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Vertical distribution of sediment concentration*

Sai-hua HUANG†, Zhi-lin SUN†‡, Dan XU, Shan-shan XIA

(*Department of Hydraulic and Ocean Engineering, Zhejiang University, Hangzhou 310058, China*)

†E-mail: ivyhuang21@163.com; oceansun@zju.edu.cn

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Abstract: A simple formula is proposed to predict the vertical distribution of a suspended load concentration in a 2D steady turbulent flow. The proposed formula significantly improves the well-known Rouse formula where sediment concentration has an infinitely large value at the channel bottom and a zero value at the water surface. Based on this formula and the logarithmic velocity profile, a theoretical elementary function for the transport rate of a suspended load is developed. This equation improves the Einstein equation in which the unit-width suspended sediment discharge must be solved by numerical integration and a contradiction between the lower limit of the integral and that of velocity distribution exists.

Key words: Sediment concentration, Vertical distribution, Suspended load, Sediment discharge
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INTRODUCTION

The vertical distribution of equilibrium of a suspended load concentration in a 2D steady turbulent flow is an important subject in the mechanics of sediment transport, which is mainly applied to a boundary condition in mathematical models (Charafi and Sadok, 2000; Orton and Kineke, 2001; Krzyk and Cetina, 2003; White and Deleersnijder, 2007; Smaoui and Boughanim, 2007) and the analysis of data measured in the field and the laboratory (Williams *et al.*, 2000; Graf and Cellino, 2002; Jiang and Law, 2004; Wren and Bennett, 2005). Since Rouse proposed a famous formula based on the turbulent diffusion theory, many formulas have been developed sequentially (Zhang and Xie, 1993). Some of them are based on different theories, such as gravity theory, mixture theory (Mctigue, 1981), as well as stochastic theory (Bechteler, 1987). Wright and Parker (2004) recently proposed predictors of the near-bed concentration and the suspended load transport rate in sand-bed rivers. Wilson (2005) studied the rapid increase in suspended sediment at high bed shear. Although characterized by several deficiencies, the Rouse formula is still the most famous and is extensively cited by many works on the mechanics of sediment transport. No other formula has vied for its outstanding position. The main reason is that other formulas, when compared to the Rouse formula, lose some of their theoretical significance or lack the simplicity and fail to simultaneously deal with the following basic requirements: (1) Sediment concentration at the water surface is a small quantity rather than zero; (2) Sediment concentration at the bed surface is a finite quantity instead of infinite; (3) The formula must possess a simple structure and reasonable precision; (4) The formula should be based on a differential equation to form an adequate theoretical foundation.

Moreover, the unit-width transport rate of the suspended load derived from the previous formulas for vertical concentration can only be expressed as a numerical integral or an approximate integral rather than as an elementary function (Umeyama, 1992; 1999; Wright and Parker, 2004). Such formulas are not convenient for practical calculation. Therefore the development of a theoretical formula to express the suspended load concentration profile, which can meet

‡ Corresponding author
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the requirements mentioned above, is of great theoretical and practical significance.

DIFFERENTIAL EQUATION AND ITS BOUNDARY CONDITION

The motion of a suspended load in turbulent flow is governed by the law of mass conservation. When turbulent diffusion is considered, the differential continuity equation of a suspended load can be written in the following tensor form:

$$\frac{\partial C}{\partial t}+\frac{\partial \boldsymbol{V}_j C}{\partial \boldsymbol{X}_j}=\frac{\partial}{\partial \boldsymbol{X}_j}\left(\varepsilon_{sj}\frac{\partial C}{\partial \boldsymbol{X}_j}\right), \tag{1}$$

where C is the concentration of suspended load; $\boldsymbol{X}_j$ denotes ordinate component (x, y, z); $\boldsymbol{V}_j$ is velocity component (u, $v-\omega$, w) corresponding to $\boldsymbol{X}_j$; ω is the settling velocity of sediment in quiescent water; ε_{sj} is a diffusion coefficient of sediment in $\boldsymbol{X}_j$ direction.

