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Nonlinear stochastic optimal bounded control of hysteretic systems with actuator saturation*

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Abstract: A modified nonlinear stochastic optimal bounded control strategy for random excited hysteretic systems with actuator saturation is proposed. First, a controlled hysteretic system is converted into an equivalent nonlinear nonhysteretic stochastic system. Then, the partially averaged Itô stochastic differential equation and dynamical programming equation are established, respectively, by using the stochastic averaging method for quasi non-integrable Hamiltonian systems and stochastic dynamical programming principle, from which the optimal control law consisting of optimal unbounded control and bang-bang control is derived. Finally, the response of optimally controlled system is predicted by solving the Fokker-Planck-Kolmogorov (FPK) equation associated with the fully averaged Itô equation. Numerical results show that the proposed control strategy has high control effectiveness and efficiency.

Key words: Hysteretic system, Stochastic averaging, Optimal control, Dynamical programming, Actuator saturation
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INTRODUCTION

Civil engineering structures often exhibit hysterestic characteristics when they are subjected to severe dynamic loading. The intelligent materials in structural control such as piezoelectric ceramics or electrorheological and magnetorheological (ER/MR) dampers also exhibit hysterestic. Because of the complicated behavior of hysteretic systems, the active control of hysteretic structures is a significant and difficult research subject and has been extensively investigated by various methods in the past few decades, including polynomial control (Yang *et al.*, 1996), sliding mode control (Yang *et al.*, 1995), etc. However, these controls are deterministic and independent of dynamic loading. Since the dynamic loading for civil engineering structures is usually random, stochastic optimal control of hysteretic structures would be more appealing and reasonable. In recent years, a nonlinear stochastic optimal control strategy has been proposed for quasi-Hamiltonian systems (Zhu *et al.*, 2001; Zhu, 2006) based on the stochastic averaging method (Zhu *et al.*, 1997; 2002; Zhu and Yang, 1997) and stochastic dynamical programming principle (Fleming and Rishel, 1975; Yong and Zhou, 1999). The strategy has been applied to hysteretic systems (Zhu *et al.*, 2000; Ying and Zhu, 2003; Cheng *et al.*, 2006).

Due to the stochastic nature of dynamical loading, it is conceivable that the required control force may exceed the capacity of the actuator, which will result in actuator saturation. The actuator saturation may cause deterioration of the control performance. Thus, in the field of nonlinear stochastic control, the actuator saturation problem must be considered.

In the paper, a nonlinear stochastic optimal con-

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trol strategy is developed for hysteretic systems with actuator saturation. The optimal control law consists of optimal unbounded control and optimal bang-bang control, i.e., the control force is of the form of optimal unbounded control if the required control force is less than the control bounds; it is equal to control bound, otherwise. A controlled hysteretic column under both horizontal and vertical random ground excitations is worked out. The numerical results show that this modified stochastic optimal control strategy has high control effectiveness and efficiency.

FORMULATION OF PROBLEM

The bounded feedback control of a single degree-of-freedom (DOF) strong nonlinear hysteretic column subjects to both horizontal and vertical random ground excitations. The equation of motion is of the form

$$\begin{cases} \ddot{X}+2\zeta\dot{X}+(\alpha-k_1-k_2\eta(t))X+(1-\alpha)Z=\xi(t)+u, \\ X(t_0)=X_0, \\ |u|\le b_u, \end{cases} \quad (1)$$

where X is non-dimensional displacement; ζ is viscous damping ratio; k_1, k_2 are constants; α is ratio of stiffness after yield to stiffness before yield; u is weak feedback control force which is bounded due to actuator saturation; $\eta(t)$ and $\xi(t)$ are vertical and horizontal ground acceleration excitations, and are modeled as Gaussian white noises with intensities $2D_1$ and $2D_2$, respectively; Z is the hysteretic component of the restoring force, the Bouc-Wen model (Bouc, 1967; Wen, 1976) of which is governed by the following 1D differential equation:

$$\dot{Z}=A\dot{X}-\beta\dot{X}|Z|^n-\gamma|\dot{X}|Z|Z|^{n-1}, \quad (2)$$

where A, β, γ and n are hysteretic parameters. System Eq.(1) without control has been studied by Lin and Cai (1995) and that with unbounded control by Zhu *et al*.(2000) and Cheng *et al*.(2006).

