Journal of Zhejiang University SCIENCE A
ISSN 1673-565X (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Science Letters:

On numerical calculation in symplectic approach for elasticity problems*

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Received Dec. 20, 2007; revision accepted Apr. 1, 2008; published online Apr. 11, 2008

Abstract: The symplectic approach proposed and developed by Zhong *et al.* in 1990s for elasticity problems is a rational analytical method, in which ample experience is not needed as in the conventional semi-inverse method. In the symplectic space, elasticity problems can be solved using the method of separation of variables along with the eigenfunction expansion technique, as in traditional Fourier analysis. The eigensolutions include those corresponding to zero and nonzero eigenvalues. The latter group can be further divided into α- and β-sets. This paper reformulates the form of β-set eigensolutions to achieve the stability of numerical calculation, which is very important to obtain accurate results within the symplectic frame. An example is finally given and numerical results are compared and discussed.

Key words: Symplectic approach, Eigenfunction, Numerical stability, Elasticity problems
doi:10.1631/jzus.A0720124 **Document code:** A **CLC number:** TU4

INTRODUCTION

Plane elasticity problems have been studied extensively, and most are solved using stress function through a trial-and-error procedure (Timoshenko and Goodier, 1970; Barber, 1992). Recently, Ding *et al.*(2007) and Huang *et al.*(2007) successfully extended the stress function method to analyze functionally graded anisotropic beams with arbitrary material inhomogeneity along the beam depth. But the successful application of this method to a concrete problem usually requires skillful experience because there is no unified treatment for problems with different boundary conditions, geometric shapes, or material properties. It is interesting to note that Jiang and Ding (2005) obtained some analytical solutions to the problems of 2D orthotropic cantilever beams by assuming appropriate forms of displacements.

To avoid the difficulty, Zhong proposed the symplectic approach for elasticity problems (Zhong, 1995; Zhong and Yao, 2002). In the symplectic space, the method of separation of variables can be employed and the adjoint symplectic orthogonality of eigenfunctions can be utilized to obtain rational solutions that are similar to the conventional Fourier series. Because it is not necessary to use the Saint-Venant's principle in the analysis, highly accurate local stress and displacement distributions (e.g. at the clamped end of a cantilever beam) can be obtained. Stephen *et al.* applied the symplectic approach to the problems of prismatic rod, curved and pre-twisted beams as well as 1D repetitive structures, and showed that the approach can lead to deep understandings of the physical problems (Stephen, 2004; 2006; Stephen and Ghosh, 2005; Stephen and Zhang, 2006). The symplectic approach has also been applied to solve the problems of piezoelectric media (Gu *et al.*, 1995; Xu *et al.*, 2005).

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* Project supported by the National Natural Science Foundation of China (Nos. 10725210 and 10432030), the Specialized Research Fund for the Doctoral Program of Higher Education (No. 20060335107) and the Program for New Century Excellent Talents in University, MOE, China (No. NCET-05-05010)

Recently, Leung and Zheng (2007) applied the symplectic approach to the problems of a straight plane orthotropic beam subjected to different loads. They considered several particular cases to illustrate the capability of symplectic approach in obtaining accurate local stress distributions. It is noted that the expansion series in the symplectic approach consists of eigensolutions corresponding to zero and nonzero eigenvalues. The nonzero ones can be further divided into two sets, i.e., the α- and β-sets, according to the sign of real part of the eigenvalue (or imaginary part when the real part is zero). On the argument that the two sets affect different ends of the beam, Leung and Zheng (2007) discarded the β-set in their calculation of stresses at the clamped end of the beam. However, this is incorrect from a strict mathematics point of view, because for a finite beam (Example 2 in their paper) the complete solution of the problem should contain all eigensolutions, though the β-set has much smaller effect on the local stress distributions at the clamped end when the span-to-thickness ratio of the beam is large.

When a large number of eigensolutions are involved to obtain accurate results, enormous difference of magnitude exists between the α- and β-sets eigensolutions. This may lead to serious numerical problem when the calculations are performed on a computer. To overcome the difficulty, we present a novel form of the β-set eigensolutions in the symplectic expansion series. The analogy tactic has been employed by Leung (1988; 1990) for the integration of beam eigenfunctions. Numerical example is given to illustrate that using the novel form can effectively avoid the numerical problem, yielding results as accurately as wanted.

