. Our method is to regard the Marcinkiewicz integral operator as a vector valued singular integral. We also use another characterization of the Triebel-Lizorkin space which makes our approach more clear.

Key words: Marcinkiewicz integral, Triebel-Lizorkin spaces, Fourier transforms
doi:10.1631/jzus.2007.A2037 **Document code:** A **CLC number:** O174.4

INTRODUCTION

The Marcinkiewicz integral operator is defined by

$$\mu_\Omega(f)(x)=\left(\int_0^\infty |F_t(x)|^2 t^{-3}\mathrm{d}t\right)^{1/2}, \tag{1}$$

where

$$F_t(x)=\int_{|y|<t}\frac{\Omega(y)}{|y|^{n-1}}f(x-y)\mathrm{d}y,\quad \Omega\in L^1(S^{n-1}).$$

The boundedness of μ_Ω on L^p spaces has been widely investigated. Assuming $\Omega\in\mathrm{Lip}(S^{n-1})$, Stein (1958) proved the boundedness of μ_Ω on L^p, $1<p\le 2$. Using the BCP method, Benedek *et al.*(1962) extended this result to $1<p<+\infty$ when $\Omega\in C^1(S^{n-1})$. Recently this result was further extended to the case that $\Omega\in H^1(S^{n-1})$ (Ding *et al.*, 2000; Xu *et al.*, 2003). In this paper, we shall prove the boundedness of μ_Ω on the Triebel-Lizorkin spaces $F_p^{\alpha,q}$ with $0<\alpha<1$ and $1<p$, $q<\infty$ under the condition that Ω is of L^1-Dini type. Note that μ_Ω is nonlinear. For linear operators such as the singular integrals, the boundedness on the Triebel-Lizorkin spaces has been thoroughly investigated and the results can be found in (Chen *et al.*, 2002; 2003; 2005; Chen and Zhang, 2004).

Let us first recall the definition of the Triebel-Lizorkin space $F_p^{\alpha,q}$. Let $\Phi\in C_c^\infty(\mathbb{R}^n)$ such that $\mathrm{supp}(\Phi)\subset\{\xi:1/2<|\xi|<2\}$ and $\Phi(\xi)>1$ when $3/5<|\xi|<5/3$. Setting $\hat{\Psi}_k(\xi)=\Phi(2^k\xi)$, $k\in\mathbb{Z}$, we say f is in the non-homogenous Triebel-Lizorkin space $F_p^{\alpha,q}$, $\alpha\ge 0$ and $1<p$, $q<\infty$ if

$$\|f\|_{F_p^{\alpha,q}}=\|f\|_{L^p}+\left\|\left(\sum_k|2^{-k\alpha}\Psi_k*f|^q\right)^{1/q}\right\|_{L^p}<+\infty. \tag{2}$$

Noting that $F_p^{0,2}=L^p$, we can say that our theorem extends some of the results mentioned above.

A function defined on S^{n-1} is said to be L^1-Dini if

$$\int_0^1\frac{\omega_1(t)}{t}\mathrm{d}t<+\infty, \tag{3}$$

where

$$\omega_1(t)=\sup_{\|\rho\|\le t}\int_{S^{n-1}}|\Omega(\rho x')-\Omega(x')|\mathrm{d}\sigma(x')$$

* Project (No. 10601046) supported by the National Natural Science Foundation of China

and ρ is a rotation on S^{n-1} with

$$\|\rho\| = \sup\left\{|\mu - \rho\mu| : \mu \in S^{n-1}\right\}.$$

It is well known that if $\Omega(x')$ satisfies the L^1-Dini condition, then it always belongs to $L\ln^+L$ (Calderón *et al.*, 1967). Thus we know that μ_Ω is bounded on L^p. To obtain the further boundedness on $F_p^{\alpha,q}$, we shall adopt the method developed by Korry (2004) who mainly dealt with the Sobolev spaces H_p^α and showed that, under a vector valued form of Hörmander's condition, certain nonlinear operators are bounded on H_p^α. In Section 2 of this paper, we shall give some extra explanations to this method so that it works for $F_p^{\alpha,q}$. Then in Section 3 we turn to the Marcinkiewicz integral operator μ_Ω and show that its kernel verifies the vector valued Hörmander's condition. Thus we draw the conclusion that μ_Ω is bounded on $F_p^{\alpha,q}$.