In a 2D steady turbulent flow on the equilibrium bed-surface, the governing equation for suspended load concentration can be obtained by simplifying Eq.(1) and taking its first integral as

$$\omega C+\varepsilon_s\frac{\mathrm{d}C}{\mathrm{d}y}=0, \tag{2}$$

where ε_s is a diffusion coefficient of sediment in y direction.

Giving a boundary condition at the bed-surface:

$$C(0)=C_0, \tag{3}$$

then Eqs.(2) and (3) constitute a 1D boundary value statement. Here C_0 is a bed-concentration of suspended sediment for the equilibrium case, $C_0=C_{0*}$, which can be calculated from an entrainment function defined at the bed-surface. Based on his stochastic model of sediment exchange, Sun (2003) proposed an entrainment function for the kth fraction of non-uniform sediment:

$$C_{0*k}=C_{0*}(k)P_{0k}=M_0P_{0k}\frac{F(\cdot)}{1+F(\cdot)}, \tag{4a}$$

where

$$F(\cdot)=\frac{10^{-5}\tau_{*k}^2 D_*^{1.84}\alpha_k\beta_k}{(1-\alpha_{0k})(1-\beta_k)(1+\alpha_{0k}\beta_k)}, \tag{4b}$$

where C_{0*k} is the sediment-transport capacity for the kth fraction of non-uniform sediment; $C_{0*}(k)$ is the possible sediment-transport capacity for the kth fraction of non-uniform sediment; M_0 is the density coefficient of bed material, approximately equal to 0.4; P_{0k} is the percentage of the kth fraction of bed material and equal to unit for uniform sediment; D_* is the dimensionless mean grain size of bed material; τ_{*k} is the dimensionless shear stress of flow acting on the kth fraction of non-uniform sediment; α_k, α_{0k} and β_k are the probabilities of initial motion, non-ceasing motion and suspended motion, respectively. In the case of uniform sediment, all the subscript k can be omitted and then $C_{0*}(k)=C_{0*}$.

DETERMINATION OF DIFFUSION COEFFICIENT

Because the diffusion equation of the suspended load transport is derived by the Reynolds-averaging of the equation of mass conservation, people naturally think that the turbulent momentum exchange coefficient ε can be a substitute for the sediment diffusion coefficient ε_s when they solve the diffusion equation. As Zhang and Xie (1993) pointed out, to assume that the diffusion coefficient ε_s is equal to the momentum exchange coefficient ε results in distortion of sediment concentration at the water-surface and at the bed-surface because of the difference in characteristics of the two diffusions and the effect of the suspended load on the flow structure.

Einstein and Chien (1954) derived a differential equation to express the motion of suspended load from the viewpoint of the equilibrium exchange of the suspended load between upper and lower layers of flow as follows:

$$\omega C+\frac{1}{2}lv\frac{\mathrm{d}C}{\mathrm{d}y}=0, \tag{5}$$

where l is the mixing length, v the turbulent velocity.

Obviously Eq.(5) is consistent with the diffusion equation in form. This shows that sediment diffusion coefficient ε_s is related to both the mixing length and the turbulent velocity. Although further stochastic processing of random variable v is required, this approach is useful in understanding the physical mechanism of the diffusion phenomenon. We will further analyze the physical component of the sediment diffusion coefficient by analogy with Einstein and Chien (1954)'s approach.

A suspended load in turbulent flow moves in consequence of gravity and upward swirls of turbulent flow. Sediment flux through a unit-area is equal to the sum of the gravity settling flux and diffusive flux due to vertical turbulent flow, i.e.,

$$\Omega(y)=\Omega_{\mathrm{g}}(y)+\Omega_{\mathrm{T}}(y), \tag{6}$$

where $\Omega(y)$ is the vertical sediment flux; $\Omega_g(y)$ is the sediment flux due to gravity; $\Omega_T(y)$ is the sediment flux due to turbulence, $\Omega_T(y)=\Omega_{Td}(y)+\Omega_{Tu}(y)$.

