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Optimization of time-delayed feedback control of seismically excited building structures*

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Abstract: An optimization method for time-delayed feedback control of partially observable linear building structures subjected to seismic excitation is proposed. A time-delayed control problem of partially observable linear building structure under horizontal ground acceleration excitation is formulated and converted into that of completely observable linear structure by using separation principle. The time-delayed control forces are approximately expressed in terms of control forces without time delay. The control system is then governed by Itô stochastic differential equations for the conditional means of system states and then transformed into those for the conditional means of modal energies by using the stochastic averaging method for quasi-Hamiltonian systems. The control law is assumed to be modal velocity feedback control with time delay and the unknown control gains are determined by the modal performance indices. A three-storey building structure is taken as example to illustrate the proposal method and the numerical results are confirmed by using Monte Carlo simulation.

Key words: Time-delayed feedback control, Stochastic averaging method, Hamiltonian, Earthquake
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INTRODUCTION

Civil engineering structures, including existing and future buildings, towers and bridges, must be adequately protected from a variety of random actions such as earthquakes, winds and traffic load. Thus, the stochastic control of civil engineering structures attracts much attention of civil engineers. The mathematical theory of stochastic optimal control has been quite well developed (Fleming and Soner, 1993; Yong and Zhou, 1999). In engineering field, however, only the linear quadratic Gaussian (LQG) control strategy has been widely used until recently. In the last few years, a nonlinear stochastic optimal control strategy has been proposed for quasi-Hamiltonian systems with external and parametric stochastic excitations (Zhu and Ying, 1999; Zhu *et al*., 2001) based on the stochastic averaging method for quasi-Hamiltonian systems (Zhu *et al*., 1997) and the stochastic dynamical programming principle (Fleming and Soner, 1993; Yong and Zhou, 1999).

In the implementation of control, time delay is usually unavoidable due to the time spent in measuring and estimating the system state, calculating and executing the control forces, etc. This time delay may cause unsynchronized application of the control forces and this unsynchronization may not only deteriorate the control performance but also cause instability of the systems. Hence, the problem of time delays in the active control of structural systems has gained considerable attention and has been investigated by several researchers. Most works view the time delays in the feedback loop as a detrimental factor and seek ways to eliminate or reduce their presumed deleterious effect. Several methods for compensation/nullification have been developed to

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date, such as, Taylor series (Choksy, 1962) and Pade approximations (Sain *et al*., 1992), the recursive response method (Agrawal and Yang, 2000), the Smith predictor method (Marshall, 1974) and predictive response methods. The time-delayed controlled systems under deterministic excitation have been studied by many researchers (Agrawal and Yang 1997; Hu and Wang, 2002). A modified fuzzy sliding mode control was presented to deal with the time-delay effect (Liao *et al*., 2002). The optimal control method in discrete form to deal with the time-delay effect for active vibration control of linear time-delay systems has been developed (Cai and Huang, 2002). The principle of time-delayed control design for the active control of structures and the control design strategy have been proposed (Udwadia *et al*., 2007). A stochastic averaging method for quasi-integrable Hamiltonian systems with time-delayed feedback control under Gaussian white noise excitation has been developed (Liu and Zhu, 2007).

In the present paper, an optimal control method for partially observable linear building structures under earthquake excitations is proposed based on the stochastic averaging method for quasi-integrable Hamiltonian systems. The control law is assumed to be time-delayed velocity feedback control which is transformed into control without time delay. By using the modal performance index, the optimal control forces gains are obtained without solving the dynamical programming equation. The method is illustrated with linear building structures equipped with control devices and sensors. The proposed method is illustrated with three-storey linear building structures equipped with control devices and sensors.

TRANSFORMING PARTIALLY OBSERVABLE CONTROL STRUCTURES INTO COMPLETELY OBSERVABLE CONTROL STRUCTURES

For an n-storey building structure equipped with control devices and sensors under horizontal ground acceleration excitations, the equation of the system is of the form

$$\boldsymbol{M}\ddot{\boldsymbol{X}}+\boldsymbol{C}\dot{\boldsymbol{X}}+\boldsymbol{K}\boldsymbol{X}=-\boldsymbol{M}\boldsymbol{\Gamma}\ddot{x}_{\mathrm{g}}+\boldsymbol{P}\boldsymbol{U}(t-\tau), \quad (1)$$

where $\boldsymbol{X}=[X_1,X_2,...,X_n]^{\mathrm{T}}, X_i$ is the horizontal displacement of ith floor relative to the ground; $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}$ are the $n\times n$ mass, damping and stiffness matrices of the system, respectively; $\boldsymbol{\Gamma}$ is an n-dimensional unit vector; $\ddot{x}_{\mathrm{g}}$ is the horizontal ground acceleration; $\boldsymbol{U}(t-\tau)$ is a k-dimensional time-delayed feedback control force vector, with k being the number of control devices; $\boldsymbol{P}$ is an $n\times k$ control device placement matrix.