A CLASS OF BOUNDED OPERATORS ON $F_p^{\alpha,q}$

We first recall a fundamental result of BCP. Let F be a Banach space and $K(x)$ be a strongly measurable, locally integrable function from $\mathbb{R}^n\backslash\{0\}$ to F. An operator U is called a BCP operator if

(1) There is a $p_0\in(1,+\infty)$ such that for any $f \in L^{p_0}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \|U(f)\|_F^{p_0} \mathrm{d}x \le C\int_{\mathbb{R}^n} |f|^{p_0} \mathrm{d}x. \tag{4}$$

(2) For every continuous function with compact support supp(f)$\subset\mathbb{R}^n$, $U(f)$ coincides with $K*f(x)$ outside supp(f)$\subset\mathbb{R}^n$ and there exists a constant $M\ge0$ such that

$$\int_{|x|>2|y|} \|K(x-y) - K(x)\|_F \,\mathrm{d}x \le M, \ \forall y \ne 0. \tag{5}$$

Under these assumptions, U can be extended to a bounded operator from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n,F)$ for every $p_0\in(1,+\infty)$ (Benedek *et al.*, 1962).

Theorem 1 Let U be a BCP operator and $Tf=\|U(f)\|_F$. If T commutes with translations, say $\tau_yT=T\tau_y$ for any $y\in\mathbb{R}^n$, then it is bounded on $F_p^{\alpha,q}$ when $1<p$, $q<\infty$ and $0<\alpha<1$.

The proof follows essentially the method shown in (Korry, 2004), so we shall only sketch it briefly and point out some differences.

I. Let B be the unit ball in $\mathbb{R}^n$, $f(x_1,x_2,x_3)$ be an F-valued function on $E=\mathbb{R}^n\times(0,+\infty)\times B$ and $\vec{p}=(p_1,p_2,p_3)$, $1<p_i<+\infty$. We say $f\in L^{\vec{p}}(E,F)$ if

$$\|f\|_{L^{\vec{p}}(E,F)} = \left(\int_{\mathbb{R}^n}\left(\int_0^\infty\left(\int_B \|f(x_1,x_2,x_3)\|_F^{p_3}\,\mathrm{d}x_3\right)^{\frac{p_2}{p_3}}\right)^{\frac{p_1}{p_2}}\right)^{\frac{1}{p_1}} < +\infty. \tag{6}$$

Defining $\tilde{U}f(x_1,x_2,x_3)=U(f_{x_2,x_3})(x_1)$, since U is a BCP operator, we can extend $\tilde{U}$ to a bounded operator from $L^{\vec{p}}(E,\mathbb{R})$ to $f\in L^{\vec{p}}(E,F)$ [Lemma 2, (Korry, 2004)].

II. Setting

$$S_r(f)(x) = \left(\int_0^\infty\left(\int_B\left|\frac{f(x+th)-f(x)}{t^\alpha}\right|^r \mathrm{d}h\right)^{q/r}\frac{\mathrm{d}t}{t}\right)^{1/q}, \tag{7}$$

we claim that the L^p norm of $S_r(f)(x)$ and $S_l(f)(x)$ are equivalent whenever $1<r<\min(p, q)$. The proof of the claim is a bit complicated but still follows the way of Lemma 1 in (Korry, 2004). So we omit it here. The essence of this claim is making r a bit larger than 1 so that we can use the conclusion of I where p_3 must be strictly greater than 1.

Now we are ready to prove Theorem 1. Recall that $F_p^{\alpha,q}$ has another equivalent norm

$$\|f\|_{L^p} + \left\|\left(\int_0^\infty\left(\int_B\left|\frac{f(x+th)-f(x)}{t^\alpha}\right|\mathrm{d}h\right)^q\frac{\mathrm{d}t}{t}\right)^{1/q}\right\|_{L^p} \tag{8}$$

when $1<p$, $q<+\infty$ and $0<\alpha<1$ (Triebel, 1983). Note that T has already been shown to be a bounded operator on L^p. By II, it remains to show