BASIC EQUATIONS

Consider the plane stress problem of a beam occupying the rectangular domain V: $0\le x\le L$, $-h\le z\le h$ as shown in Fig.1. For isotropic materials, the constitutive equations are

$$\sigma_x=\frac{E}{1-\nu^2}\left(\frac{\partial u}{\partial x}+\nu\frac{\partial w}{\partial z}\right),\ \sigma_z=\frac{E}{1-\nu^2}\left(\nu\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right),$$
$$\tau_{xz}=\frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right), \tag{1}$$

where u and w are displacements in x- and z-directions respectively, and E and ν are Young's modulus and Poisson's ratio, respectively. The governing differential equations of equilibrium are

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xz}}{\partial z}+F_x=0,$$
$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\sigma_z}{\partial z}+F_z=0. \tag{2}$$

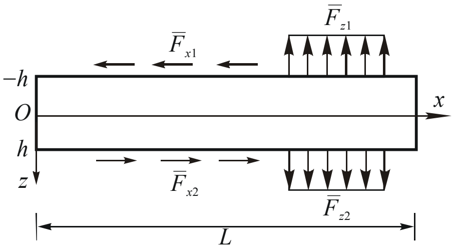

Fig.1 The plane beam problem

The following equation can be deduced from Eqs.(1) and (2) directly,

$$\frac{\partial}{\partial x}\begin{Bmatrix}u\\ w\\ \sigma_x\\ \tau_{xz}\end{Bmatrix}=$$
$$\begin{bmatrix}0 & -\nu\frac{\partial}{\partial z} & \frac{1-\nu^2}{E} & 0\\ -\frac{\partial}{\partial z} & 0 & 0 & \frac{2(1+\nu)}{E}\\ 0 & 0 & 0 & -\frac{\partial}{\partial z}\\ 0 & -E\frac{\partial^2}{\partial z^2} & -\nu\frac{\partial}{\partial z} & 0\end{bmatrix}\begin{Bmatrix}u\\ w\\ \sigma_x\\ \tau_{xz}\end{Bmatrix}+\begin{Bmatrix}0\\ 0\\ -F_x\\ -F_z\end{Bmatrix}, \tag{3}$$

which can be rewritten more compactly as

$$\dot{\boldsymbol{v}}=\boldsymbol{H}\boldsymbol{v}+\boldsymbol{q}, \tag{4}$$

where $\boldsymbol{v}=[u\ w\ \sigma_x\ \tau_{xz}]^{\mathrm{T}}$ is the state vector. Eq.(4) is usually referred to as the state equation (Kameswara Rao and Das, 1977; Ding *et al*., 2006), and is accordant with the dual equation derived by Zhong (1995) in the Hamiltonian system.

SYMPLECTIC APPROACH

This section gives a brief review of the symplectic approach for elasticity problems (Zhong, 1995; Zhong and Yao, 2002). First, we consider the homogeneous equation of Eq.(4), i.e.,

$$\dot{\boldsymbol{v}} = \boldsymbol{H}\boldsymbol{v}, \tag{5}$$

with the following homogeneous boundary conditions on the lateral surfaces:

$$z = \pm h: \ \sigma_z = \nu\sigma_x + E\frac{\partial w}{\partial z} = 0, \ \tau_{xz} = 0. \tag{6}$$

It is easy to prove that the operator matrix $\boldsymbol{H}$ is a Hamiltonian transform (operator matrix) in the symplectic geometric space.

Using the method of separation of variables, we obtain the following solution to Eq.(5)

$$\boldsymbol{v}(x,z) = \mathrm{e}^{\mu x}\boldsymbol{\psi}(z), \tag{7}$$

where μ is the eigenvalue of Hamiltonian matrix $\boldsymbol{H}$, and $\boldsymbol{\psi}(z)$ is the corresponding eigenvector. According to (Zhong, 1995; Zhong and Yao, 2002), we know that, if μ is the eigenvalue of $\boldsymbol{H}$, $-\mu$ is also an eigenvalue. Therefore we can divide the eigenvalues, the number of which is infinite, into two groups, i.e., zero eigenvalues and nonzero eigenvalues. The latter group is further divided into α- and β-sets. These are summarized as follows,