Introducing the following modal transformation

$$\boldsymbol{X}=\boldsymbol{\Phi}\boldsymbol{Q}, \quad (2)$$

where $\boldsymbol{\Phi}$ is the $n\times n$ real modal matrix of the structure which has been normalized to yield $\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{\Phi}=\boldsymbol{I}$, and $\boldsymbol{I}$ is an $n\times n$ unit matrix. Then Eq.(1) becomes

$$\ddot{\boldsymbol{Q}}+2\boldsymbol{\zeta}\boldsymbol{\Omega}\dot{\boldsymbol{Q}}+\boldsymbol{\Omega}^2\boldsymbol{Q}=\boldsymbol{u}(t-\tau)-\boldsymbol{G}_{\mathrm{p}}\ddot{x}_{\mathrm{g}}, \quad (3)$$

where $\boldsymbol{Q}=[Q_1,Q_2,...,Q_m]^{\mathrm{T}}, Q_i$ is the generalized displacement of ith mode; $2\boldsymbol{\zeta}\boldsymbol{\Omega}=\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{\Omega}$ with $\boldsymbol{\zeta}$=diag[ζ_i] and ζ_i is the damping ratio of ith mode; $\boldsymbol{\Omega}^2=\mathrm{diag}(\omega_i^2)=\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{\Phi}$ and ω_i is the natural frequency of the ith mode; $\boldsymbol{G}_{\mathrm{p}}=\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{\Gamma}$; $\boldsymbol{u}(t-\tau)=\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{U}(t-\tau)$ and u_i is the time-delayed generalized control force of the ith mode. $\ddot{x}_{\mathrm{g}}$ is a wide-band stationary random process with rational power spectral density as:

$$S_{\ddot{x}_{\mathrm{g}}}(\omega,\tau)=\sigma^2(\tau)\sum_{j=0}^{1}b_j\omega^{2j}\Big/\sum_{i=0}^{2}a_j\omega^{2j}, \quad (4)$$

which can be regarded as the output of the following linear filter to unit intensity Gaussian white noise process $W(t)$:

$$\begin{aligned}&\ddot{x}_{\mathrm{g}}=d_1\frac{\mathrm{d}f}{\mathrm{d}t}+d_0f,\\&\frac{\mathrm{d}^2f}{\mathrm{d}t^2}+\frac{c_1}{c_2}\frac{\mathrm{d}f}{\mathrm{d}t}+\frac{c_0}{c_2}f=\frac{1}{c_2}\sigma(\tau)W(t),\end{aligned} \quad (5)$$

where

$$a_0=\omega_{\mathrm{g}}^4,\ a_1=4\zeta_{\mathrm{g}}^2\omega_{\mathrm{g}}^2-2\omega_{\mathrm{g}}^2,\ a_2=1,\ b_0=\omega_{\mathrm{g}}^4,$$
$$b_1=4\zeta_{\mathrm{g}}^2\omega_{\mathrm{g}}^2,\ c_0^2=a_0,\ c_1^2=2c_0c_2,\ c_2^2=a_2,$$
$$d_0^2=b_0,\ d_1^2=b_1.$$

Eq.(3) and Eq.(5) can be combined and converted into the following Itô stochastic differential equation:

$$\mathrm{d}\boldsymbol{Z} = (\boldsymbol{AZ} + \boldsymbol{Bu})\mathrm{d}t + \boldsymbol{C}\mathrm{d}B(t), \tag{6}$$

where $\boldsymbol{Z} = [\boldsymbol{Q}^{\mathrm{T}}, \dot{\boldsymbol{Q}}^{\mathrm{T}}, \boldsymbol{F}_1^{\mathrm{T}}]^{\mathrm{T}}$; $\boldsymbol{F}_1 = [f_1, f_2]^{\mathrm{T}}$; $B(t)$ is the standard Wiener process; and

$$\boldsymbol{A} = \begin{bmatrix} 0 & \boldsymbol{I}_n & 0 & 0 \\ -\boldsymbol{\Omega}^2 & -2\zeta\boldsymbol{\Omega} & -\boldsymbol{G}_{\mathrm{p}}d_0 & -\boldsymbol{G}_{\mathrm{p}}d_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -c_0/c_2 & -c_1/c_2 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ \boldsymbol{I}_n \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sigma(t)/c_2 \end{bmatrix}. \tag{7}$$

