Journal of Zhejiang University SCIENCE A
ISSN 1673-565X (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Exponential synchronization of general chaotic delayed neural networks via hybrid feedback*

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Received June 23, 2007; revision accepted Nov. 26, 2007; published online Jan. 10, 2008

Abstract: This paper investigates the exponential synchronization problem of some chaotic delayed neural networks based on the proposed general neural network model, which is the interconnection of a linear delayed dynamic system and a bounded static nonlinear operator, and covers several well-known neural networks, such as Hopfield neural networks, cellular neural networks (CNNs), bidirectional associative memory (BAM) networks, recurrent multilayer perceptrons (RMLPs). By virtue of Lyapunov-Krasovskii stability theory and linear matrix inequality (LMI) technique, some exponential synchronization criteria are derived. Using the drive-response concept, hybrid feedback controllers are designed to synchronize two identical chaotic neural networks based on those synchronization criteria. Finally, detailed comparisons with existing results are made and numerical simulations are carried out to demonstrate the effectiveness of the established synchronization laws.

Key words: Exponential synchronization, Hybrid feedback, Drive-response conception, Linear matrix inequality (LMI), Chaotic neural network model
doi:10.1631/jzus.A071336 **Document code:** A **CLC number:** TP183

INTRODUCTION

Since Aihara *et al.*(1990) firstly introduced chaotic neural network model to simulate the chaotic behavior of biological neurons, chaotic neural networks have been successfully applied in combinational optimization (Kwok and Smith, 2000), associative memory (Tan and Ali, 2001), secure communication (Milanovic and Zaghloul, 1996), chemical biology (Han *et al*., 1995), and so on. Research on the synchronization of chaotic neural networks has broadened considerably in the last few years. Synchronization in chaotic systems has found many applications. It was used to understand self-organization behavior in the brain as well as in ecological systems, and has been applied to secure communications (Lu and Leeuwen, 2006). A wide variety of approaches have been proposed for the synchronization of the chaotic neural networks with or without delays which include linear and nonlinear feedback control, adaptive design control, impulsive control method, and invariant manifold method, among many others (Cheng *et al*., 2005; Lu and Leeuwen, 2006; Zhang *et al*., 2006; Zhou *et al*., 2006; Lu and Cao, 2007; Sun and Cao, 2007; Sun *et al*., 2007).

However, to our best knowledge, the aforementioned methods and many other existing synchronization methods give the existing conditions of controllers, but hardly provide the solution algorithms of the controller parameters. On the other hand, there does not seem to be much (if any) study on the exponential synchronization under the designated convergence rate for chaotic neural networks based on the linear matrix inequality (LMI) approach. It is well known that LMI approach has been successfully used in stability analysis of dynamic neural networks (Cao

* Project supported in part by the National Natural Science Foundation of China (No. 60504024), the Specialized Research Fund for the Doctoral Program of Higher Education, China (No. 20060335022), the Natural Science Foundation of Zhejiang Province (No. Y106010), China, and the "151 Talent Project" of Zhejiang Province (Nos. 05-3-1013 and 06-2-034), China

and Wang, 2005; Zhang and Liu, 2005; Liu, 2006). The main advantage of the LMI-based approaches is that the LMI-based conditions can be solved numerically using the interior-point method (Boyd *et al.*, 1994; Nesterov and Nemirovsky, 1994) or other existing software (Gahinet *et al.*, 1995). Therefore, here we will combine the LMI approach and Lyapunov-Krasovskii functional to investigate the exponential synchronization problem of chaotic neural networks. Distinct from previous investigations, our study focuses on the synthesis of exponential synchronization controller under the designated convergence rate, and the solvability of the synchronization problem about chaotic neural networks with time delays based on LMI approach.

Synchronization or control problem of chaotic neural networks has been extensively studied based on the common neural network model in some literature (Cheng *et al.*, 2005; Lu and Leeuwen, 2006; Zhang *et al.*, 2006; Zhou *et al.*, 2006; Lu and Cao, 2007; Sun and Cao, 2007; Sun *et al.*, 2007). However, although this common model can unify several neural networks such as Hopfield neural networks, cellular neural networks (CNNs), bidirectional associative memory (BAM) networks, it cannot describe neural networks composed of multi-layer structure such as RMLPs (Barabanov and Prokhorov, 2002) or including time-varying parameters such as Cohen-Grossberg neural networks (CGNNs) (Cohen and Grossberg, 1983). Inspired by the standard neural network model (SNNM) in (Zhang and Liu, 2005; Liu, 2006), we put forward a general chaotic neural network, which is the interconnection of a linear delayed dynamic system and a bounded static nonlinear operator. Most chaotic systems with (or without) time delays can be transformed into this general chaotic neural network to be synchronization synthesized in a unified way.

NOTATION AND PRELIMINARIES

Throughout this paper, the following notations will be used: $\mathbb{R}^n$ denotes n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices, $\boldsymbol{I}$ denotes identity matrix of appropriate order, and $*$ denotes the symmetric parts. $\lambda_{\rm M}(\boldsymbol{A})$ and $\lambda_{\rm m}(\boldsymbol{A})$ denote the maximal and minimal eigenvalues of a square matrix $\boldsymbol{A}$, respectively. $||\boldsymbol{x}||$ denotes the Euclid norm of the vector $\boldsymbol{x}$, and $||\boldsymbol{A}||$ denotes the induced norm of $\boldsymbol{A}$, that is, $\|\boldsymbol{A}\|=\sqrt{\lambda_{\rm M}(\boldsymbol{A}^{\rm T}\boldsymbol{A})}$. The notations $\boldsymbol{X}>\boldsymbol{Y}$ and $\boldsymbol{X}\geq\boldsymbol{Y}$, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are matrices of the same dimension, mean that the matrix $\boldsymbol{X}-\boldsymbol{Y}$ is positive definite and positive semi-definite, respectively. If $\boldsymbol{X}\in\mathbb{R}^p$ and $\boldsymbol{Y}\in\mathbb{R}^q$, $\boldsymbol{C}(\boldsymbol{X};\ \boldsymbol{Y})$ denotes the space of all continuous functions mapping $\mathbb{R}^p\rightarrow\mathbb{R}^q$.