Gravity flux related to settling velocity is a deterministic variable while diffusion flux associated with turbulent velocity is a stochastic variable. Therefore the stochastic processing of vertical diffusion flux due to turbulence is necessary to summate gravity flux and diffusion flux. According to the previous research (Asaeda *et al.*, 1989), vertical turbulent velocity is a normal stochastic function:

$$f[v(y)]=\frac{1}{\sqrt{2\pi}\sigma_{\mathrm{v}}}\exp\left[-0.5\left(\frac{v(y)}{\sigma_{\mathrm{v}}}\right)^2\right], \tag{7}$$

where $v(y)$ is the vertical turbulent velocity; σ_v is the standard deviation of vertical velocity, a function of elevation y above the bottom.

Assuming a horizontal unit-area-section at y above the bottom, we analyze the vertical flux of a suspended load through this unit-area-section within the neighborhood of water layer ($y-bl$, $y+bl$). Providing that downward flux comes from the upper layer and upward flux comes from the lower layer (Fig.1), sediment concentrations of the upper and lower layers are expanded as a series at point y and then first-order approximations are taken: $C+bl\mathrm{d}C/\mathrm{d}y$ and $C-bl\mathrm{d}C/\mathrm{d}y$, where b is a proportionality coefficient.

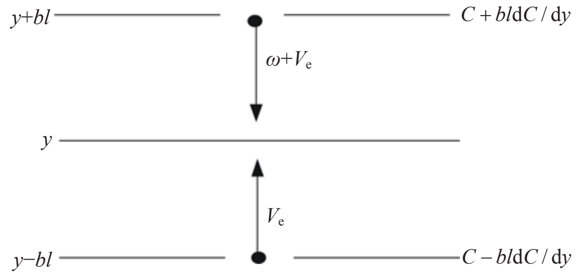

Fig.1 Schematic pattern of sediment flux through flow layer

Generally speaking, turbulent diffusion flux is mainly attributed to large-scale turbulence. Smaller eddies, i.e., high frequency turbulent eddies, have no effect on the motion of sediment particles. The diffusion motion of particles only reflects the action of low-frequency turbulence. Gravity acting on particles will make particles settle in turbulent flow. Therefore, given a turbulent velocity spectrum in the same water layer, only those eddies with instantaneous velocities larger than ω can carry sediment particles from a lower layer into an upper layer. Eddies with instantaneous velocities less than ω cannot carry particles into an upper layer. Therefore the upward sediment flux from the ($y-bl$) layer due to vertical turbulence is

$$\Omega_{\mathrm{Tu}}(y)=\frac{1}{\sqrt{2\pi}\sigma_{\mathrm{v}}}\int_{\omega}^{\infty}v\left[C-bl\frac{\mathrm{d}C}{\mathrm{d}y}\right]\exp\left[-0.5\left(\frac{v}{\sigma_{\mathrm{v}}}\right)^2\right]\mathrm{d}v$$
$$=\frac{\sigma_{\mathrm{v}}}{\sqrt{2\pi}}\left\{\exp\left[-0.5\left(\frac{\omega}{\sigma_{\mathrm{v}}}\right)^2\right]\right\}\left(C-bl\frac{\mathrm{d}C}{\mathrm{d}y}\right), \tag{8}$$

where $\Omega_{Tu}(y)$ is upward sediment flux due to turbulence.

Similarly, the downward sediment fluxes from the ($y+bl$) layer due to gravity and downward vertical turbulence are respectively

$$\Omega_{\mathrm{g}}(y)=-\omega\left(C+bl\mathrm{d}C/\mathrm{d}y\right), \tag{9}$$

$$\Omega_{\mathrm{Td}}(y)=\frac{1}{\sqrt{2\pi}\sigma_{\mathrm{v}}}\int_{-\infty}^{\omega}v\left(C+bl\frac{\mathrm{d}C}{\mathrm{d}y}\right)\exp\left[-0.5\left(\frac{v}{\sigma_{\mathrm{v}}}\right)^2\right]\mathrm{d}v$$
$$=-\frac{\sigma_{\mathrm{v}}}{\sqrt{2\pi}}\left\{\exp\left[-0.5\left(\frac{\omega}{\sigma_{\mathrm{v}}}\right)^2\right]\right\}\left(C+bl\frac{\mathrm{d}C}{\mathrm{d}y}\right), \tag{10}$$

where $\Omega_g(y)$ is sediment flux due to gravity; $\Omega_{Td}(y)$ is downward sediment flux due to turbulence.