$$(\alpha)\ \mu_i,\ \mathrm{Re}[\mu_i]<0 \text{ or } \mathrm{Re}[\mu_i]=0\cap\mathrm{Im}[\mu_i]<0,\ \ i=1,2,\ldots, \tag{8a}$$

$$(\beta)\ \mu_{-i}=-\mu_i, \tag{8b}$$

$$(\gamma)\ \mu_0=0. \tag{8c}$$

The zero eigenvalues are a special set because the eigensolutions do not decrease with x, as can be seen from Eq.(7). These eigensolutions actually correspond to the classical solutions of Saint-Venant's problem. The six eigenvectors corresponding to the zero eigenvalues are (Zhong, 1995)

$$\boldsymbol{\psi}_{\mathrm{s}}^{(0)} = [1,0,0,0]^{\mathrm{T}}, \quad \boldsymbol{\psi}_{\mathrm{a}}^{(0)} = [0,1,0,0]^{\mathrm{T}},$$

$$\boldsymbol{\psi}_{\mathrm{s}}^{(1)} = [0,-\nu z,E,0]^{\mathrm{T}}, \ \boldsymbol{\psi}_{\mathrm{a}}^{(1)} = [-z,0,0,0]^{\mathrm{T}},$$

$$\boldsymbol{\psi}_{\mathrm{a}}^{(2)} = [0,\nu z^2/2+c,-Ez,0]^{\mathrm{T}},$$

$$\boldsymbol{\psi}_{\mathrm{a}}^{(3)} = \left[-(1+\nu)h^2z - cz + \frac{1}{6}(2+\nu)z^3, 0, 0, \frac{1}{2}E(z^2-h^2)\right]^{\mathrm{T}}, \tag{9}$$

where the superscripts 0 and i indicate the fundamental eigenvector and the ith eigenvector of Jordan normal form, respectively, while subscripts s and a denote the symmetric and antisymmetric deformations, respectively. The eigensolutions of zero eigenvalues can be obtained as

$$\begin{aligned}
&\boldsymbol{v}_{\mathrm{s}}^{(0)} = \boldsymbol{\psi}_{\mathrm{s}}^{(0)},\ \boldsymbol{v}_{\mathrm{a}}^{(0)} = \boldsymbol{\psi}_{\mathrm{a}}^{(0)},\ \boldsymbol{v}_{\mathrm{s}}^{(1)} = \boldsymbol{\psi}_{\mathrm{s}}^{(1)} + z\boldsymbol{\psi}_{\mathrm{s}}^{(0)},\\
&\boldsymbol{v}_{\mathrm{a}}^{(1)} = \boldsymbol{\psi}_{\mathrm{a}}^{(1)} + z\boldsymbol{\psi}_{\mathrm{a}}^{(0)},\ \boldsymbol{v}_{\mathrm{a}}^{(2)} = \boldsymbol{\psi}_{\mathrm{a}}^{(2)} + z\boldsymbol{\psi}_{\mathrm{a}}^{(1)} + z^2\boldsymbol{\psi}_{\mathrm{a}}^{(0)}/2,\\
&\boldsymbol{v}_{\mathrm{a}}^{(3)} = \boldsymbol{\psi}_{\mathrm{a}}^{(3)} + z\boldsymbol{\psi}_{\mathrm{a}}^{(2)} + z^2\boldsymbol{\psi}_{\mathrm{a}}^{(1)}/2 + z^3\boldsymbol{\psi}_{\mathrm{a}}^{(0)}/6.
\end{aligned} \tag{10}$$

The general solution to the eigenequation, which governs the eigenvector $\boldsymbol{\psi}(z)$ for nonzero eigenvalue μ, is

$$\boldsymbol{\psi}(z) = \begin{bmatrix} A_1 & B_1 & C_1 & D_1 \\ B_2 & A_2 & D_2 & C_2 \\ A_3 & B_3 & C_3 & D_3 \\ B_4 & A_4 & D_4 & C_4 \end{bmatrix} \begin{Bmatrix} \cos(\mu z) \\ \sin(\mu z) \\ z\sin(\mu z) \\ z\cos(\mu z) \end{Bmatrix}. \tag{11}$$