In this paper, we consider the following novel chaotic delayed neural network model (Zhang and Liu, 2005; Liu, 2006):

$$\begin{cases}\dot{\boldsymbol{x}}(t)=\boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{A}_d\boldsymbol{x}(t-\tau)+\boldsymbol{B}_p\phi(\xi(t)),\\ \xi(t)=\boldsymbol{C}_q\boldsymbol{x}(t)+\boldsymbol{C}_{qd}\boldsymbol{x}(t-\tau)+\boldsymbol{D}_p\phi(\xi(t)),\end{cases}\tag{1}$$

with the initial condition function

$$\boldsymbol{x}(t)=\boldsymbol{\varpi}(t),\ \forall t\in[-\tau,0],\tag{2}$$

where $\boldsymbol{x}(t)\in\mathbb{R}^n$ is the state vector associated with the neurons, $\boldsymbol{A}\in\mathbb{R}^{n\times n}$, $\boldsymbol{A}_d\in\mathbb{R}^{n\times n}$, $\boldsymbol{B}_p\in\mathbb{R}^{n\times L}$, $\boldsymbol{C}_q\in\mathbb{R}^{L\times n}$, $\boldsymbol{C}_{qd}\in\mathbb{R}^{L\times n}$, and $\boldsymbol{D}_p\in\mathbb{R}^{L\times L}$ are the corresponding state-space matrices, $\phi\in\boldsymbol{C}(\mathbb{R}^L;\ \mathbb{R}^L)$ is nonlinear activation functions satisfying $\boldsymbol{\phi}(\boldsymbol{0})=\boldsymbol{0}$, $\boldsymbol{\xi}\in\mathbb{R}^L$ is the input vector of ϕ, $\tau>0$ is the transmission delay, $\boldsymbol{\varpi}(\cdot)$ is the given continuous function on $[-\tau, 0]$, $L\in\mathbb{R}$ is the total number of neurons in the hidden layers and output layer of the neural network.

In this paper, we assume that the activation functions in Eq.(1) are monotonically non-decreasing and globally Lipschitz. That is, there exist non-negative scalar q_i and positive scalar h_i such that

$$q_i\leq\frac{\phi_i(\alpha)-\phi_i(\beta)}{\alpha-\beta}\leq h_i,\quad i=1,\ \ldots,\ L\tag{3}$$

for arbitrary α, $\beta\in\mathbb{R}$.

Remark 1 The novel chaotic delayed neural network model (1) unifies several well-known dynamic neural networks with or without delays such as Hopfield neural networks, CNNs, BAM networks (Zhang and Liu, 2005), RMLP, etc. In Section 4, we will illustrate that these neural network models are special examples of Eq.(1). On the other hand, the system (1) reduces to a neural network without delay if $\tau=0$ or $\boldsymbol{A}_d=\boldsymbol{0}$ and $\boldsymbol{C}_{qd}=\boldsymbol{0}$.

It has been demonstrated by Gilli (1993) that dynamic neural networks can exhibit some chaotic behaviors. Therefore, if the state-space matrices and the delay parameter τ are suitably chosen, the system (1) will also display a chaotic behavior. So herein we are concerned with the synchronization problem of the system (1). Based on the drive-response concept for synchronization of coupled chaotic systems, which was initially proposed by Pecora and Carroll (1990), the corresponding response of system (1) is given in the following form:

$$\begin{cases} \dot{\boldsymbol{y}}(t) = \boldsymbol{A}\boldsymbol{y}(t) + \boldsymbol{A}_d\boldsymbol{y}(t-\tau) + \boldsymbol{B}_p\phi(\boldsymbol{\zeta}(t)) + \boldsymbol{u}(t), \\ \boldsymbol{\zeta}(t) = \boldsymbol{C}_q\boldsymbol{y}(t) + \boldsymbol{C}_{qd}\boldsymbol{y}(t-\tau) + \boldsymbol{D}_p\phi(\boldsymbol{\zeta}(t)), \end{cases} \tag{4}$$

with the initial condition function

$$\boldsymbol{y}(t) = \boldsymbol{\sigma}(t), \ \forall t \in [-\tau, 0], \tag{5}$$

where $\boldsymbol{y}(t)\in\mathbb{R}^n$ is the state vector, $\boldsymbol{\sigma}(\cdot)$ is the given continuous function on $[-\tau, 0]$, $\boldsymbol{u}(t)\in\mathbb{R}^n$ is the state feedback controller given to achieve the exponential synchronization between drive-response system.

In order to investigate the problem of exponential synchronization between the systems (1) and (4), we define the synchronization error signal as $\boldsymbol{e}(t)=\boldsymbol{x}(t)-\boldsymbol{y}(t)$, where $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ are the state variables of drive system (1) and response system (4), respectively. Then, the error dynamical system between (1) and (4) is given as follows:

$$\begin{cases} \dot{\boldsymbol{e}}(t) = \boldsymbol{A}\boldsymbol{e}(t) + \boldsymbol{A}_d\boldsymbol{e}(t-\tau) + \boldsymbol{B}_p\psi(\boldsymbol{\eta}(t)) - \boldsymbol{u}(t), \\ \boldsymbol{\eta}(t) = \boldsymbol{C}_q\boldsymbol{e}(t) + \boldsymbol{C}_{qd}\boldsymbol{e}(t-\tau) + \boldsymbol{D}_p\psi(\boldsymbol{\eta}(t)), \end{cases} \tag{6}$$

where $\boldsymbol{e}(t)\in\mathbb{R}^n$, and $\boldsymbol{\eta}(t)=\boldsymbol{\xi}(t)-\boldsymbol{\zeta}(t)$, $\psi(\boldsymbol{\eta}(t))=\phi(\boldsymbol{\xi}(t))-\phi(\boldsymbol{\zeta}(t))=\phi(\boldsymbol{\eta}(t)+\boldsymbol{\zeta}(t))-\phi(\boldsymbol{\zeta}(t))$, therefore $\psi(\boldsymbol{0})=\boldsymbol{0}$. Since all the $\phi_i(\cdot)$ are globally Lipschitz, $\psi_i(\cdot)$ satisfy the sector conditions, i.e., for each $i=1, \ldots, L$

$$q_i \le \psi_i(\eta_i(t))/\eta_i(t) \le h_i \ \text{or} \\ [\psi_i(\eta_i(t))-q_i\eta_i(t)]\cdot[\psi_i(\eta_i(t))-h_i\eta_i(t)] \le 0. \tag{7}$$

If the state variables of the drive system (1) are used to drive the response system (4), the control input vector with hybrid feedback is designed as follows:

$$\boldsymbol{u}(t) = \boldsymbol{K}_1\boldsymbol{e}(t) + \boldsymbol{K}_2\boldsymbol{e}(t-\tau) + \boldsymbol{K}_3\psi(\boldsymbol{\eta}(t)), \tag{8}$$

where $\boldsymbol{K}_1$, $\boldsymbol{K}_2$ and $\boldsymbol{K}_3\in\mathbb{R}^{n\times n}$ are feedback gain parameters to be scheduled. With the control law Eq.(8), the error dynamics can be expressed as follows:

$$\begin{cases} \dot{\boldsymbol{e}}(t) = (\boldsymbol{A}-\boldsymbol{K}_1)\boldsymbol{e}(t) + (\boldsymbol{A}_d-\boldsymbol{K}_2)\boldsymbol{e}(t-\tau) \\ \qquad\quad + (\boldsymbol{B}_p-\boldsymbol{K}_3)\psi(\boldsymbol{\eta}(t)), \\ \boldsymbol{\eta}(t) = \boldsymbol{C}_q\boldsymbol{e}(t) + \boldsymbol{C}_{qd}\boldsymbol{e}(t-\tau) + \boldsymbol{D}_p\psi(\boldsymbol{\eta}(t)). \end{cases} \tag{9}$$

Since $\psi(\boldsymbol{0})=\boldsymbol{0}$, the system (9) admits a trivial solution $\boldsymbol{e}(t)\equiv\boldsymbol{0}$.