$$\|S_r(Tf)\|_{L^p} \le C\|S_r(f)\|_{L^p}, \tag{9}$$

with $\vec{p}=(r,q,p)$ and $1<r<\min(p, q)$. Given $f\in F_p^{\alpha,q}$, and setting

$$g(x,t,h)=\frac{f(x+th)-f(x)}{t^{\alpha+1/q}}, \tag{10}$$

we have

$$\frac{Tf(x+th)-Tf(x)}{t^{\alpha+1/q}}=\frac{T(\tau_{th}f)(x)-Tf(x)}{t^{\alpha+1/q}} \le T\left(\frac{\tau_{th}f(x)-f(x)}{t^{\alpha+1/q}}\right)=\left\|\tilde{U}g(x,t,h)\right\|_F. \tag{11}$$

It follows that

$$\begin{aligned}&\| S_r(Tf)\|_{L^p}\\ &=\left\|\left(\int_0^\infty\left(\int_B\left|\frac{Tf(x+th)-Tf(x)}{t^{\alpha+1/q}}\right|^r \mathrm{d}h\right)^{q/r}\mathrm{d}t\right)^{1/q}\right\|_{L^p}\\ &\le\left\|\left(\int_0^\infty\left(\int_B\left\|\tilde{U}g(x,t,h)\right\|_F^r \mathrm{d}h\right)^{q/r}\mathrm{d}t\right)^{1/q}\right\|_{L^p}\\ &=\left\|\tilde{U}g(x,t,h)\right\|_{L^{\vec{p}}(F)}.\end{aligned} \tag{12}$$

Now we are in a position to apply conclusion I and get that the above term is no larger than a constant times

$$\|g\|_{L^{\vec{p}}(F)}=\|S_r(f)(x)\|_{L^p}. \tag{13}$$

Remark 1 The g-function operator defined by

$$g_\phi(f)(x)=\left(\int_0^\infty|\phi_t*f(x)|^2\frac{\mathrm{d}t}{t}\right)^{1/2} \tag{14}$$

verifies the assumption of Theorem 1 with $F=L^2((0,+\infty),\mathrm{d}t/t)$ and $U(f)=\phi_t*f(x)$ whenever ϕ is a standard L-P function. The Hardy-Littlewood maximal function can also be modified to satisfy the hypothesis of our theorem [Corollary I, (Korry, 2004)]. So they are all bounded operators on $F_p^{\alpha,q}$, when $1<p$, $q<\infty$ and $0<\alpha<1$.

APPLICATION TO MARCINKIEWICZ INTEGRALS

Setting $\phi(x)=\chi_B(x)\Omega(x)/|x|^{n-1}$, we can regard μ_Ω as a g-function operator with this kernel. Since ϕ is no longer an L-P function, we cannot use Remark 1 directly. Instead we shall prove that $U: f\mapsto\phi_t*f$ is a BCP operator. Thus by Theorem 1, the boundedness of μ_Ω on $F_p^{\alpha,q}$ follows:

Theorem 2 Suppose $\Omega(x)$ satisfies $\int\Omega(x')\mathrm{d}\sigma=0$ and the L^1-Dini condition. Then μ_Ω is bounded on $F_p^{\alpha,q}$ for $1<p$, $q<+\infty$ and $0<\alpha<1$.

Proof By Theorem 1, we only need to prove that, there exists some constant M such that

$$\int_{|x|>2|y|}\left(\int_0^\infty\left|\phi_t(x-y)-\phi_t(x)\right|^2\frac{\mathrm{d}t}{t}\right)^{1/2}\mathrm{d}x<M \tag{15}$$

for any $y\neq0$. In fact one has already known the equivalence between the L^1-Dini condition and the normal scalar valued Hömander's condition when dealing with the singular integrals. Now write