The objective of control is expressed in terms of minimizing a performance index, which depends on control time interval. For finite time-interval control, the performance index is of the form

$$J_1(u)=E\left[\int_0^{t_f} f_1(X(s),\dot{X}(s),u(s))\mathrm{d}s+g(X(t_f),\dot{X}(t_f))\right]; \quad (3)$$

and for an infinite time-interval ergodic control, the index is

$$J_2(u)=\lim_{T\to\infty}\frac{1}{T}\int_0^T f_2(X(s),\dot{X}(s),u(s))\mathrm{d}s, \quad (4)$$

where $E[\cdot]$ denotes an expectation operation; t_f is the terminal time of control; $f_1(X,\dot{X},u)$, $f_2(X,\dot{X},u)$ are cost functions; $g(X,\dot{X})$ is the terminal cost. Eqs.(1), (2), (3) or (4) constitute the mathematical formulation of the optimal bounded control problem for a stochastically excited hysteretic column with actuator saturation.

PARTIALLY AVERAGED SYSTEM

The controlled hysteretic column subject to random excitations is a very complicated system. To apply the stochastic averaging method, it is necessary first to convert hysteretic system Eq.(1) into the following equivalent nonlinear non-hysteretic system:

$$\ddot{X}+[2\zeta+2\zeta_1(H)]\dot{X}+\partial U/\partial X=\xi(t)+u+k_2X\eta(t), \quad (5)$$

where $H=\dot{x}^2/2+U(x)$ is the total energy of the system, and U is the equivalent potential energy. The nonlinear damping coefficient $2\zeta_1(H)$ can be expressed as

$$2\zeta_1(H)=\frac{A_r}{2\int_{-a}^{a}(2H-2U)\mathrm{d}x}, \quad (6)$$

where A_r is the area of hysteretic loop representing the energy dissipated by the hysteretic force; a is the amplitude of displacement related to H by $H=U(\pm a)$. The expressions for area A_r and potential energy $U(x)$ depend on the values of hysteresis parameters. In the case $\beta=\gamma$, $n=A=1$, for example, they are

$$A_r=(1-\alpha)[-2x_0/\gamma+(a-x_0)^2], \quad (7)$$

$$U(x)=\begin{cases}(\alpha-k_1)x^2/2+(1-\alpha)(x+x_0)^2/2, \\ \qquad -a\le x\le -x_0; \\ (\alpha-k_1)x^2/2+(1-\alpha)[1-\mathrm{e}^{-2\gamma(x+x_0)}]^2\big/(8\gamma^2), \\ \qquad -x_0\le x\le a,\end{cases} \tag{8}$$

where x_0 represents the residual hysteresis displacement. The quantities a and x_0 for certain H can be obtained by solving the following equations:

$$\begin{cases}2\gamma(a-x_0)=1-\mathrm{e}^{-2\gamma(a+x_0)}, \\ 2H-(\alpha-k_1)a^2=(1-\alpha)(a-x_0)^2.\end{cases} \tag{9}$$

By applying the stochastic averaging method for quasi non-integrable Hamiltonian systems (Zhu and Yang, 1997) to system Eq.(5), the following partially averaged Itô equation for total energy H is derived:

$$\mathrm{d}H=\left[m(H)+\left\langle u\frac{\partial H}{\partial \dot{X}}\right\rangle\right]\mathrm{d}t+\sigma(H)\mathrm{d}B(t), \tag{10}$$

where $B(t)$ is a standard Wiener process; $m(H)$ and $\sigma(H)$ are the averaged drift and diffusion coefficients, respectively, which can be expressed as

$$\begin{cases}m(H)=\dfrac{1}{T(H)}\left[-4\zeta\displaystyle\int_{-a}^{a}\sqrt{2H-2U(x)}\mathrm{d}x-A_r+\right. \\ \qquad\left.2k_2^2D_2\displaystyle\int_{-a}^{a}\frac{x^2}{\sqrt{2H-2U(x)}}\mathrm{d}x\right]+D_1, \\ \sigma^2(H)=\dfrac{2}{T(H)}\displaystyle\int_{-a}^{a}(2D_1+2k_2^2D_2x^2)\sqrt{2H-2U(x)}\mathrm{d}x, \\ T(H)=2\displaystyle\int_{-a}^{a}\frac{1}{\sqrt{2H-2U(x)}}\mathrm{d}x.\end{cases} \tag{11}$$

To be consistent with partially averaged Eq.(10), performance index Eqs.(3) and (4) are also partially averaged, i.e., Eqs.(3) and (4) are replaced with

$$J_3(u)=E\left[\int_0^{t_f}f_3(H(s),\langle u(s)\rangle)\mathrm{d}s+g(H(t_f))\right], \tag{12}$$

$$J_4(u)=\lim_{T\to\infty}\frac{1}{T}\int_0^{T}f_4(H(s),\langle u(s)\rangle)\mathrm{d}s, \tag{13}$$

respectively. Eqs.(10) and (12) or (13) constitute the mathematical formulation of the optimal nonlinear control problem of the partially averaged hysteretic system with actuator saturation.