Suppose that l absolute accelerations $\boldsymbol{a}=[a_1, a_2, \ldots, a_l]^{\mathrm{T}}$ at different floors are measured with noises. They can be expressed in terms of $\ddot{\boldsymbol{X}}$ and $\ddot{x}_{\mathrm{g}}$ as follows:

$$[\boldsymbol{a}] = \boldsymbol{P}_0[\ddot{\boldsymbol{X}} + \boldsymbol{\Gamma}\ddot{x}_{\mathrm{g}}] + \boldsymbol{\sigma}_0 \boldsymbol{w}_l(t), \tag{8}$$

where $\boldsymbol{P}_0$ is an $l\times n$ sensor placement matrix, $\boldsymbol{w}_l(t)$ is an l-dimensional vector of independent unit Gaussian white noises; $\boldsymbol{\sigma}_0$ is an $l\times l$ measurement accuracy matrix. Eq.(8) can be converted into the following Itô stochastic differential equation:

$$\mathrm{d}\boldsymbol{V} = (\boldsymbol{D}_0\boldsymbol{Z} + \boldsymbol{C}_0\boldsymbol{u})\mathrm{d}t + \boldsymbol{\sigma}_0\mathrm{d}\boldsymbol{B}_l(t), \tag{9}$$

where $\mathrm{d}\boldsymbol{V}/\mathrm{d}t=[a_1, a_2, \ldots, a_l]^{\mathrm{T}}$; $\boldsymbol{B}_l(t)$ is an l-dimensional vector of the standard Wiener processes, which are independent of $B(t)$; and

$$\begin{aligned} \boldsymbol{D}_0 &= \boldsymbol{P}_0\left[-\boldsymbol{\Phi\Omega}^2 \quad -\boldsymbol{\Phi}2\zeta\boldsymbol{\Omega} \quad \boldsymbol{0} \quad \boldsymbol{0}\right], \\ \boldsymbol{C}_0 &= \boldsymbol{P}_0\boldsymbol{\Phi}, \end{aligned} \tag{10}$$

are measurement matrices. Eq.(9) is the observation equation of the control problem.

According to the separation principle (Bensoussan, 1992), the partially observed stochastic optimal control problem can be converted into a completely observed stochastic optimal control problem. Let $\hat{\boldsymbol{Z}}(t)$ be the conditional mean of the system state $\boldsymbol{Z}(t)$. $\hat{\boldsymbol{Z}}(t)$ is governed by the following Itô equation:

$$\mathrm{d}\hat{\boldsymbol{Z}} = (\boldsymbol{A}\hat{\boldsymbol{Z}} + \boldsymbol{Bu})\mathrm{d}t + \boldsymbol{F}_{\mathrm{e}}(t)\boldsymbol{\sigma}_0\mathrm{d}\hat{\boldsymbol{B}}_l(t), \tag{11}$$

where

$$\begin{aligned} &\hat{\boldsymbol{Z}} = [\hat{\boldsymbol{Q}}^{\mathrm{T}}, \dot{\hat{\boldsymbol{Q}}}^{\mathrm{T}}, \boldsymbol{F}_1^{\mathrm{T}}]^{\mathrm{T}}, \\ &\boldsymbol{F}_{\mathrm{e}}(t) = \boldsymbol{R}_{\mathrm{e}}(t)\boldsymbol{D}_0^{\mathrm{T}}\boldsymbol{S}_0^{-1}, \end{aligned} \tag{12}$$

and $\boldsymbol{R}_{\mathrm{e}}(t)$ is the covariance matrix of error $\bar{\boldsymbol{Z}} = \boldsymbol{Z} - \hat{\boldsymbol{Z}}$, which satisfies the following Riccati matrix differential equation:

$$\dot{\boldsymbol{R}}_{\mathrm{e}} = \boldsymbol{AR}_{\mathrm{e}} + \boldsymbol{R}_{\mathrm{e}}\boldsymbol{A}^{\mathrm{T}} - \boldsymbol{R}_{\mathrm{e}}\boldsymbol{D}_0^{\mathrm{T}}\boldsymbol{S}_0^{-1}\boldsymbol{D}_0\boldsymbol{R}_{\mathrm{e}} + \boldsymbol{S}_{\mathrm{c}}, \tag{13}$$

where $\boldsymbol{S}_0 = \boldsymbol{\sigma}_0\boldsymbol{\sigma}_0^{\mathrm{T}}$, $\boldsymbol{S}_{\mathrm{c}} = \boldsymbol{C}_{\mathrm{c}}\boldsymbol{C}_{\mathrm{c}}^{\mathrm{T}}$. Eq.(11) is the one for the completely observed stochastic optimal control problem.

PARTIALLY AVERAGED CONTROL SYSTEMS WITH TIME-DELAYED CONTROL

According to (Liu and Zhu, 2007), the solutions to Eq.(11) can be expressed by

$$\begin{aligned} &\hat{Q}_i(t) = A_i\cos\Phi_i(t), \\ &\dot{\hat{Q}}_i(t) = -A_i\omega_i\sin\Phi_i(t), \\ &\Phi_i(t) = \Theta_i(t) + \Gamma_i(t). \end{aligned} \tag{14}$$