Remark 2 Many synchronization methods can be seen as special cases of Eq.(8). While $\boldsymbol{K}_2=\boldsymbol{K}_3=\boldsymbol{0}$, the control law $\boldsymbol{u}(t)$ is the linear combinations of the differences between the states of Eqs.(1) and (4), i.e., $\boldsymbol{u}(t)=\boldsymbol{K}_1\boldsymbol{e}(t)$, which should be referred to as a state feedback controller. This state feedback method is common in the literature about synchronization of chaotic systems. While $\boldsymbol{K}_3=\boldsymbol{0}$, the controller includes time-delay state feedback term $\boldsymbol{K}_2\boldsymbol{e}(t-\tau)$. As discussed in (Wang *et al*., 2001; Liao and Chen, 2003; Sun *et al*., 2007), such a time-delayed feedback term could be employed to tackle more general systems and achieve less conservative analysis results than those results without using the delayed feedback term. While $\boldsymbol{K}_1=\boldsymbol{K}_2=\boldsymbol{0}$, the control law $\boldsymbol{u}(t)$ is set to be linear combinations of the differences between the output of Eqs.(1) and (4), i.e., $\boldsymbol{u}(t)=\boldsymbol{K}_3\psi(\boldsymbol{\eta}(t))$, which should be referred to as an output feedback controller. Synchronization via output feedback is not considered in most literature except (Lu and Leeuwen, 2006). However, this is important because in real systems only output signals can be measured.

Before stating the main results, we first need the following definition:

Definition 1 (Cheng *et al*., 2005) The drive system (1) and the response system (4) are said to be exponentially synchronized if, for a suitably designed feedback controller, there exist constants $\lambda\ge1$ and $\gamma>0$ such that $\|\boldsymbol{x}(t)-\boldsymbol{y}(t)\|\le\lambda\|\boldsymbol{x}(0)-\boldsymbol{y}(0)\|\exp(-\gamma t)$, for any $t\ge0$. Moreover, the constant γ is defined as the exponential synchronization rate.

In fact, the exponential synchronization problem considered in this paper is to determine the control input $\boldsymbol{u}(t)$ to synchronize the two identical chaotic delayed neural networks exponentially with the same system parameters, but different initial conditions. It is clear that, if the trivial solution of the controlled error dynamical system (9) is exponentially stable, the exponential synchronization between the drive system (1) and the response system (4) can be realized.

MAIN RESULTS

We have cast the problem of global exponential synchronization as a stability problem of the system (9). This means that we can use Lyapunov functionals and estimation techniques similar to those in (Liu, 2007) to develop theoretical conditions for global exponential synchronization. With these, we can provide design rules for the controller feedback gain matrices $\boldsymbol{K}_1$, $\boldsymbol{K}_2$ and $\boldsymbol{K}_3$ to ensure global synchronization.

Theorem 1 If there exist positive definite matrices $\boldsymbol{P}$, $\boldsymbol{\Gamma}$, diagonal semi-positive definite matrix $\boldsymbol{\Sigma}$, a positive scalar γ, and feedback gain matrices $\boldsymbol{K}_1$, $\boldsymbol{K}_2$ and $\boldsymbol{K}_3$ in error dynamical system (9) such that the following nonlinear matrix inequality

$$\boldsymbol{G}=\begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{G}_{12} & \boldsymbol{G}_{13} \\ * & \boldsymbol{G}_{22} & \boldsymbol{G}_{23} \\ * & * & \boldsymbol{G}_{33} \end{bmatrix} < \boldsymbol{0} \tag{10}$$

holds, then the drive system (1) and the response system (4) can be globally exponentially synchronized. The submatrices of $\boldsymbol{G}$ are:

$$\boldsymbol{G}_{11}=(\boldsymbol{A}-\boldsymbol{K}_1)^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}(\boldsymbol{A}-\boldsymbol{K}_1)+2\gamma\boldsymbol{P}+\boldsymbol{\Gamma},$$
$$\boldsymbol{G}_{12}=\boldsymbol{P}(\boldsymbol{A}_d-\boldsymbol{K}_2),\ \boldsymbol{G}_{13}=\boldsymbol{P}(\boldsymbol{B}_p-\boldsymbol{K}_3)+\boldsymbol{C}_q^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{H})\boldsymbol{\Sigma},$$
$$\boldsymbol{G}_{22}=-\exp(-2\gamma\tau)\boldsymbol{\Gamma},\ \boldsymbol{G}_{23}=\boldsymbol{C}_{qd}^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{H})\boldsymbol{\Sigma},$$
$$\boldsymbol{G}_{33}=\boldsymbol{D}_p^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{H})\boldsymbol{\Sigma}+\boldsymbol{\Sigma}(\boldsymbol{Q}+\boldsymbol{H})\boldsymbol{D}_p-2\boldsymbol{T},$$

where

$$\boldsymbol{Q}=\mathrm{diag}(q_1,q_2,\cdots,q_L),\ \ \boldsymbol{H}=\mathrm{diag}(h_1,h_2,\cdots,h_L).$$

Proof For the error dynamical system (9), we define a positive definite Lyapunov-Krasovskii functional as

$$V(\boldsymbol{e}(t))=\int_{-\tau}^{0}\exp(2\gamma(t+\theta))\boldsymbol{e}^{\mathrm{T}}(t+\theta)\boldsymbol{\Gamma}\boldsymbol{e}(t+\theta)\mathrm{d}\theta + \exp(2\gamma t)\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}(t), \tag{11}$$

where $\boldsymbol{P}=\boldsymbol{P}^{\mathrm{T}}>\boldsymbol{0}$, $\boldsymbol{\Gamma}=\boldsymbol{\Gamma}^{\mathrm{T}}>\boldsymbol{0}$, and $\gamma>0$. The derivative of $V(\boldsymbol{e}(t))$ along the solution of the system (9) is