OPTIMAL CONTROL FORCE

Based on the stochastic dynamical programming principle (Fleming and Rishel, 1975), the following dynamical programming equation can be established

$$\frac{\partial V}{\partial t}=-\min_{u\in U}\left\{\frac{1}{2}\sigma^2(H)\frac{\mathrm{d}^2V}{\mathrm{d}H^2}+f_3(H,\langle u\rangle)+\left[m(H)+\left\langle u\frac{\partial H}{\partial \dot{X}}\right\rangle\right]\frac{\mathrm{d}V}{\mathrm{d}H}\right\}, \tag{14}$$

for the partially averaged controlled system Eq.(10) with performance index Eq.(12), or

$$\lambda=\min_{u\in U}\left\{\frac{1}{2}\sigma^2(H)\frac{\mathrm{d}^2V}{\mathrm{d}H^2}+f_4(H,\langle u\rangle)+\left[m(H)+\left\langle u\frac{\partial H}{\partial \dot{X}}\right\rangle\right]\frac{\mathrm{d}V}{\mathrm{d}H}\right\}, \tag{15}$$

for system Eq.(10) with performance index Eq.(13), where $V=V(H,t)$ is value function, $\lambda=\lim_{T\to\infty}\frac{1}{T}\int_0^T f_4(H,\langle u^*\rangle)\mathrm{d}s$ is optimal average cost and U denotes the domain of control force u. For symmetric bounded control forces,

$$U:\ |u|\le b_u,\ \ b_u>0. \tag{16}$$

The sufficient and necessary condition for minimization of the right-hand side of Eqs.(14) or (15) subject to the constraints in Eq.(16) yields the optimal control force u^*. For example, let function f_4 be of the form

$$\begin{cases}f_4(H,\langle u\rangle)=f_c(H)+R\langle u^2\rangle, \\ f_c(H)=s_0+s_1H+s_2H^2+s_3H^3,\end{cases} \tag{17}$$

where $f_c(H)\ge 0$ and R is a positive-definite symmetric constant. The optimal control law is determined as follows:

$$u^*=\begin{cases}-\dfrac{1}{2R}\dfrac{\mathrm{d}V}{\mathrm{d}H}\dot{X}, & \left|\dfrac{1}{2R}\dfrac{\mathrm{d}V}{\mathrm{d}H}\dot{X}\right|<b_\mathrm{u};\\ -b_\mathrm{u}\,\mathrm{sgn}(\dot{X}), & \left|\dfrac{1}{2R}\dfrac{\mathrm{d}V}{\mathrm{d}H}\dot{X}\right|\geq b_\mathrm{u}.\end{cases} \tag{18}$$

Inserting u^* in Eq.(18) into dynamical programming Eq.(15) for replacing u, the final dynamical programming equation is obtained as follows:

$$\lambda=\frac{1}{2}\sigma^2(H)\frac{\mathrm{d}^2V}{\mathrm{d}H^2}+m(H)\frac{\mathrm{d}V}{\mathrm{d}H}+f_\mathrm{c}(H)+m'(H), \tag{19}$$