Amplitudes $A_i(t)$ and phase $\Gamma_i(t)$ are slowly varying processes. For small τ, $A_i(t-\tau)$ and $\Gamma_i(t-\tau)$ can be approximated by $A_i(t)$ and $\Gamma_i(t)$, respectively. Then the following approximate expressions can be obtained:

$$\begin{aligned} \dot{\hat{Q}}_i(t-\tau) &= -A_i(t-\tau)\omega_i\sin\Phi_i(t-\tau) \\ &\doteq A_i(t)\omega_i\sin[\omega_i(t-\tau) + \Gamma_i(t)] \\ &= -A_i(t)\omega_i\{\sin[\omega_i t + \Gamma_i(t)]\cos\omega_i\tau \\ &\quad - \cos[\omega_i t + \Gamma_i(t)]\sin\omega_i\tau\} \\ &= \dot{\hat{Q}}_i\cos\omega_i\tau + \hat{Q}_i(t)\omega_i\sin\omega_i\tau. \end{aligned} \tag{15}$$

ω_i is the natural frequency of the control system Eq.(11). For linear system, velocity feedback control is efficient (Phohomsiri *et al*., 2006). Thus we take

$$u_i = -k_i \dot{\hat{Q}}_i (t-\tau),\quad i=1,\ 2,\ ...,\ n, \tag{16}$$

where k_i are unknown control gains, which will be determined in next step. Substituting Eq.(15) into Eq.(16), the control forces can be expressed by using system state estimate without time delay

$$u_i = -k'_{1i}\hat{Q}_i - k'_{2i}\dot{\hat{Q}}_i,\ k'_{1i} = k_i\omega_i \sin\omega_i\tau,\ k'_{2i} = k_i \cos\omega_i\tau. \tag{17}$$

Let

$$\hat{H}_i = (\dot{\hat{Q}}_i^2 + \omega_i'^2 \hat{Q}_i^2)/2, \tag{18}$$

where $\omega_i' = (\omega_i^2 + k_i\omega_i \sin\omega_i\tau)^{1/2}$ and $\hat{H}_i$ represents the conditional mean of the ith modal energy. Introducing the transformations:

$$\begin{aligned} &\omega_i'\hat{Q}_i = \sqrt{2\hat{H}_i}\cos\theta_i,\ \dot{\hat{Q}}_i = -\sqrt{2\hat{H}_i}\sin\theta_i, \\ &\theta_i = \omega_i' t + \varphi_i, \qquad i=1,\ 2,\ ...,\ m, \end{aligned} \tag{19}$$

the Itô equations for $\hat{H}_i$, φ_i can be obtained from the equations for $\hat{Q}_i$, $\dot{\hat{Q}}_i$ in Eq.(11) by using Itô differential rule as follows:

$$\begin{aligned} \mathrm{d}\hat{H}_i = \Bigg[&-(4\zeta_i\omega_i + 2k_i\cos\omega_i\tau)\hat{H}_i \sin^2\theta_i \\ &+\frac{1}{2}\sum_{i=1}^{m}(\omega_i'^2 (\boldsymbol{F}_\mathrm{e}\boldsymbol{\sigma}_0)^2_{i,k} + (\boldsymbol{F}_\mathrm{e}\boldsymbol{\sigma}_0)^2_{n+i,k})\Bigg]\mathrm{d}t \\ &+\sqrt{2\hat{H}_i}[\omega_i'(\boldsymbol{F}_\mathrm{e}\boldsymbol{\sigma}_0)_{i,k}\cos\theta_i - (\boldsymbol{F}_\mathrm{e}\boldsymbol{\sigma}_0)_{n+i,k}\sin\theta_i]\mathrm{d}B_k(t) \\ &+\sqrt{2\hat{H}_i}\sin\theta_i[\boldsymbol{G}_\mathrm{p}]_i \mathrm{d}\hat{\dot{x}}_\mathrm{g}(t). \end{aligned} \tag{20}$$

By applying the stochastic averaging method for quasi-integrable Hamiltonian systems (Zhu *et al*., 1997), the following averaged Itô equation can be obtained:

$$\mathrm{d}\hat{H}_i = m_i(\hat{H}_i)\mathrm{d}t + \sigma_i(\hat{H}_i)\mathrm{d}\overline{B}_i(t),\ i=1,\ 2,...,\ m, \tag{21}$$

where $\overline{B}_i(t)$ are independent unit Wiener processes.

$$\begin{aligned} &m_i(\hat{H}_i) = -(4\zeta_i\omega_i + 2k_i\cos\omega_i\tau)\hat{H}_i + \sigma_i^2(\hat{H}_i)/(2\hat{H}_i), \\ &\sigma_i^2(\hat{H}_i) = \hat{H}_i\sum_{i=1}^{m}(\omega_i'^2(\boldsymbol{F}_\mathrm{e}\boldsymbol{\sigma}_0)^2_{i,k} + (\boldsymbol{F}_\mathrm{e}\boldsymbol{\sigma}_0)^2_{n+i,k}) \\ &\qquad + \hat{H}_i[\boldsymbol{G}_\mathrm{p}]_i^2 S_{\hat{\ddot{x}}_\mathrm{g}}(\omega_i',\tau), \end{aligned} \tag{22}$$