$$\begin{aligned}\frac{\mathrm{d}V(\boldsymbol{e}(t))}{\mathrm{d}t}&=2\gamma\exp(2\gamma t)\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}(t)+\\
&2\exp(2\gamma t)\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}[(\boldsymbol{A}-\boldsymbol{K}_1)\boldsymbol{e}(t)+(\boldsymbol{A}_d-\boldsymbol{K}_2)\boldsymbol{e}(t-\tau)+\\
&(\boldsymbol{B}_p-\boldsymbol{K}_3)\psi(\boldsymbol{\eta}(t))]+\exp(2\gamma t)\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{\Gamma}\boldsymbol{e}(t)-\\
&\exp(2\gamma(t-\tau))\boldsymbol{e}^{\mathrm{T}}(t-\tau)\boldsymbol{\Gamma}\boldsymbol{e}(t-\tau)\\
&=\exp(2\gamma t)\big\{\boldsymbol{e}^{\mathrm{T}}(t)[(\boldsymbol{A}-\boldsymbol{K}_1)^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}(\boldsymbol{A}-\boldsymbol{K}_1)+\\
&2\gamma\boldsymbol{P}+\boldsymbol{\Gamma}]\boldsymbol{e}(t)+\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}(\boldsymbol{A}_d-\boldsymbol{K}_2)\boldsymbol{e}(t-\tau)+\\
&\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}(\boldsymbol{B}_p-\boldsymbol{K}_3)\psi(\boldsymbol{\eta}(t))+\boldsymbol{e}^{\mathrm{T}}(t-\tau)(\boldsymbol{A}_d-\boldsymbol{K}_2)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{e}(t)-\\
&\exp(-2\gamma\tau)\boldsymbol{e}^{\mathrm{T}}(t-\tau)\boldsymbol{\Gamma}\boldsymbol{e}(t-\tau)+\psi^{\mathrm{T}}(\boldsymbol{\eta}(t))(\boldsymbol{B}_p-\boldsymbol{K}_3)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{e}(t)\big\}\\
&=\exp(2\gamma t)\begin{bmatrix}\boldsymbol{e}(t)\\ \boldsymbol{e}(t-\tau)\\ \psi(\boldsymbol{\eta}(t))\end{bmatrix}^{\mathrm{T}}\boldsymbol{R}_0\begin{bmatrix}\boldsymbol{e}(t)\\ \boldsymbol{e}(t-\tau)\\ \psi(\boldsymbol{\eta}(t))\end{bmatrix},\end{aligned} \tag{12}$$

where

$$\boldsymbol{R}_0=\begin{bmatrix}\boldsymbol{N} & \boldsymbol{P}(\boldsymbol{A}_d-\boldsymbol{K}_2) & \boldsymbol{P}(\boldsymbol{B}_p-\boldsymbol{K}_3)\\ (\boldsymbol{A}_d-\boldsymbol{K}_2)^{\mathrm{T}}\boldsymbol{P} & -\exp(-2\gamma\tau)\boldsymbol{\Gamma} & \boldsymbol{0}\\ (\boldsymbol{B}_p-\boldsymbol{K}_3)^{\mathrm{T}}\boldsymbol{P} & \boldsymbol{0} & \boldsymbol{0}\end{bmatrix},$$

and

$$\boldsymbol{N}=(\boldsymbol{A}-\boldsymbol{K}_1)^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}(\boldsymbol{A}-\boldsymbol{K}_1)+2\gamma\boldsymbol{P}+\boldsymbol{\Gamma}.$$

The sector conditions (7) can be rewritten as follows:

$$\psi_i^2(\eta_i(t))-\psi_i(\eta_i(t))(q_i+h_i)\eta_i(t)+q_ih_i\eta_i^2(t)\le 0.$$

Since $q_ih_i\eta_i^2(t)\ge 0$, we can obtain

$$\psi_i^2(\eta_i(t))-\psi_i(\eta_i(t))(q_i+h_i)\eta_i(t)\le 0,$$

which is equivalent to

$$\begin{aligned}&2\psi_i^2(\eta_i(t))-2\psi_i(\eta_i(t))(q_i+h_i)\boldsymbol{C}_{q,i}\boldsymbol{e}(t)-\\
&2\psi_i(\eta_i(t))(q_i+h_i)\boldsymbol{C}_{qd,i}\boldsymbol{e}(t-\tau)-\\
&2\psi_i(\eta_i(t))(q_i+h_i)\boldsymbol{D}_{p,i}\psi(\boldsymbol{\eta}(t))\le 0,\end{aligned} \tag{13}$$

where $\boldsymbol{C}_{q,i}$ denotes the ith row of $\boldsymbol{C}_q$, $\boldsymbol{C}_{qd,i}$ denotes the ith row of $\boldsymbol{C}_{qd}$, $\boldsymbol{D}_{p,i}$ denotes the ith row of $\boldsymbol{D}_p$. We rewrite inequality (13) in matrix notation as follows:

$$\begin{bmatrix} \boldsymbol{e}(t) \\ \boldsymbol{e}(t-\tau) \\ \psi_1(\eta_1(t)) \\ \vdots \\ \psi_{i-1}(\eta_{i-1}(t)) \\ \psi_i(\eta_i(t)) \\ \psi_{i+1}(\eta_{i+1}(t)) \\ \vdots \\ \psi_L(\eta_L(t)) \end{bmatrix}^{\mathrm{T}} \boldsymbol{R}_i \begin{bmatrix} \boldsymbol{e}(t) \\ \boldsymbol{e}(t-\tau) \\ \psi_1(\eta_1(t)) \\ \vdots \\ \psi_{i-1}(\eta_{i-1}(t)) \\ \psi_i(\eta_i(t)) \\ \psi_{i+1}(\eta_{i+1}(t)) \\ \vdots \\ \psi_L(\eta_L(t)) \end{bmatrix} \le \boldsymbol{0}, \quad (14)$$

where i=1, …, L and $\boldsymbol{R}_i$ is shown below.

In the expression of $\boldsymbol{R}_i$, $s_i=q_i+h_i$, $d_{p,i,j}$ is the entry of the matrix $\boldsymbol{D}_p$ at the ith row and jth column. By the S-procedure (Boyd *et al.*, 1994), if there exist $\tau_i\ge 0$ (i=1, 2, …, L), such that the inequality (15) (shown at the bottom of this page) holds. In inequality (15), $\boldsymbol{\Sigma}=\mathrm{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_L)$, and $\boldsymbol{\Sigma}\ge\boldsymbol{0}$, then $\boldsymbol{R}_0<\boldsymbol{0}$, that is, $\mathrm{d}V(\boldsymbol{e}(t))/\mathrm{d}t\le\boldsymbol{0}$, therefore, $V(\boldsymbol{e}(t))\le V(\boldsymbol{e}(0))$. However,

$$V(\boldsymbol{e}(0)) = \boldsymbol{e}^{\mathrm{T}}(0)\boldsymbol{P}\boldsymbol{e}(0) + \int_{-\tau}^{0}\exp(2\gamma\theta)\boldsymbol{e}^{\mathrm{T}}(\theta)\boldsymbol{\Gamma}\boldsymbol{e}(\theta)\mathrm{d}\theta$$
$$\le \lambda_{\mathrm{M}}(\boldsymbol{P})\|\boldsymbol{e}(0)\|^2 + \lambda_{\mathrm{M}}(\boldsymbol{\Gamma})\int_{-\tau}^{0}\exp(2\gamma\theta)\|\boldsymbol{e}(0)\|^2\,\mathrm{d}\theta$$
$$\le \lambda_{\mathrm{M}}(\boldsymbol{P})\|\boldsymbol{e}(0)\|^2 + \lambda_{\mathrm{M}}(\boldsymbol{\Gamma})\|\boldsymbol{e}(0)\|^2\frac{\|\boldsymbol{\Omega}\|^2}{\|\boldsymbol{e}(0)\|^2}$$
$$\cdot\int_{-\tau}^{0}\exp(2\gamma\theta)\mathrm{d}\theta$$
$$=\left[\lambda_{\mathrm{M}}(\boldsymbol{P}) + \lambda_{\mathrm{M}}(\boldsymbol{\Gamma})\frac{\|\boldsymbol{\Omega}\|^2}{\|\boldsymbol{e}(0)\|^2}\frac{1-\exp(-2\gamma\tau)}{2\gamma}\right]\|\boldsymbol{e}(0)\|^2,$$