Thus sediment flux through the unit-horizontal-section at y is

$$\begin{aligned}\Omega(y) &= -(\omega+V_e)\left(C+bl\frac{dC}{dy}\right)+V_e\left(C-bl\frac{dC}{dy}\right)\\ &= -\left[\omega C+bl(\omega+2V_e)\frac{dC}{dy}\right],\end{aligned} \tag{11}$$

where

$$V_e=\frac{\sigma_v}{\sqrt{2\pi}}\exp\left[-0.5(\omega/\sigma_v)^2\right], \tag{12}$$

where V_e is called the effective laden-sediment velocity. Grass (1971), using hydrogen bubble flow tracers and medium high-speed motion photography, obtained the vertical components of turbulence intensity. His results show that σ_v hardly varies in the vertical direction. Thus we assume V_e to be independent of y.

It should be noted that a non-statistical-processed stochastic variable should not be taken as a deterministic variable, nor should a deterministic variable be subject to unnecessary statistical processing. Conforming to this logic, gravity flux and diffusion flux will be discussed separately in this paper.

Under the circumstances of sediment exchange equilibrium, $\Omega(y)$=0, therefore

$$\omega C+bl(\omega+2V_e)dC/dy=0, \tag{13}$$

if

$$\varepsilon_s=bl(\omega+2V_e). \tag{14}$$

Then Eq.(13) reduces to a diffusion equation of suspended load, Eq.(2). It is shown in Eq.(14) that the diffusion coefficient ε_s, is related not only to the flow fluctuation parameter σ_v and the mixing length l, but also to the settling velocity of particles. This relationship can be explained as an effect of sediment on mass exchange due to turbulence. The difference between flow momentum exchange and sediment exchange results from gravity settling of sediment particles.

SOLUTION OF DIFFERENTIAL EQUATION

To derive the mixing length l, Jasmund-Nikurads's logarithmic velocity profile (Bogardi, 1974) is used:

$$\frac{u}{U_*}=\frac{1}{\kappa}\ln\left(1+\frac{y}{k_s}\right), \tag{15}$$

where u is velocity of flow at y; U_* is shear velocity; κ is von Karman constant; k_s is roughness height.

Taking the derivative of u with respect to y from Eq.(15) as

$$\frac{du}{dy}=\frac{U_*}{\kappa(y+k_s)}. \tag{16}$$

According to Prandtl's mixing theory, the mixing length can be obtained as

$$l=\sqrt{\frac{\tau_0}{\rho}\left(1-\frac{y}{H}\right)}\Bigg/\frac{du}{dy}=\kappa(y+k_s)\left(1-\frac{y}{H}\right)^{0.5}, \tag{17}$$

where H is the depth of flow.

If Larsen (1980)'s assumption is adapted that

$$b=(1-y/H)^{-0.5}, \tag{18a}$$

then

$$bl=\kappa(y+k_s). \tag{18b}$$

As a result,

$$\varepsilon_s=bl(\omega+2V_e)=\beta_*\kappa U_*(y+k_s), \tag{19a}$$

where

$$\beta_*=(\omega+2V_e)/U_*. \tag{19b}$$

When Eq.(19) is substituted into Eq.(2), and its integral is taken, one obtains

$$\ln C(y)=\ln[1/(y+k_s)]^Z+const. \tag{20}$$

Using the boundary condition $C(0)=C_{0*}$, the vertical distribution of sediment concentration is described as

$$C(y) = C_{0*}[k_s/(y+k_s)]^Z, \tag{21}$$

where suspension index Z is defined as

$$Z = \omega/(\beta_* \kappa U_*). \tag{22}$$

RESULT ANALYSIS

Taking into account that the formula for equilibrium bed-concentration represents a statistical-average relation, the value of C_{0*} calculated from such a formula will not be exactly equal to the value which exists but cannot be measured at present. To verify reasonability of the formula structure, Eq.(21) should be changed into a form in which the sediment concentration C_a at the lowest measured point a, is taken as a reference concentration