where

$$\begin{cases}m'(H)=\dfrac{2}{T(H)}\left[-b_\mathrm{u}(X_\mathrm{cr2}-X_\mathrm{cr1})\dfrac{\mathrm{d}V}{\mathrm{d}H}+\displaystyle\int_{X_\mathrm{cr1}}^{-x_0}\dfrac{b_\mathrm{u}^2R}{\sqrt{2H-2V_1}}\mathrm{d}x\right.\\ \quad+\displaystyle\int_{-x_0}^{X_\mathrm{cr2}}\dfrac{b_\mathrm{u}^2R}{\sqrt{2H-2V_2}}\mathrm{d}x-\dfrac{1}{4R}\left(\dfrac{\mathrm{d}V}{\mathrm{d}H}\right)^2\displaystyle\int_{-a}^{X_\mathrm{cr1}}\sqrt{2H-2V_1}\mathrm{d}x\\ \quad\left.-\dfrac{1}{4R}\left(\dfrac{\mathrm{d}V}{\mathrm{d}H}\right)^2\displaystyle\int_{X_\mathrm{cr2}}^{a}\sqrt{2H-2V_2}\mathrm{d}x\right],\\ V_1=(\alpha-k_1)x^2/2+(1-\alpha)(x+x_0)^2/2,\\ V_2=(\alpha-k_1)x^2/2+(1-\alpha)[1-\mathrm{e}^{-2\gamma(x+x_0)}]^2/(8\gamma^2),\\ \lambda=s_0+D_1\left.\dfrac{\mathrm{d}V}{\mathrm{d}H}\right|_{H=0}.\end{cases} \tag{20}$$

For certain H, X_cr1, X_cr2 can be obtained by the following equations:

$$\begin{cases}H=\dfrac{1}{2}\left(\dfrac{2Rb_\mathrm{u}}{\mathrm{d}V/\mathrm{d}H}\right)^2+\dfrac{1}{2}(\alpha-k_1)X_\mathrm{cr1}^2\\ \quad+\dfrac{1}{2}(1-\alpha)(X_\mathrm{cr1}^2+x_0)^2,\ -a\leq X_\mathrm{cr1}\leq -x_0;\\ H=\dfrac{1}{2}\left(\dfrac{2Rb_\mathrm{u}}{\mathrm{d}V/\mathrm{d}H}\right)^2+\dfrac{1}{2}(\alpha-k_1)X_\mathrm{cr2}^2\\ \quad+\dfrac{1}{8\gamma^2}(1-\alpha)\left[1-\mathrm{e}^{-2\gamma(X_\mathrm{cr2}+x_0)}\right]^2,\ 0\leq X_\mathrm{cr2}\leq a.\end{cases} \tag{21}$$

Solving the final dynamical programming Eq.(19) yields $\mathrm{d}V/\mathrm{d}H$ and optimal control forces u^*. Substituting the optimal control forces u^* into Eq.(10) to replace u, completing the averaging of the terms with u^*, the following fully averaged Itô equation can be obtained:

$$\mathrm{d}H=\overline{m}(H)\mathrm{d}t+\sigma(H)\mathrm{d}B(t), \tag{22}$$

where

$$\overline{m}(H)=m(H)+\left\langle u^*\frac{\partial H}{\partial\dot{X}}\right\rangle=m(H)-b_\mathrm{u}(X_\mathrm{cr2}-X_\mathrm{cr1})\frac{\mathrm{d}V}{\mathrm{d}H}$$
$$-\frac{1}{2R}\frac{\mathrm{d}V}{\mathrm{d}H}\int_{-a}^{X_\mathrm{cr1}}\sqrt{2H-2V_1}\mathrm{d}x-\frac{1}{2R}\frac{\mathrm{d}V}{\mathrm{d}H}\int_{X_\mathrm{cr2}}^{a}\sqrt{2H-2V_2}\mathrm{d}x. \tag{23}$$

Solving the reduced Fokker-Planck-Kolmogorov equation associated with the fully averaged Itô Eq.(22), the following exact stationary probability density $p_\mathrm{c}(H)$ can be obtained:

$$p_\mathrm{c}(H)=C_\mathrm{c}\exp\left[-\int_0^H\frac{-2\overline{m}(y)+\mathrm{d}\sigma^2(y)/\mathrm{d}y}{\sigma^2(y)}\mathrm{d}y\right]. \tag{24}$$

The exact stationary probability density $p_\mathrm{u}(H)$ of uncontrolled system can be obtained from Eq.(24) by replacing $\overline{m}(y)$ with $m(y)$ in Eq.(11).