where

$$\|\boldsymbol{\Omega}\| = \sup_{-\tau\le\theta\le 0}\|\boldsymbol{e}(\theta)\|,$$

$$V(\boldsymbol{e}(t)) \ge \exp(2\gamma t)\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}(t) \ge \exp(2\gamma t)\lambda_{\mathrm{m}}(\boldsymbol{P})\|\boldsymbol{e}(t)\|^2,$$

therefore the convergence rates of the error states between the drive system (1) and the response system (4) are

$$\|\boldsymbol{e}(t)\| = \sqrt{\frac{\lambda_{\mathrm{M}}(\boldsymbol{P})}{\lambda_{\mathrm{m}}(\boldsymbol{P})} + \frac{\lambda_{\mathrm{M}}(\boldsymbol{\Gamma})}{\lambda_{\mathrm{m}}(\boldsymbol{P})}\frac{\|\boldsymbol{\Omega}\|^2}{\|\boldsymbol{e}(0)\|^2}\frac{1-\exp(-2\gamma\tau)}{2\gamma}}$$
$$\cdot\|\boldsymbol{e}(0)\|^2\exp(-\gamma t). \quad (16)$$

From Definition 1, it is concluded that the drive system (1) and the response system (4) are globally exponentially synchronized. This completes the proof.

$$\boldsymbol{R}_i = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & 0 & \cdots & 0 & -\boldsymbol{C}_{q,i}^{\mathrm{T}}s_i & 0 & \cdots & 0 \\ \boldsymbol{0} & \boldsymbol{0} & 0 & \cdots & 0 & -\boldsymbol{C}_{qd,i}^{\mathrm{T}}s_i & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & -d_{p,i,1}s_i & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -d_{p,i,i-1}s_i & 0 & \cdots & 0 \\ -s_i\boldsymbol{C}_{q,i} & -s_i\boldsymbol{C}_{qd,i} & -s_id_{p,i,1} & \cdots & -s_id_{p,i,i-1} & 2-2s_id_{p,i,i} & s_id_{p,i,i+1} & \cdots & s_id_{p,i,L} \\ 0 & 0 & 0 & \cdots & 0 & d_{p,i,i+1}s_i & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & d_{p,i,L}s_i & 0 & 0 & 0 \end{bmatrix}.$$

$$\boldsymbol{R}_0 - \sum_{i=1}^{L}\varepsilon_i\boldsymbol{R}_i = \begin{bmatrix} (\boldsymbol{A}-\boldsymbol{K}_1)^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A}-\boldsymbol{K}_1) + 2\gamma\boldsymbol{P} + \boldsymbol{\Gamma} & \boldsymbol{P}(\boldsymbol{A}_d-\boldsymbol{K}_2) & \boldsymbol{P}\boldsymbol{B}_p \\ (\boldsymbol{A}_d-\boldsymbol{K}_2)^{\mathrm{T}}\boldsymbol{P} & -\exp(-2\gamma\tau)\boldsymbol{\Gamma} & \boldsymbol{0} \\ \boldsymbol{B}_p^{\mathrm{T}}\boldsymbol{P} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$$
$$-\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{C}_q^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{H})\boldsymbol{\Sigma} \\ \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{C}_{qd}^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{H})\boldsymbol{\Sigma} \\ -\boldsymbol{\Sigma}(\boldsymbol{Q}+\boldsymbol{H})\boldsymbol{C}_q & -\boldsymbol{\Sigma}(\boldsymbol{Q}+\boldsymbol{H})\boldsymbol{C}_q & 2\boldsymbol{\Sigma} - \boldsymbol{D}_p^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{H})\boldsymbol{\Sigma} - \boldsymbol{\Sigma}(\boldsymbol{Q}+\boldsymbol{H})\boldsymbol{D}_p \end{bmatrix} \quad (15)$$
$$=\begin{bmatrix} (\boldsymbol{A}-\boldsymbol{K}_1)^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A}-\boldsymbol{K}_1) + 2\gamma\boldsymbol{P} + \boldsymbol{\Gamma} & \boldsymbol{P}(\boldsymbol{A}_d-\boldsymbol{K}_2) & \boldsymbol{P}(\boldsymbol{B}_p-\boldsymbol{K}_3) + \boldsymbol{C}_q^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{H})\boldsymbol{\Sigma} \\ (\boldsymbol{A}_d-\boldsymbol{K}_2)^{\mathrm{T}}\boldsymbol{P} & -\exp(-2\gamma\tau)\boldsymbol{\Gamma} & \boldsymbol{C}_{qd}^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{H})\boldsymbol{\Sigma} \\ (\boldsymbol{B}_p-\boldsymbol{K}_3)^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{\Sigma}(\boldsymbol{Q}+\boldsymbol{H})\boldsymbol{C}_q & \boldsymbol{\Sigma}(\boldsymbol{Q}+\boldsymbol{H})\boldsymbol{C}_q & \boldsymbol{D}_p^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{H})\boldsymbol{\Sigma} + \boldsymbol{\Sigma}(\boldsymbol{Q}+\boldsymbol{H})\boldsymbol{D}_p - 2\boldsymbol{\Sigma} \end{bmatrix}$$
$$=\boldsymbol{G}<\boldsymbol{0}.$$

Eq.(10) in Theorem 1 is nonlinear matrix inequality with respect to the parameters $\boldsymbol{P}$, $\boldsymbol{\Gamma}$, $\boldsymbol{\Sigma}$, γ, $\boldsymbol{K}_1$, $\boldsymbol{K}_2$, and $\boldsymbol{K}_3$. There are no efficient algorithms and computer software to solve this inequality. To address this problem, we will derive some exponential synchronization conditions for the systems (1) and (4), which are represented in the form of LMIs, which can be efficiently solved using existing algorithms.