$S_{\hat{\ddot{x}}_\mathrm{g}}(\omega_i',\tau)$ is the evolutionary spectral density of the conditional mean $\hat{\ddot{x}}_\mathrm{g}$ of $\ddot{x}_\mathrm{g}$, and can be obtained from the last two equations of Eq.(11) as follows:

$$\begin{aligned} S_{\hat{\ddot{x}}_\mathrm{g}}(\omega_i,\tau) = \sigma^2(\tau)\left|\sum_{i=0}^{1} d_i(\mathrm{j}\omega)^i\right|^2 \sum_{k=1}^{l}\sum_{i=1}^{2}\left|\sum_{i_1=i}^{2} c_{i_1}(\mathrm{j}\omega)^{i_1-1}\right|^2 \\ \cdot(\boldsymbol{F}_\mathrm{e}(t)\boldsymbol{\sigma}_1)^2_{2n+i,k} \Bigg/ \left|\sum_{i=0}^{2} c_i(\mathrm{j}\omega)^i\right|^2,\quad \mathrm{j}=\sqrt{-1}. \end{aligned} \tag{23}$$

The FPK equation associated with averaged Itô Eq.(21) is

$$\frac{\partial \rho(\hat{\boldsymbol{H}})}{\partial t} = -\sum_{i=1}^{n}\left[\frac{\partial[m_i(\hat{H}_i)\rho(\hat{\boldsymbol{H}})]}{\partial \hat{H}_i} + \frac{\partial^2[\sigma_i^2(\hat{H}_i)\rho(\hat{\boldsymbol{H}})]}{2\partial^2 \hat{H}_i}\right]. \tag{24}$$

In the stationary case, the analytical solution of Eq.(24) can be obtained as follows:

$$\begin{aligned} &p(\hat{\boldsymbol{H}}) = c\prod_{i=1}^{n} p_i(\hat{H}_i), \\ &p_i(\hat{H}_i) = c_i \exp\left\{-(4\zeta_i\omega_i + 2k_i\cos\omega_i\tau)\Bigg/\left[\left[\sum_{i=1}^{m}(\omega_i'^2(\boldsymbol{F}_\mathrm{e}\boldsymbol{\sigma}_0)^2_{i,k}\right.\right.\right. \\ &\qquad \left.\left.\left. + (\boldsymbol{F}_\mathrm{e}\boldsymbol{\sigma}_0)^2_{n+i,k}) + [\boldsymbol{G}_\mathrm{p}]_i^2 S_{\hat{\ddot{x}}_\mathrm{g}}(\omega_i',\tau)\right] H_i\right]\right\}, \end{aligned} \tag{25}$$

where c_i are normalization constants. The stationary probability density of each modal displacement and velocity of the controlled structure is then

$$\rho_i(\hat{Q}_i,\dot{\hat{Q}}_i) = \left.\frac{\rho_i(\hat{H}_i)}{T_i(\hat{H}_i)}\right|_{H_i=(\dot{\hat{Q}}_i^2+\omega_i'^2\hat{Q}_i^2)/2}, \tag{26}$$

where $T_i(\hat{H}_i) = \omega_i'/(2\pi)$.

DETERMINING THE OPTIMUM CONTROL GAINS

Since Eq.(11) is separable, we can only control m dominant modes among n modes and each mode is controlled with control force. The control objective is to minimize the response. So the performance index is of the form

$$J_i = E[\hat{H}_i] + R_i E[u_i^2],\ i = 1,\ 2,\ ...,\ m, \quad (27)$$

where

$$\begin{aligned} E[\hat{H}_i] &= \int_0^{\infty} \hat{H}_i \rho(\hat{H}_i) \mathrm{d}H_i, \\ E[u_i^2] &= \int_{-\infty}^{\infty} (k'_{1i}\hat{Q}_i)^2 \int_{-\infty}^{\infty} \rho(\hat{Q}_i, \dot{\hat{Q}}_i) \mathrm{d}\dot{\hat{Q}}_i \mathrm{d}\hat{Q}_i + \\ &\int_{-\infty}^{\infty} (k'_{2i}\dot{\hat{Q}}_i)^2 \int_{-\infty}^{\infty} \rho(\hat{Q}_i, \dot{\hat{Q}}_i) \mathrm{d}\hat{Q}_i \mathrm{d}\dot{\hat{Q}}_i + \\ &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2k'_{1i}k'_{2i}\hat{Q}_i\dot{\hat{Q}}_i \rho(\hat{Q}_i, \dot{\hat{Q}}_i) \mathrm{d}\hat{Q}_i \mathrm{d}\dot{\hat{Q}}_i. \end{aligned} \quad (28)$$

Substituting Eq.(28) into Eq.(27) yields the optimal gains k_{1i} and k_{2i}. R_i are selected as the balanced parameters which adjust the amplitude of the control force.