$$C(y) = C_a \left(\frac{a+k_s}{y+k_s} \right)^Z. \tag{23}$$

Eq.(23) is verified with Vanoni (1946) experimental data, as shown in Fig.2 and Table 1. Good agreement between the theoretical curves and the measured data indicates that the structure of Eq.(21) is reasonable. The Rouse curve is also plotted in Fig.2. Obviously Eq.(21) avoids deficiencies in the Rouse formula. At the water-surface when $y=h$, $C(h) = C_{0*}(k_s/(h+k_s))^Z \neq 0$. At the bottom when $y=0$, $C(0) = C_{0*}$, which is less than M_0 according to Eq.(4), instead of ∞. The present formula is also simple and convenient for practical use.

Eq.(21) shows that roughness of bed-surface k_s affects the vertical gradient of sediment concentration in the near-bed flow layer. It seems that the larger k_s is, the smaller the concentration gradient becomes. Engelund (1965) let k_s be $2D_{65}$, whereas van Rijn (1982) used $3D_{90}$. We believe that taking k_s as $3D_m$ is appropriate, where D_m is the mean diameter of bed material.

According to Eq.(15) and Eq.(21), let $\eta=y/h$ as relative depth of flow and $\eta_0=k_s/h$ as relative roughness, then a predictor for suspended load discharge can be developed:

Table 1 Vanoni (1946)'s experimental condition (a=0.05h)

Figure No.	D_m (mm)	D_*	τ_*	ω/U_*	h	C_a
a	0.160	4.019	1.391	0.260	0.148	2.800
b	0.160	4.019	1.419	0.258	0.151	6.950
c	0.160	4.019	1.432	0.257	0.158	9.250
d	0.160	4.019	1.495	0.251	0.158	8.400
e	0.160	4.019	1.437	0.256	0.158	8.400
f	0.160	4.019	1.368	0.263	0.158	19.60
g	0.160	4.019	1.380	0.262	0.146	14.00
h	0.160	4.019	1.555	0.246	0.164	9.090
i	0.160	4.019	0.792	0.345	0.080	5.250
j	0.160	4.019	0.775	0.349	0.164	3.020
k	0.160	4.019	0.578	0.404	0.082	0.995
l	0.100	2.512	1.067	0.157	0.141	4.250
m	0.100	2.512	0.546	0.219	0.072	1.200
n	0.100	2.512	1.081	0.156	0.072	3.420
o	0.130	3.266	1.048	0.228	0.090	6.750

D_m is mean diameter of bed material; D_* is dimensionless grain size

$$\begin{aligned} q_s &= \int_0^h uC(y)\,dy \\ &= 2.5U_*C_0h\int_0^1 \ln\left(1+\frac{\eta}{\eta_0}\right)\left(1+\frac{\eta}{\eta_0}\right)^{-Z} d\eta \\ &= \begin{cases} 1.25\, U_*C_0k_s\ln^2(1+1/\eta_0), & Z=1, \\ 2.5U_*C_0k_s(1-Z)^{-2}\Big[(1-Z)(1+1/\eta_0)^{1-Z} \\ \quad \cdot\ln(1+1/\eta_0)+1-(1+1/\eta_0)^{1-Z}\Big], & Z\neq 1. \end{cases} \end{aligned} \tag{24}$$

Eq.(24) is a theoretical relation taking the form of an elementary function that can be used to calculate the unit-width transport rate of the suspended load. This equation will improve the Einstein approach where the unit suspended sediment discharge must be solved by numerical integration. It also avoids the contradiction between the lower integral limit and that of velocity distribution in the Einstein method.