Theorem 2 If there exist positive definite matrices $\boldsymbol{X}$ and $\boldsymbol{S}$, diagonal semi-positive definite matrix $\boldsymbol{T}$, a positive scalar γ, and matrices $\boldsymbol{Y}_1$, $\boldsymbol{Y}_2$ and $\boldsymbol{Y}_3$, which satisfy the following matrix inequality:

$$\boldsymbol{M}=\begin{bmatrix}\boldsymbol{M}_{11} & \boldsymbol{M}_{12} & \boldsymbol{M}_{13}\\ * & \boldsymbol{M}_{22} & \boldsymbol{M}_{23}\\ * & * & \boldsymbol{M}_{33}\end{bmatrix}<\boldsymbol{0}, \tag{17}$$

then the drive system (1) and the response system (4) can be synchronized with an exponential synchronization rate of γ. Moreover, the feedback gains are obtained as $\boldsymbol{K}_1=\boldsymbol{Y}_1\boldsymbol{X}^{-1}$, $\boldsymbol{K}_2=\boldsymbol{Y}_2\boldsymbol{X}^{-1}$, and $\boldsymbol{K}_3=\boldsymbol{Y}_3\boldsymbol{T}^{-1}$. The submatrices of $\boldsymbol{M}$ are:

$$\begin{aligned}&\boldsymbol{M}_{11}=(\boldsymbol{AX}-\boldsymbol{Y}_1)^{\mathrm{T}}+\boldsymbol{AX}-\boldsymbol{Y}_1+2\gamma\boldsymbol{X}+\boldsymbol{S},\\ &\boldsymbol{M}_{12}=\boldsymbol{A}_d\boldsymbol{X}-\boldsymbol{Y}_2,\ \boldsymbol{M}_{13}=\boldsymbol{B}_p\boldsymbol{T}-\boldsymbol{Y}_3+\boldsymbol{XC}_q^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{H}),\\ &\boldsymbol{M}_{22}=-\exp(-2\gamma\tau)\boldsymbol{S},\ \boldsymbol{M}_{23}=\boldsymbol{XC}_{qd}^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{H}),\\ &\boldsymbol{M}_{33}=\boldsymbol{TD}_p^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{H})+(\boldsymbol{Q}+\boldsymbol{H})\boldsymbol{D}_p\boldsymbol{T}-2\boldsymbol{T},\end{aligned}$$

where

$$\boldsymbol{Q}=\mathrm{diag}(q_1,q_2,\cdots,q_L),\ \boldsymbol{H}=\mathrm{diag}(h_1,h_2,\cdots,h_L).$$

Proof Pre- and post-multiplying the left-hand side matrix of Eq.(10) by diag($\boldsymbol{P}^{-1}$, $\boldsymbol{P}^{-1}$, $\boldsymbol{\Sigma}^{-1}$), Eq.(10) is equivalent to

$$\begin{bmatrix}\boldsymbol{P}_{11} & \boldsymbol{P}_{12} & \boldsymbol{P}_{13}\\ * & \boldsymbol{P}_{22} & \boldsymbol{P}_{23}\\ * & * & \boldsymbol{P}_{33}\end{bmatrix}<\boldsymbol{0}, \tag{18}$$

where

$$\begin{aligned}&\boldsymbol{P}_{11}=\boldsymbol{P}^{-1}(\boldsymbol{A}-\boldsymbol{K}_1)^{\mathrm{T}}+(\boldsymbol{A}-\boldsymbol{K}_1)\boldsymbol{P}^{-1}+2\gamma\boldsymbol{P}^{-1}+\boldsymbol{P}^{-1}\boldsymbol{\Gamma}\boldsymbol{P}^{-1},\\ &\boldsymbol{P}_{12}=(\boldsymbol{A}_d-\boldsymbol{K}_2)\boldsymbol{P}^{-1},\ \boldsymbol{P}_{22}=-\exp(-2\gamma\tau)\boldsymbol{P}^{-1}\boldsymbol{\Gamma}\boldsymbol{P}^{-1},\\ &\boldsymbol{P}_{13}=(\boldsymbol{B}_p-\boldsymbol{K}_3)\boldsymbol{\Sigma}^{-1}+\boldsymbol{P}^{-1}\boldsymbol{C}_q^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{H}),\\ &\boldsymbol{P}_{23}=\boldsymbol{P}^{-1}\boldsymbol{C}_{qd}^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{H}),\\ &\boldsymbol{P}_{33}=\boldsymbol{\Sigma}^{-1}\boldsymbol{D}_p^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{H})+(\boldsymbol{Q}+\boldsymbol{H})\boldsymbol{D}_p\boldsymbol{\Sigma}^{-1}-2\boldsymbol{\Sigma}^{-1}.\end{aligned}$$

Let $\boldsymbol{X}=\boldsymbol{P}^{-1}$, $\boldsymbol{Y}_1=\boldsymbol{K}_1\boldsymbol{X}$, $\boldsymbol{Y}_2=\boldsymbol{K}_2\boldsymbol{X}$, $\boldsymbol{T}=\boldsymbol{\Sigma}^{-1}$, $\boldsymbol{Y}_3=\boldsymbol{K}_3\boldsymbol{T}$, and $\boldsymbol{S}=\boldsymbol{P}^{-1}\boldsymbol{\Gamma}\boldsymbol{P}^{-1}$, Eq.(18) is rewritten as Eq.(17). Hence, if Eq.(17) holds, Eq.(10) is also satisfied. According to Theorem 1, we can judge that the drive system (1) and the response system (4) are globally exponentially synchronized. The proof of Theorem 2 is thus completed.

ILLUSTRATIVE EXAMPLES

In this section, we investigate the problem of global exponential synchronization for two chaotic delayed neural networks: CNN and RMLP, and compare our results with those in other literature in some detail. The simulation results demonstrate that our approaches are not only very convenient to implement in practice, but also further extend the ideas and techniques presented in recent literature.

Example 1 Consider the following delayed CNN (Gilli, 1993; Cheng *et al.*, 2005):

$$\begin{aligned}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=&-\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}1+\pi/4 & 20\\ 0.1 & 1+\pi/4\end{bmatrix}\begin{bmatrix}f(x_1(t))\\ f(x_2(t))\end{bmatrix}\\ &+\begin{bmatrix}-13\sqrt{2}\pi/40 & 0.1\\ 0.1 & -13\sqrt{2}\pi/40\end{bmatrix}\begin{bmatrix}f(x_1(t-1))\\ f(x_2(t-1))\end{bmatrix},\end{aligned} \tag{19}$$

where $f(x_i(t))=(|x_i(t)+1|-|x_i(t)-1|)/2$, i=1, 2. As shown in Fig.1, the CNN Eq.(19) possesses a chaotic behavior (Cheng *et al.*, 2005; Zhou *et al.*, 2006). Now the response chaotic delayed CNN is designed as follows:

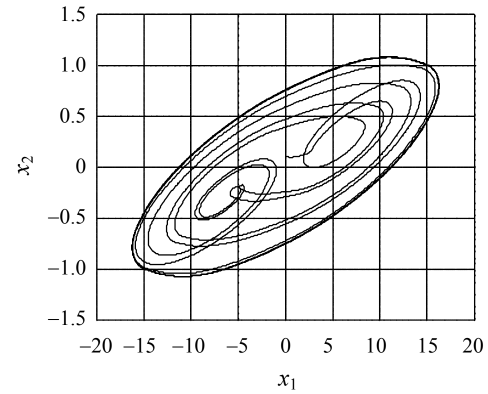

Fig.1 The chaotic behavior of the delayed CNN Eq.(19) with the initial condition $[x_1(s)\ x_2(s)]^{\mathrm{T}}$=[0.1 0.1]$^{\mathrm{T}}$ for −1≤s≤0, in Example 1

$$\begin{bmatrix}\dot{y}_1(t)\\ \dot{y}_2(t)\end{bmatrix}=-\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix}+\begin{bmatrix}1+\pi/4 & 20\\ 0.1 & 1+\pi/4\end{bmatrix}\begin{bmatrix}f(y_1(t))\\ f(y_2(t))\end{bmatrix}+$$
$$\begin{bmatrix}-13\sqrt{2}\pi/40 & 0.1\\ 0.1 & -13\sqrt{2}\pi/40\end{bmatrix}\begin{bmatrix}f(y_1(t-1))\\ f(y_2(t-1))\end{bmatrix}+\boldsymbol{u}(t). \quad (20)$$