NUMERICAL RESULTS AND DISCUSSION

To illustrate the proposed control method, the optimal control of a three-storey experimental building structure model given in Appendix A and Fig.A1 is studied. Three actuators are placed on three storeys of the building. The size of the actuators is limited to provide maximum control forces of 1000 kN. The actuators placed on the first and second storeys will produce equal and opposite control forces. So the control force device placement matrix is chosen as

$$\boldsymbol{P} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}. \quad (29)$$

The observation equation is of the form of Eq.(8). Three accelerometers are placed on storeys 1, 2 and 3 and each contains noise of 0.002. The horizontal ground acceleration $\ddot{x}_g$ is taken to be a stationary wide-band random process which can be obtained by using Eq.(5), where c_0=625, c_1=12.5, c_2=1, d_0=625, d_1=12.5, $\sigma(\tau)$=0.015. The power spectral density of $\ddot{x}_g$ is shown in Fig.1 by using solid line. For comparison, the power spectral densities of the El Centro, Hachinohe, Kobe and Northridge earthquakes obtained through the history records are also shown in Fig.1. The observation equation is in the form of Eq.(8) and three accelerometers are placed on levels 1, 2 and 3. Each of the measured response contains noise of 0.002.

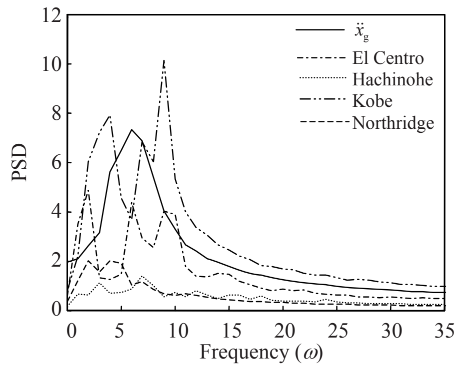

Fig.1 The power spectral density of (PSD) the earthquake excitation

We control 2 dominant modes among 3 modes. The time-delayed control force vector is of the form

$$\bar{\boldsymbol{u}} = -\boldsymbol{k}\hat{\boldsymbol{Z}}(t-\tau), \quad (30)$$

where $\boldsymbol{k} = \mathrm{diag}[0\ \ k_1\ \ 0\ \ k_2\ \ 0\ \ 0]$ and the corresponding mode control force vector without time delay can be determined by using Eq.(17) as follows:

$$\boldsymbol{u} = -\boldsymbol{k}'\boldsymbol{Z}(t), \quad (31)$$

where $\boldsymbol{k}' = \mathrm{diag}[k'_{11}\ \ k'_{12}\ \ k'_{21}\ \ k'_{22}\ \ 0\ \ 0]$. Optimal control gains $\boldsymbol{k}$ and $\boldsymbol{k}'$ can be obtained by using the method discussed in the last section. In this example,

$$\tau = 0.04,\quad k_1 = 4.8,\ \ k_2 = 3.8,\ \ k'_{11} = 18.52,$$
$$k'_{21} = 4.42,\ \ k'_{12} = 95.16,\ \ k'_{22} = 1.67.$$

From Eq.(9) and Eq.(11), we obtain the equation for the conditional mean of system state

$$\dot{\hat{\boldsymbol{Z}}} = (\boldsymbol{A}\hat{\boldsymbol{Z}} + \boldsymbol{B}\boldsymbol{u}) + \boldsymbol{F}_{\mathrm{e}}(t)(\dot{\boldsymbol{V}} - \boldsymbol{D}_0\hat{\boldsymbol{Z}} - \boldsymbol{C}_0\boldsymbol{u}), \quad (32)$$

where

$$u = -k'\hat{Z}. \tag{33}$$

Substituting Eq.(33) into Eq.(32) leads to controlled equation:

$$\dot{\hat{Z}} = (A - Bk' - F_e D_0 + F_e C_0 k')\hat{Z} + F_e \dot{V}. \tag{34}$$

The control is implemented in MATLAB simulink with the block illustrated in Fig.2.

The control forces vector $U(t-\tau)$ applied on the building structure can be obtained in the form $u(t-\tau) = \Phi^T P U(t-\tau)$ as follows:

$$U(t-\tau) = (\Phi^T P)^{-1} u(t-\tau). \tag{35}$$

Figs.3a and 3b show the analytical probability densities (solid line) and the simulation results (dots ● and ○). It is seen that the analytical results and simulation results are in good agreement. Figs.4~5 show the response and control force of the structure subjected to the El Centro earthquake excitation. It is seen that the response of the system is greatly reduced and that the control forces are bounded.

The maximal response and control force of the uncontrolled and controlled three-story building structure under El Centro, Hachinohe, Northridge and Kobe earthquake excitations are given in Table 1, where x_u^{max}, $\dot{x}_u^{max}$, $\ddot{x}_u^{max}$ are the maximal displacement, velocity and acceleration of the uncontrolled system, x_c^{max}, $\dot{x}_c^{max}$, $\ddot{x}_c^{max}$ are the maximal displacement, velocity and acceleration of the controlled system, respectively, and u^{max} is the maximal control force.