The statistics of the stationary responses of the controlled and uncontrolled system Eq.(1) can be obtained from the stationary probability densities $p_\mathrm{c}(H)$ and $p_\mathrm{u}(H)$. For instance, the joint probability density, the mean square value of the displacement, and the mean square value of optimal control force u^* are

$$\begin{cases}p_\mathrm{c}(\dot{x},x)=p_\mathrm{c}(H)/T(H)\big|_{H=H(\dot{x},x)},\\ E[X_\mathrm{c}^2]=\displaystyle\int_0^\infty x^2p_\mathrm{c}(\dot{x},x)\mathrm{d}x\mathrm{d}\dot{x},\\ E[u_\mathrm{c}^{*2}]=\displaystyle\int_0^\infty u_\mathrm{c}^{*2}p_\mathrm{c}(H)\mathrm{d}H.\end{cases} \tag{25}$$

The control effectiveness and efficiency of the proposed control can be obtained by using the following definitions:

$$K=\frac{E[X_\mathrm{u}^2]-E[X_\mathrm{c}^2]}{E[X_\mathrm{c}^2]},\ \mu=\frac{K}{E[u^{*2}]}, \tag{26}$$

where $E[X_\mathrm{u}^2]$ is the mean-square displacement of an uncontrolled system. K represents the percentage reduction in the mean square displacement of the optimal controlled systems while μ is the relative reduction per unit of the normalized mean-square control.

OPTIMAL BOUNDED POLYNOMIAL CONTROL

For comparison, the optimal bounded polynomial control strategy is considered (Yang *et al*., 1996). Let the performance index for the infinite time control problem be

$$J=\int_0^{\infty}(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{S}_1\boldsymbol{Q}+\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{P}_2\boldsymbol{Q}\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{S}_2\boldsymbol{Q}+\boldsymbol{u}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{u})\mathrm{d}\tau, \quad (27)$$

where

$$\boldsymbol{Q}=\begin{Bmatrix}\dot{X}\\ -2\zeta\dot{X}-(\alpha-k_1)X-(1-\alpha)Z\end{Bmatrix},\quad \boldsymbol{u}=\begin{Bmatrix}0\\ u\end{Bmatrix},$$

$$\boldsymbol{R}=\begin{Bmatrix}0&0\\0&R\end{Bmatrix},\qquad \boldsymbol{S}_1=\begin{Bmatrix}s_{11}&0\\0&s_{12}\end{Bmatrix}, \quad (28)$$

$$\boldsymbol{S}_2=\begin{Bmatrix}s_{21}&0\\0&s_{22}\end{Bmatrix},\qquad \boldsymbol{P}_2=\begin{Bmatrix}P_{21}&0\\0&P_{22}\end{Bmatrix}.$$

In this case, suppose the optimal value function is of the form

$$V=\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{Q}+\frac{1}{2}(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{P}_2\boldsymbol{Q})^2;\ \boldsymbol{P}_1=\begin{Bmatrix}P_{11}&0\\0&P_{12}\end{Bmatrix}. \quad (29)$$

By solving the associated Hamilton-Jacobi-Bellman equation (Yang *et al*., 1996), the optimal control force subjected to the constraints in Eq.(16) is obtained

$$u^*=\begin{cases}F, & |F|<b_{\mathrm{u}};\\ -b_{\mathrm{u}}\,\mathrm{sgn}(\dot{X}), & |F|\ge b_{\mathrm{u}},\end{cases} \quad (30)$$

where

$$\begin{aligned}F=&-2\zeta P_{12}[2\zeta'\dot{X}+(\alpha-k_1)X+(1-\alpha)Z]/R\\ &-2\zeta P_{22}[2\zeta''\dot{X}+(\alpha-k_1)X+(1-\alpha)Z] \quad (31)\\ &\cdot[P_{21}\dot{X}^2+P_{22}(2\zeta\dot{X}+(\alpha-k_1)X+(1-\alpha)Z)^2]/R,\end{aligned}$$

$2\zeta'=2\zeta+P_{11}/(2\zeta P_{12})$, $2\zeta''=2\zeta+P_{21}/(2\zeta P_{22})$.

Then the statistics of the responses of the optimal bounded polynomial controlled system are obtained.

NUMERICAL RESULTS

Numerical results were obtained for the following parameter values: ζ=0.1, k_1=0.04, k_2=0.1, γ=0.5, α=0.9, D_1=0.3, D_2=0.1, R=0.1, s_1=s_3=0, s_2=1.0, b_{u}=1.5, dV(0)/dH=3.5 unless otherwise mentioned. Table 1 shows that the proposed control strategy performs very well in the entire range of parameter values used. The control effectiveness and control efficiency are very high for various control parameters, intensities of excitation and hysteresis parameters. Figs.1~2 are numerical results for system Eq.(1) without parametric excitation (D_2=0) by using the proposed control and the optimal bounded polynomial control. It is seen that the control effectiveness and efficiency of the proposed control strategy are higher than the optimal bounded polynomial control.