We convert the CNN Eq.(20) into the system (4), where

$$\boldsymbol{y}=[y_1(t)\quad y_2(t)]^{\mathrm{T}},\ \boldsymbol{A}=\mathrm{diag}(-1,-1),\ \boldsymbol{A}_d=\boldsymbol{0}_{3\times 3},$$
$$\boldsymbol{B}_p=\begin{bmatrix}1+\pi/4 & 20 & -13\sqrt{2}\pi/40 & 0.1\\ 0.1 & 1+\pi/4 & 0.1 & -13\sqrt{2}\pi/40\end{bmatrix},$$
$$\boldsymbol{C}_q=\begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix},\ \boldsymbol{C}_{qd}=\begin{bmatrix}0&0\\0&0\\1&0\\0&1\end{bmatrix},$$

$\boldsymbol{D}_p=\boldsymbol{0}_{2\times 2}$, $\boldsymbol{Q}=\boldsymbol{0}_{4\times 4}$, $\boldsymbol{H}=\boldsymbol{I}_{4\times 4}$, $\phi_i(x_i)=f_i(x_i)$, $i=1, 2$.

Output feedback controller, where $\boldsymbol{K}_1=\boldsymbol{K}_2=\boldsymbol{0}$ in Eq.(8), is used to synchronize the delayed CNNs Eqs.(19) and (20), where $\boldsymbol{u}(t)\in\mathbb{R}^2$ and $\boldsymbol{K}_3\in\mathbb{R}^{2\times 4}$. According to Theorem 2, solving the LMI Eq.(17) with the constraints: $Y_1=0$, $Y_2=0$, and $\gamma=0.95$, we obtain the solutions of (17) and the feedback gain as:

$$\boldsymbol{X}=\begin{bmatrix}0.4765 & 0.0000\\ 0.0000 & 0.4765\end{bmatrix},\ \boldsymbol{S}=\begin{bmatrix}0.0351 & 0.0000\\ 0.0000 & 0.0351\end{bmatrix},$$
$$\boldsymbol{T}=\mathrm{diag}(8.9943,\ 8.9943,\ 24.3168,\ 24.3168),$$
$$\boldsymbol{Y}_3=\begin{bmatrix}16.5350 & 179.8865 & -35.1120 & 2.4317\\ 0.8994 & 16.5350 & 2.4317 & -35.1120\end{bmatrix},$$
$$\boldsymbol{K}_3=\begin{bmatrix}1.8384 & 20.0000 & -1.4439 & 0.1000\\ 0.1000 & 1.8384 & 0.1000 & -1.4439\end{bmatrix}.$$

Fig.2 depicts the synchronization errors of the state variables between the drive system (19) and the response (20) with the initial condition $[x_1(s)\ x_2(s)]^{\mathrm{T}}=[0.1\ 0.1]^{\mathrm{T}}$ and $[0.2\ -0.2]^{\mathrm{T}}$, for $-1\le s\le 0$, respectively. Comparing our method with that in (Cheng *et al.*, 2005), we can see that the latter state feedback gain is obtained by trial, while our controller parameters can be solved by MATLAB LMI Control Toolbox (Gahinet *et al.*, 1995). In addition, if the states are unmeasurable, the state feedback method in (Cheng *et al.*, 2005) cannot be applied in engineering. On the other hand, we can get the well-designed controller according to our requirement of performance, or implementation in practice. It is flexible and easy to change some parameters in Theorem 2, such as γ, $\boldsymbol{K}_1$, $\boldsymbol{K}_2$, and $\boldsymbol{K}_3$, to satisfy such requirement.

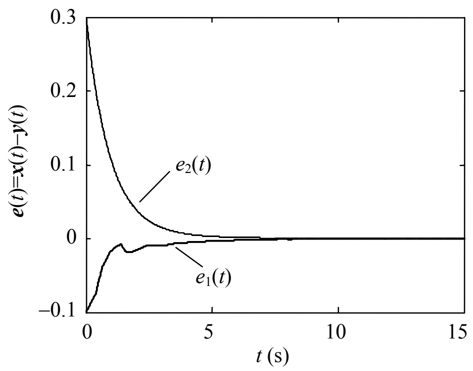

Fig.2 The waveform of synchronization errors $e_1(t)=x_1(t)-y_1(t)$ and $e_2(t)=x_2(t)-y_2(t)$ in Example 1

Example 2 We consider the following chaotic delayed RMLP:

$$\dot{\boldsymbol{x}}(t)=\tanh\{\boldsymbol{W}_A\tanh[\boldsymbol{V}_A\boldsymbol{x}(t)]+\boldsymbol{W}_B\tanh[\boldsymbol{V}_B\boldsymbol{x}(t-1)]\}, \quad (21)$$

with the initial value $x_1(0)=x_2(0)=x_3(0)=0.2$, where

$$\boldsymbol{W}_A=\begin{bmatrix}3&0.02&0\\0&3&0\\0&0&1\end{bmatrix},\ \boldsymbol{V}_A=\begin{bmatrix}1&0&0\\0&1.5&0\\0&0&2\end{bmatrix},$$
$$\boldsymbol{W}_B=\begin{bmatrix}-5.8&0&0.785\\0&-5.8&0.785\\0&0&-5.8\end{bmatrix},\ \boldsymbol{V}_B=\begin{bmatrix}1&0&1\\0&1.5&0\\1&1.2&2\end{bmatrix}.$$

The behavior of the chaotic RMLP Eq.(21) is shown in Fig.3. The response chaotic delayed RMLP is designed as follows:

$$\dot{\boldsymbol{y}}(t)=\tanh\{\boldsymbol{W}_A\tanh[\boldsymbol{V}_A\boldsymbol{y}(t)]+\boldsymbol{W}_B\tanh[\boldsymbol{V}_B\boldsymbol{y}(t-1)]\}+\boldsymbol{u}(t), \quad (22)$$