CONCLUSION

In the present paper, a time-delayed optimal control method for partially observable linear build-

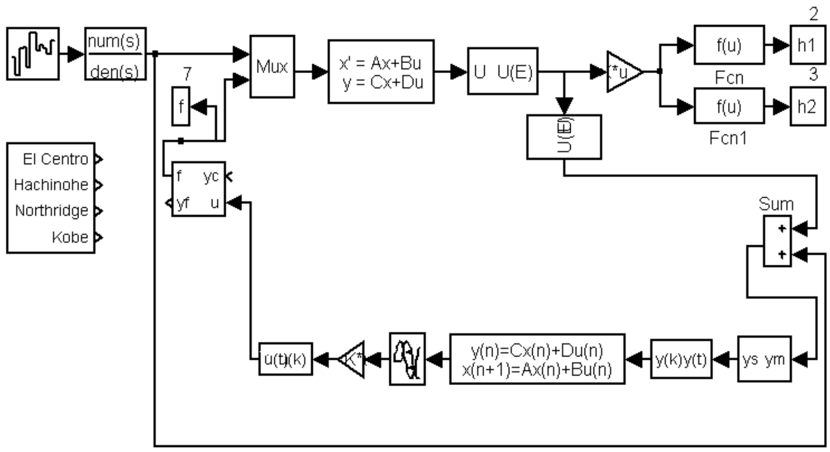

Fig.2 Simulink block diagram for vibration control

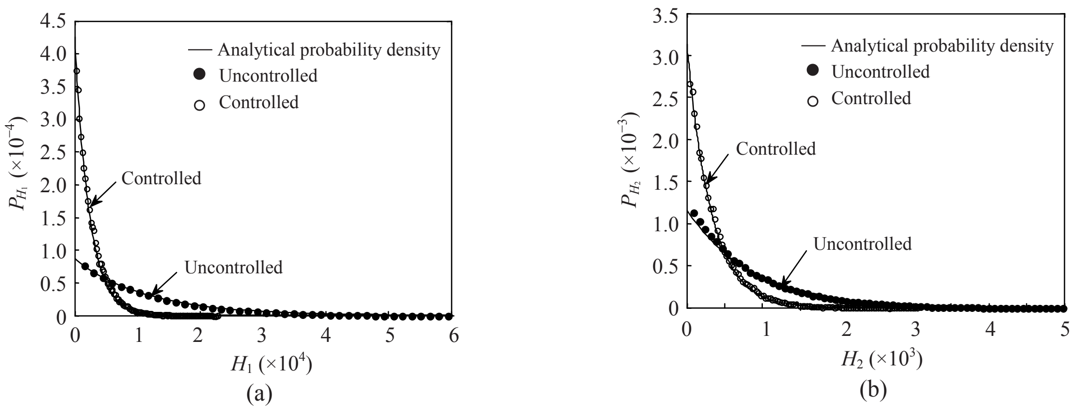

Fig.3 The probability density of H_1 (a) and H_2 (b) of the uncontrolled (solid) and controlled (dotted) systems

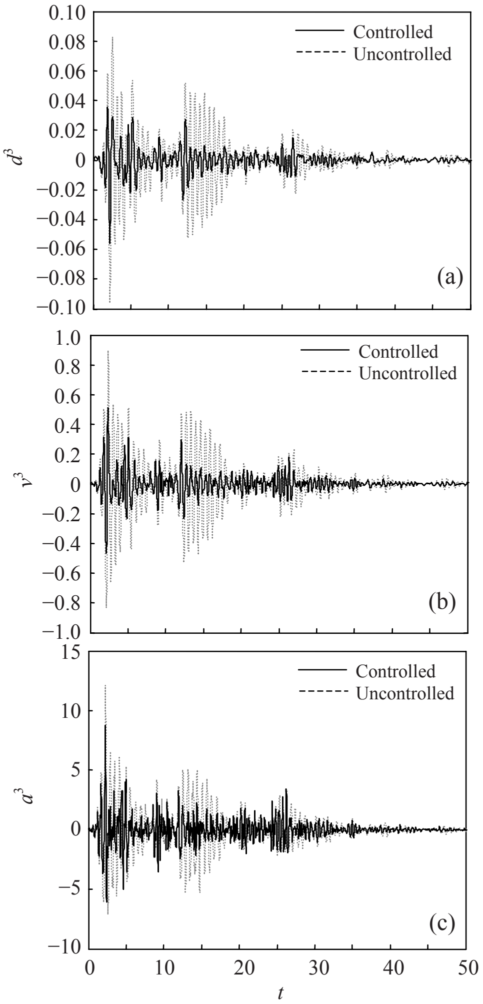

Fig.4 The roof displacement (a), roof velocity (b) and roof acceleration (c) of the uncontrolled and controlled systems