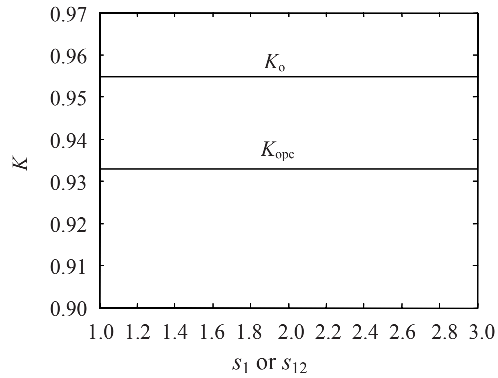

Fig.1 Comparison of control effectiveness. K_{o} for the proposed optimal control (s_2=s_3=0) and K_{opc} for the optimal bounded polynomial control (s_{11}=s_{21}=0, s_{22}=1.0)

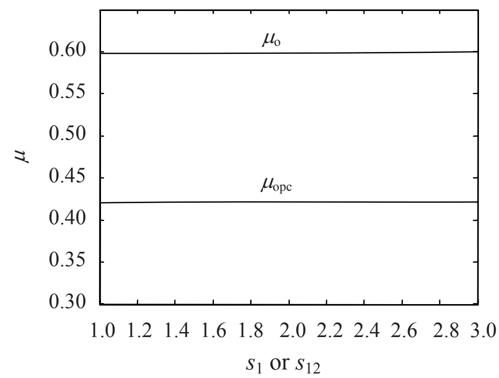

Fig.2 Comparison of control efficiency. μ_{o} for the proposed optimal control (s_2=s_3=0) and μ_{opc} for the optimal bounded polynomial control (s_{11}=s_{21}=0, s_{22}=1.0)

CONCLUSION

In the paper, an optimal nonlinear stochastic bounded control for hysteretic system with actuator saturation has been developed based on the stochastic

Table 1 Mean square value of the displacement, mean square value of optimal control force, control effectiveness and control efficiency versus parameters S_1, S_2, D_1, D_2, k_1, γ, α

Parameters		$E[Q_c^2]$	$E[Q_u^2]$	$E[u^{*2}]$	K (%)	μ
S_1	1.0	0.068192	1.517	1.59835	95.5048	0.59750
	1.4	0.068197	1.517	1.59791	95.5045	0.59770
	1.8	0.068202	1.517	1.59730	95.5042	0.59790
	2.2	0.068207	1.517	1.59552	95.5038	0.59860
	2.6	0.068213	1.517	1.59477	95.5035	0.55989
S_2	1.0	0.06817960	1.517	1.60083	95.50563	0.59660
	1.4	0.06817980	1.517	1.60074	95.50562	0.59664
	1.8	0.06817982	1.517	1.60062	95.50562	0.59668
	2.2	0.06818000	1.517	1.60172	95.50561	0.59627
	2.6	0.06818000	1.517	1.60158	95.50560	0.59632
D_1	0.2	0.033724	1.00823	1.32114	96.6550	0.73160
	0.3	0.068180	1.51700	1.60083	95.5056	0.59600
	0.4	0.114983	2.03179	1.73388	94.3410	0.54410
	0.5	0.173152	2.55095	1.79873	93.2123	0.51821
	0.7	0.319623	3.58957	1.83542	91.0959	0.49630
D_2	0.1	0.068180	1.51700	1.60083	95.5056	0.5960
	0.3	0.068243	1.53283	1.60094	95.5481	0.5968
	0.5	0.068303	1.54900	1.60110	95.5907	0.5970
	0.7	0.068386	1.56552	1.60246	95.6317	0.5968
	0.9	0.068460	1.58242	1.60221	95.6735	0.5971
k_1	0.030	0.067491	1.50060	1.60033	95.5024	0.5968
	0.035	0.067521	1.50875	1.59543	95.5247	0.5987
	0.040	0.068180	1.51700	1.60083	95.5056	0.5960
	0.045	0.069338	1.52534	1.60804	95.4543	0.5936
	0.050	0.068831	1.53377	1.60062	95.5123	0.5967
γ	0.1	0.066184	1.54048	1.59790	95.7037	0.5989
	0.3	0.066578	1.50965	1.59182	95.5898	0.6005
	0.5	0.068180	1.51700	1.60083	95.5056	0.5960
	0.7	0.069167	1.53769	1.60291	95.5019	0.5958
	0.9	0.069086	1.55959	1.60105	95.5702	0.5969
α	0.1	0.115975	3.51237	1.59311	96.6981	0.6070
	0.3	0.089864	1.74489	1.59280	94.8499	0.5955
	0.5	0.078066	1.51059	1.58957	94.8321	0.5966
	0.7	0.073167	1.46831	1.60312	95.0169	0.5927
	0.9	0.068180	1.51700	1.60083	95.5056	0.5960