We transform the delayed RMLP Eq.(22) into the system (4), where

$$\boldsymbol{A}=\boldsymbol{A}_d=\boldsymbol{0}_{3\times 3},\ \boldsymbol{B}_p=\begin{bmatrix}\boldsymbol{I}_{3\times 3} & \boldsymbol{0}_{3\times 3} & \boldsymbol{0}_{3\times 3}\end{bmatrix},$$
$$\boldsymbol{C}_q=\begin{bmatrix}\boldsymbol{0}_{3\times 3}\\ \boldsymbol{V}_A\\ \boldsymbol{0}_{3\times 3}\end{bmatrix},\ \boldsymbol{C}_{qd}=\begin{bmatrix}\boldsymbol{0}_{3\times 3}\\ \boldsymbol{0}_{3\times 3}\\ \boldsymbol{V}_B\end{bmatrix},\ \boldsymbol{D}_p=\begin{bmatrix}\boldsymbol{0}_{3\times 3} & \boldsymbol{W}_A & \boldsymbol{W}_B\\ \boldsymbol{0}_{3\times 3} & \boldsymbol{0}_{3\times 3} & \boldsymbol{0}_{3\times 3}\\ \boldsymbol{0}_{3\times 3} & \boldsymbol{0}_{3\times 3} & \boldsymbol{0}_{3\times 3}\end{bmatrix},$$

$\boldsymbol{Q}=\boldsymbol{0}_{9\times 9}$, $\boldsymbol{H}=\boldsymbol{I}_{9\times 9}$, $\phi_i(x_i)=\tanh x_i$, $i=1, 2, \ldots, 9$.

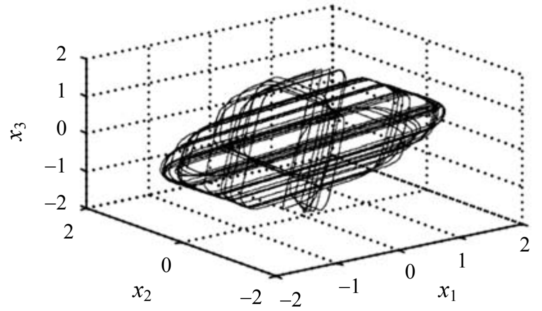

Fig.3 The chaotic behavior of the delayed RMLP Eq.(21) with the initial condition $[x_1(s)\ x_2(s)\ x_3(s)]^T=[0.2\ 0.2\ 0.2]^T$, for $-1\le s\le 0$, in Example 2

We employ the following state and time-delay state feedback controller, where $\boldsymbol{K}_3=\boldsymbol{0}$ in Eq.(8), to synchronize the delayed RMLPs Eqs.(21) and (22), where $\boldsymbol{u}(t)\in\mathbb{R}^3$, $\boldsymbol{K}_1, \boldsymbol{K}_2\in\mathbb{R}^{3\times 3}$. According to Theorem 2, solving the LMI Eq.(17) with the constraints: $\boldsymbol{Y}_3=\boldsymbol{0}$ and $\gamma=1$, we obtain the solutions of Eq.(17) and the feedback gain as:

$$\boldsymbol{X}=\begin{bmatrix} 7.5268 & 0.0990 & -3.5579 \\ 0.0990 & 2.9290 & -0.9458 \\ -3.5579 & -0.9458 & 4.2969 \end{bmatrix},$$

$$\boldsymbol{S}=\begin{bmatrix} 143.0025 & -2.3534 & 2.0586 \\ -2.3534 & 146.4412 & -1.6954 \\ 2.0586 & -1.6954 & 144.5161 \end{bmatrix},$$

$$\boldsymbol{T}=\text{diag}(81.6410, 82.5278, 73.1287, 10.6073, 10.8323, 36.9265, 2.8154, 2.9514, 3.2963),$$

$$\boldsymbol{Y}_1=\begin{bmatrix} 170.9098 & 200.8281 & -348.3251 \\ -200.1154 & 163.1545 & -98.2054 \\ 327.5560 & 85.4098 & 151.7349 \end{bmatrix},$$

$$\boldsymbol{Y}_2=\begin{bmatrix} -11.5074 & 3.2627 & 1.2881 \\ -0.8462 & -10.6887 & 6.4990 \\ 1.0834 & -3.2782 & -10.4654 \end{bmatrix},$$

$$\boldsymbol{K}_1=\begin{bmatrix} -19.5740 & 40.7102 & -88.3120 \\ -55.7637 & 37.9983 & -60.6653 \\ 109.5738 & 71.2192 & 141.7200 \end{bmatrix},$$

$$\boldsymbol{K}_2=\begin{bmatrix} -2.1660 & 0.7587 & -1.3267 \\ 0.5006 & -3.2768 & 1.2058 \\ -2.0382 & -2.5640 & -4.6877 \end{bmatrix}.$$

When the state feedback law Eq.(8) with the above $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ is put on the unforced error dynamical system between the drive system Eq.(21) and the response Eq.(22) with the initial condition $[x_1(s)\ x_2(s)\ x_3(s)]^T=[0.2\ 0.2\ 0.2]^T$ and $[-0.1\ 0.4\ -0.1]^T$, for $-1\le s\le 0$, respectively, the synchronization errors of the state variables converge to zero exponentially, which is shown in Fig.4.

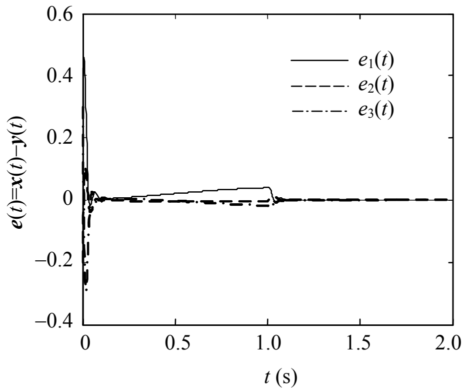

Fig.4 The waveform of synchronization errors $e_1(t)=x_1(t)-y_1(t)$, $e_2(t)=x_2(t)-y_2(t)$ and $e_3(t)=x_3(t)-y_3(t)$ in Example 2

It is worth noticing that, although some references, e.g. (Cheng *et al.*, 2005; Lu and Leeuwen, 2006; Zhang *et al.*, 2006; Zhou *et al.*, 2006; Lu and Cao, 2007; Sun and Cao, 2007; Sun *et al.*, 2007), have provided a common chaotic neural network model to describe several well-known dynamic neural networks, and given some explicit design procedures for synchronization controller of this chaotic neural network model, this model could not include RMLPs, and their approaches could not be used in synchronization synthesis of chaotic RMLPs.

CONCLUSION

In this paper, we propose a general chaotic neural network model to unify several well-known dynamic neural networks with time delays. Utilizing hybrid feedback control and LMI techniques, we have proposed several criteria to design exponential synchronization controllers of this general chaotic neural network model. Solving the LMIs by using the MATLAB LMI Control Toolbox (Gahinet *et al.*, 1995), we have obtained the feedback gain matrices of synchronization controllers in the response networks, with which the drive systems and the response systems can be exponentially synchronized under the designated convergence rate. Finally, some illustrated examples with their simulations have been utilized to demonstrate the effectiveness of the proposed methods. In addition, the design approaches can be easily extended to synthesize synchronization controllers for any chaotic systems as long as their equations can be transformed into the general model Eq.(1). It is also believed that our results should provide some practical guidelines for chaos in engineering applications.

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