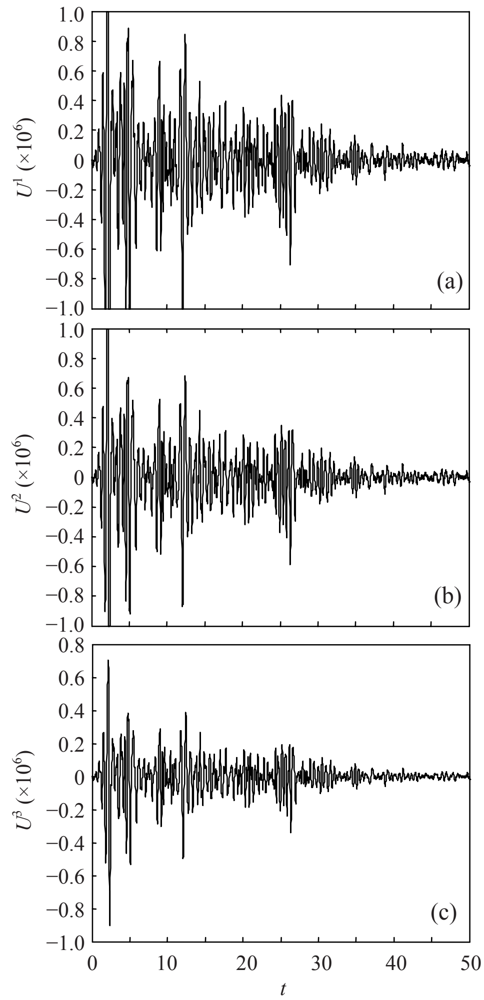

Fig.5 The control force on the first (a), second (b) and third (c) storey

Table 1 The maximal response and control force for the (uncontrolled and controlled) three-storey building structure

Earthquakes	Story	x_u^{max} (m)	x_c^{max} (m)	$\dot{x}_u^{max}$ (m/s)	$\dot{x}_c^{max}$ (m/s)	$\ddot{x}_u^{max}$ (m/s^2)	$\ddot{x}_c^{max}$ (m/s^2)	u^{max} (kN)
El Centro	1	0.0397	0.0158	0.3693	0.1803	6.2580	6.1573	1000
	2	0.0682	0.0285	0.6720	0.3707	9.7587	8.2460	1000
	3	0.0830	0.0352	0.9004	0.5111	12.0980	8.3696	703.37
Hachinohe	1	0.0183	0.0102	0.1713	0.1087	2.6896	1.6825	1000
	2	0.0317	0.0175	0.2885	0.1847	3.6619	2.4176	816.59
	3	0.0378	0.0213	0.4061	0.2367	4.5607	2.9867	460.75
Northridge	1	0.0384	0.0334	0.5750	0.4343	11.5410	10.9650	1000
	2	0.0651	0.0564	1.2311	0.9865	18.1550	16.9330	1000
	3	0.0849	0.0680	1.6790	1.3059	22.9050	21.2120	1000
Kobe	1	0.0886	0.0638	0.9327	0.5858	8.7386	7.9334	1000
	2	0.1642	0.1107	1.6218	1.0910	13.9770	14.5230	1000
	3	0.2069	0.1349	1.9512	1.3632	17.5410	14.7930	1000

ing structures has been proposed. The method consists of five steps: converting the partially observable control structures into completely observable control structures; transforming the time-delayed control forces into the control forces without time delay; performing stochastic averaging and establishing the corresponding FPK equation; solving the FPK equation and obtaining the exact stationary solutions;

combining the mean square value of the modal control force and the mean modal Hamiltonians as the performance index of the system and obtaining the expression of the optimal control gains. As an example, the proposed method has been applied to a three-storey experimental building equipped with control devices and sensors. The main advantage of the method is that the time-delayed effect can be compensated. Another advantage of the method is that it is not necessary to solve the dynamical programming equation for obtaining the optimal control law. From the result in Table 1, it is seen that the proposed method may be applied to seismic-excited building structures.

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APPENDIX A (THE THREE-STOREY BUILDING STRUCTURE MODEL)

The mass, stiffness and damping matrices of the system are:

$$\boldsymbol{M}=\begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix}\times10^5\ \text{kg},$$

$$\boldsymbol{K}=\begin{bmatrix}4&-2&0\\-2&4&-2\\0&-2&2\end{bmatrix}\times10^8\ \text{N/m},$$

$$\boldsymbol{C}=0.733\boldsymbol{M}+0.0026\boldsymbol{K}.$$

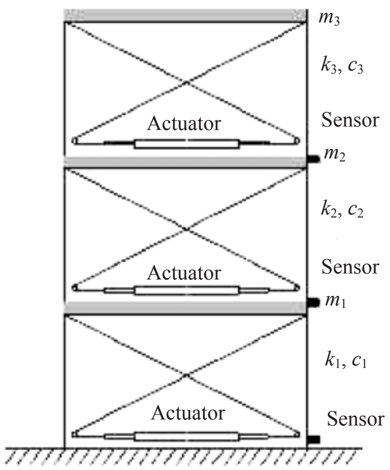

Fig.A1 The three-storey building structure model