CONCLUSION

Based on the differential equation for sediment diffusion in the 2D steady turbulent flow, a new simple formula for vertical distribution of suspended load

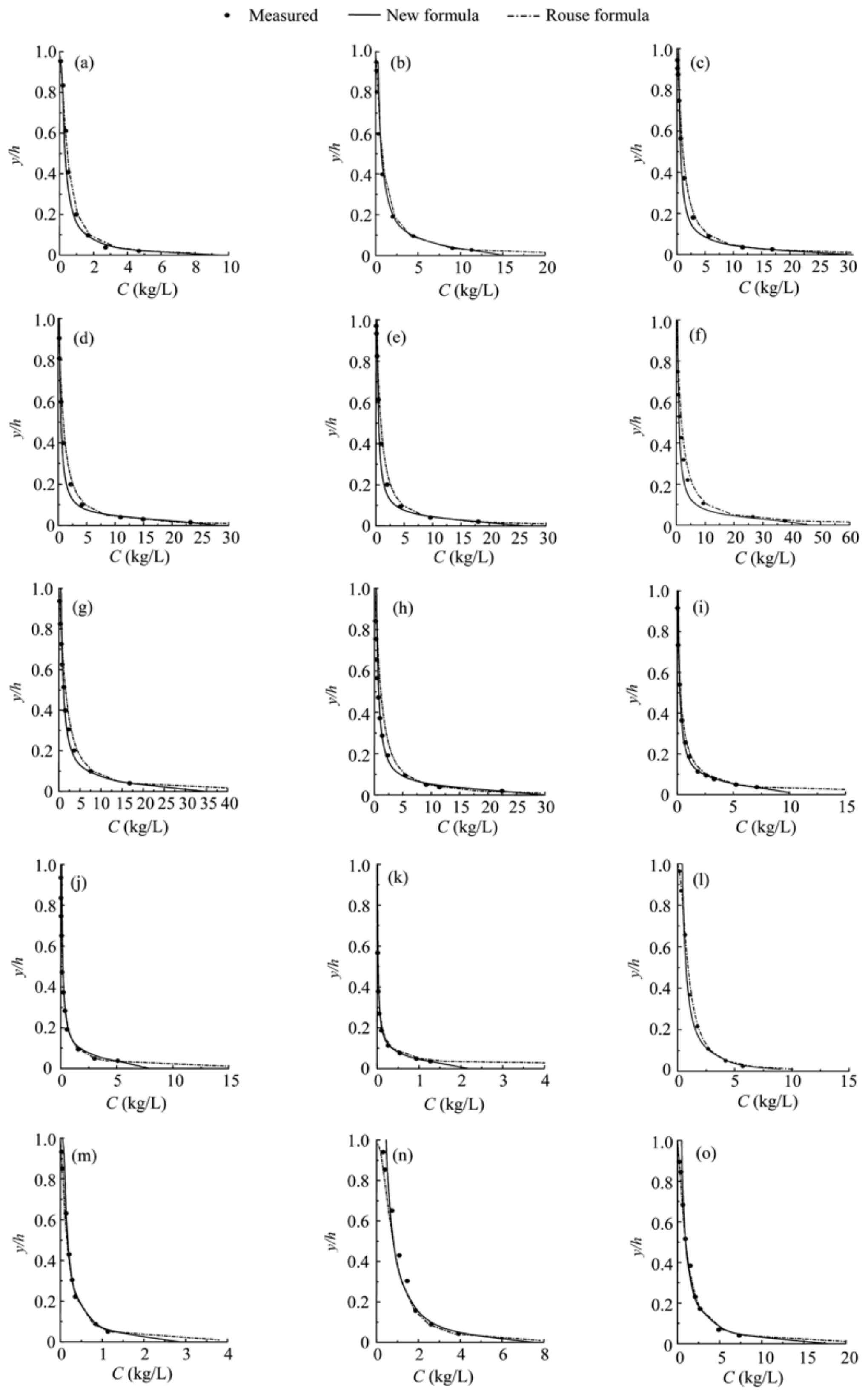

Fig.2 (a)~(o) comparison of theoretical curves with Vanoni (1946)'s experimental data

concentration is proposed, which avoids such deficiencies as the infinite value of concentration at the bottom and zero value at the surface, existing in the Rouse formula. A theoretical elementary function for suspended load discharge is also derived from the new formula and improves the famous Einstein equation.

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