$$\begin{aligned}I(x,y)&=\left(\int_0^\infty\left|\phi_t(x-y)-\phi_t(x)\right|^2\frac{\mathrm{d}t}{t}\right)^{1/2}\\ &=\left(\int_0^\infty t^{-3}\left|\chi_B\left(\frac{x-y}{t}\right)\frac{\Omega(x-y)}{|x-y|^{n-1}}-\chi_B\left(\frac{x}{t}\right)\frac{\Omega(x)}{|x|^{n-1}}\right|^2\mathrm{d}t\right)^{1/2}\\ &\le\left\{\int_0^\infty t^{-3}\left|\chi_B\left(\frac{x-y}{t}\right)\frac{\Omega(x-y)}{|x-y|^{n-1}}-\chi_B\left(\frac{x-y}{t}\right)\frac{\Omega(x)}{|x-y|^{n-1}}\right|^2\mathrm{d}t\right\}^{1/2}\\ &+\left(\int_0^\infty t^{-3}\left|\chi_B\left(\frac{x-y}{t}\right)\frac{\Omega(x)}{|x-y|^{n-1}}-\chi_B\left(\frac{x}{t}\right)\frac{\Omega(x)}{|x-y|^{n-1}}\right|^2\mathrm{d}t\right)^{1/2}\\ &+\left(\int_0^\infty t^{-3}\left|\chi_B\left(\frac{x}{t}\right)\frac{\Omega(x)}{|x-y|^{n-1}}-\chi_B\left(\frac{x}{t}\right)\frac{\Omega(x)}{|x|^{n-1}}\right|^2\mathrm{d}t\right)^{1/2}\\ &=I_1(x,y)+I_2(x,y)+I_3(x,y).\end{aligned} \tag{16}$$

Let us consider the three terms separately. Firstly,

$$\begin{aligned}I_3(x,y)&=\left|\frac{\Omega(x)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x|^{n-1}}\right|\left(\int_0^\infty t^{-3}\left|\chi_B(x/t)\right|^2\mathrm{d}t\right)^{1/2}\\ &=\left|\frac{\Omega(x)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x|^{n-1}}\right|\left(\int_{|x|}^\infty t^{-3}\mathrm{d}t\right)^{1/2}\\ &=\frac{|\Omega(x)|}{\sqrt{2}|x|}\left|\frac{1}{|x-y|^{n-1}}-\frac{1}{|x|^{n-1}}\right|.\end{aligned} \tag{17}$$

So when $|x|>2|y|$, $I_3(x,y)$ is bounded by $C_3(n)|\Omega(x)||y|/|x|^{n+1}$. Thus

$$\int_{|x|>2|y|} I_3(x,y)\mathrm{d}x \le C_3(n)\int_{|x|>2|y|}\frac{|\Omega(x)||y|}{|x|^{n+1}}\mathrm{d}x$$
$$=C_3(n)|y|\int_{2|y|}^{\infty}\int_{S^{n-1}}|\Omega(x')|\mathrm{d}\sigma(x')\frac{\mathrm{d}r}{r^2} \quad (18)$$
$$=C_3(n)|y|\|\Omega\|_{L^1}\int_{2|y|}^{\infty}\frac{\mathrm{d}r}{r^2}=C_3(n)\|\Omega\|_{L^1}.$$

Secondly, we rewrite $I_2(x, y)$ as

$$\frac{|\Omega(x)|}{|x-y|^{n-1}}\left(\int_0^{\infty}t^{-3}\left|\chi_B\left(\frac{x-y}{t}\right)-\chi_B(x/t)\right|^2\mathrm{d}t\right)^{1/2}. \quad (19)$$

Assuming $|x|\ge|x-y|$, we have

$$I_2(x,y)=\frac{|\Omega(x)|}{|x-y|^{n-1}}\left(\int_{|x-y|}^{|x|}t^{-3}\mathrm{d}t\right)^{1/2}$$
$$=\frac{|\Omega(x)|}{\sqrt{2}|x-y|^{n-1}}\left(\frac{1}{|x-y|^2}-\frac{1}{|x|^2}\right)^{1/2}, \quad (20)$$

while if $|x|<|x-y|$, the case is

$$I_2(x,y)=\frac{|\Omega(x)|}{\sqrt{2}|x-y|^{n-1}}\left(\frac{1}{|x|^2}-\frac{1}{|x-y|^2}\right)^{1/2}. \quad (21)$$

Anyway, we obtain

$$I_2(x,y)\le\frac{|\Omega(x)|}{\sqrt{2}|x-y|^{n-1}}\left|\frac{1}{|x|^2}-\frac{1}{|x-y|^2}\right|^{5/2}$$
$$\le C_2(n)|\Omega(x)|\frac{|y|^{1/2}}{|x|^{n+1/2}}. \quad (22)$$