S_1, S_2: control parameter; D_1, D_2: excitation intensity; k_1: frequency parameter; γ: hysteresis parameter; α: ratio of stiffness

averaging method and stochastic dynamical programming principle. The results for the example show that the proposed control strategy has some advantages: the dimension of the dynamical programming equation is reduced by using the stochastic averaging method and the equation has classical solution; the strategy has both advantages of optimal unbounded and bang-bang control and has high control effectiveness and control efficiency. Thus, the proposed control strategy is very promising.

References

Bouc, R., 1967. Forced Vibration of Mechanical System with Hysteresis. Proc. 4th Conf. on Non-linear Oscillation. Prague, Czechoslovakia.

Cheng, H., Zhu, W.Q., Ying, Z.G., 2006. Stochastic optimal semi-active control of hysteretic systems by using a magneto-rheological damper. *Smart Mater. Struct.*, **15**(3):711-718. [doi:10.1088/0964-1726/15/3/006]

Fleming, W.H., Rishel, R.W., 1975. Deterministic and Stochastic Optimal Control. Springer, New York.

Lin, Y.K., Cai, G.Q., 1995. Probabilistic Structural Dynamics: Advanced Theory and Application. McGraw-Hill, New

York.

Wen, Y.K., 1976. Method for random vibration of hysteretic systems. *J. Engrg. Mech. Div., ASCE*, **102**:249-263.

Yang, J.N., Wu, J.C., Agrawal, A.K., 1995. Sliding mode control for nonlinear and hysteretic structures. *J. Eng. Mech.*, **121**(12):1330-1339. [doi:10.1061/(ASCE)0733-9399(1995)121:12(1330)]

Yang, J.N., Agrawal, A.K., Chen, S., 1996. Optimal polynomial control for seismically excited non-linear and hysteretic structures. *Earthquake Eng. Struct. Dyn.*, **25**(11): 1211-1230. [doi:10.1002/(SICI)1096-9845(199611)25:11 <1211::AID-EQE609>3.3.CO;2-V]

Ying, Z.G., Zhu, W.Q., 2003. Stochastic optimal control of hysteretic systems under externally and parametrically random excitations. *Acta Mech. Solid. Sini.*, **16**(1):61-66. [doi:10.1007/PL00013119]

Yong, J.M., Zhou, X.Y., 1999. Stochastic Control, Hamiltonian Systems and HJB Equations. Springer, New York.

Zhu, W.Q., 2006. Nonlinear stochastic dynamics and control in Hamiltonian formulation. *ASME Applied Mechanics Reviews*, **59**(4):230-248. [doi:10.1115/1.2193137]

Zhu, W.Q., Yang, Y.Q., 1997. Stochastic averaging of quasi non-integrable Hamiltonian systems. *ASME Journal of Applied Mechanics*, **64**:157-164.

Zhu, W.Q., Huang, Z.L., Yang, Y.Q., 1997. Stochastic averaging of quasi integrable Hamiltonian systems. *ASME Journal of Applied Mechanics*, **64**:975-984.

Zhu, W.Q., Ying, Z.G., Ni, Y.Q., Ko, J.M., 2000. Optimal nonlinear stochastic control of hysteretic systems. *Journal of Engineering Mechanics*, **126**(10):1027-1032. [doi:10.1061/(ASCE)0733-9399(2000)126:10(1027)]

Zhu, W.Q., Ying, Z.G., Soong, T.T., 2001. An optimal nonlinear feedback control strategy for randomly excited structural systems. *Nonlinear Dynamics*, **24**(1):31-51. [doi:10.1023/A:1026527404183]

Zhu, W.Q., Huang, Z.L., Suzuki, Y., 2002. Stochastic averaging and Lyapunov exponent of quasi partially integrable Hamiltonian system. *International Journal of Nonlinear Mechanics*, **37**(3):419-437. [doi:10.1016/S0020-7462 (01)00018-X]