By the same argument with that of $I_3(x, y)$, we reach

$$\int_{|x|>2|y|}I_2(x,y)\mathrm{d}x\le C_2(n)\|\Omega\|_{L^1}. \quad (23)$$

Finally we turn to $I_1(x,y)$, and the L^1-Dini condition will be used here.

$$I_1(x,y)$$
$$=\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x-y|^{n-1}}\right|\left(\int_0^{\infty}t^{-3}\left|\chi_B\left(\frac{x-y}{t}\right)\right|^2\mathrm{d}t\right)^{1/2}$$
$$=\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x-y|^{n-1}}\right|\left(\int_{|x-y|}^{\infty}t^{-3}\mathrm{d}t\right)^{1/2}$$
$$=\frac{\Omega(x-y)-\Omega(x)}{\sqrt{2}|x-y|^n}. \quad (24)$$

By simple calculus, when $|x|>2|y|$, we have

$$|(x-y)'-x'|\le 4|y|/|x|. \quad (25)$$

And consequently

$$\int_{|x|>2|y|}I_1(x,y)\mathrm{d}x$$
$$\le C_1(n)\int_{|x|>2|y|}\frac{|\Omega(x-y)-\Omega(x)|}{|x|^n}\mathrm{d}x$$
$$=C_1(n)\int_{2|y|}^{\infty}\left(\int_{S^{n-1}}\left|\Omega((rx'-y)')-\Omega(x')\right|\mathrm{d}\sigma(x')\right)\frac{\mathrm{d}r}{r}$$
$$\le C_1(n)\int_{2|y|}^{\infty}\omega_1(2|y|/r)\frac{\mathrm{d}r}{r}$$
$$=C_1(n)\int_0^2\frac{\omega_1(s)}{s}\mathrm{d}s<+\infty. \quad (26)$$

Remark 2 Recently, we have managed to extend Theorem 2 to the case that $\Omega\in H^1(S^{n-1})$ by incorporating a rotation method and the atomic decomposition of $H^1(S^{n-1})$.

References

Benedek, A., Calderón, A.P., Panzone, R., 1962. Convolution operators on Banach space valued functions. *PNAS*, **48**(3):356-365. [doi:10.1073/pnas.48.3.356]

Calderón, A.P., Weiss, M., Zygmund, A., 1967. On the existence of singular integrals. *Proc. Symp. Pure Math.*, **10**:56-73.

Chen, J.C., Fan, D.S., Ying, Y.M., 2002. Singular integral operators on function spaces. *J. Math. Anal. Appl.*, **276**(2):691-708. [doi:10.1016/S0022-247X(02)00419-5]

Chen, J.C., Fan, D.S., Ying, Y.M., 2003. Certain operators with rough singular kernels. *Can. J. Math.*, **55**:504-532.

Chen, Q.L., Zhang, Z.F., 2004. Boundedness of a class of super singular integral operators and the associated commutators. *Sci. China Ser. A*, **47**(6):842-853 (in Chinese). [doi:10.1360/03ys0099]

Chen, D.N., Chen, J.C., Fan, D.S., 2005. Rough singular integral operators on Hardy-Sobolev spaces. *Appl. Math. J. Chin. Univ. Ser. B*, **20**(1):1-9. [doi:10.1007/s11766-005-0030-8]

Ding, Y., Fan, D.S., Pan, Y.B., 2000. L^p-boundedness of Marcinkiewicz integrals with Hardy space function kernels. *Acta. Math. Sinica*, **16**:593-600. [doi:10.1007/PL00011567]

Korry, S., 2004. A class of bounded operators on Sobolev spaces. *Arch. Math.*, **82**(1):40-50. [doi:10.1007/s00013-003-0416-x]

Stein, E.M., 1958. On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz. *Trans. Amer. Math. Soc.*, **88**:430-466. [doi:10.2307/1993226]

Triebel, H., 1983. Theory of Function Spaces. Birkhäuser-Verlag, Basel-Boston, p.108.

Xu, H., Chen, J.C., Ying, Y.M., 2003. A note on Marcinkiewicz integrals with H^1 kernels. *Acta. Math. Scientia*, **23**:133-138.