The general solution can be divided into symmetric part (B_i=D_i=0) and antisymmetric part (A_i=C_i=0) as well. For the symmetric part, the homogeneous boundary conditions on lateral surfaces lead to the following equation satisfied by the nonzero eigenvalues

$$2\mu_i h + \sin(2\mu_i h) = 0, \quad i=1,2,\ldots; \tag{12a}$$

and for the antisymmetric part, we get

$$2\mu_i h - \sin(2\mu_i h) = 0, \ i=1,2,\ldots \tag{12b}$$

Meanwhile, the ratio between the coefficients A_i and C_i or B_i and D_i can be obtained. Thus the eigenvectors $\boldsymbol{\psi}_i$ can be determined and the eigensolutions are

$$\boldsymbol{v}_i = \mathrm{e}^{\mu_i x}\boldsymbol{\psi}_i,\ \boldsymbol{v}_{-i} = \mathrm{e}^{-\mu_i x}\boldsymbol{\psi}_{-i}, \tag{13}$$

for the α- and β-sets, respectively.

Any two of the eigenvectors ψ_i of nonzero eigenvalues satisfy the adjoint symplectic orthogonality relationships, and they are all orthogonal to the eigenvectors of zero eigenvalues in the symplectic sense. In other words, when $\mu_i+\mu_j\neq0$, we have

$$\int_{-h}^{h}\boldsymbol{\psi}_i^{\mathrm{T}}\boldsymbol{J}\boldsymbol{\psi}_j\,\mathrm{d}x = 0, \tag{14}$$

where $\boldsymbol{J}$ is the unit symplectic matrix. It is easy to prove that the adjoint symplectic orthogonality relationships are valid between the six eigenvectors of zero eigenvalues in Eq.(9) if we take $c=-(2/5+\nu/2)h^2$ in $\psi_{\mathrm{a}}^{(3)}$. Therefore, for the beam with nonzero boundary conditions only at its two ends, the complete solution can be expanded as

$$\boldsymbol{v} = \sum_{i=0}^{1} m_{0\mathrm{s}}^{(i)}\boldsymbol{v}_{\mathrm{s}}^{(i)} + \sum_{i=0}^{3} m_{0\mathrm{a}}^{(i)}\boldsymbol{v}_{\mathrm{a}}^{(i)} + \sum_{i=1}^{\infty}(m_i\boldsymbol{v}_i + m_{-i}\boldsymbol{v}_{-i}), \tag{15}$$

where $m_{0\mathrm{s}}^{(i)}$, $m_{0\mathrm{a}}^{(i)}$, m_i and m_{-i} are constants determined by a linear system of equations resulted from the Hamiltonian variational principle in which Eq.(15) and the boundary conditions are accounted for. In practice, the infinite summation in Eq.(15) should be truncated and an approximate solution is obtained as

$$\boldsymbol{v} \approx \sum_{i=0}^{1} m_{0\mathrm{s}}^{(i)}\boldsymbol{v}_{\mathrm{s}}^{(i)} + \sum_{i=0}^{3} m_{0\mathrm{a}}^{(i)}\boldsymbol{v}_{\mathrm{a}}^{(i)} + \sum_{i=1}^{N}(m_i\boldsymbol{v}_i + m_{-i}\boldsymbol{v}_{-i}), \tag{16}$$

where N is an integer, which is determined according to the required precision of the results.

REFORMULATING AND NUMERICAL VERIFICATION

As we can see, the eigensolution $\boldsymbol{v}_{-i}$ in the conventional expansion of Eq.(15) has the factor $\mathrm{e}^{-\mu_i L}$, which becomes very large when i is large. This will lead to numerical problem due to the limited precision of the floating-point representation of a numerical quantity imposed by the digital computers (Pestel and Leckie, 1963). Leung and Zheng (2007) avoided the difficulty by discarding the β-set eigensolutions in their calculation. However, for a finite beam, this treatment is mathematically incorrect, and all eigensolutions should be taken into consideration.

In order to overcome the numerical difficulty, we rewrite Eq.(15) as

$$\begin{aligned}\boldsymbol{v} &= \sum_{i=0}^{1} m_{0\mathrm{s}}^{(i)}\boldsymbol{v}_{\mathrm{s}}^{(i)} + \sum_{i=0}^{3} m_{0\mathrm{a}}^{(i)}\boldsymbol{v}_{\mathrm{a}}^{(i)} + \sum_{i=1}^{\infty}(m_i\boldsymbol{v}_i + m_{-i}\,\mathrm{e}^{-\mu_i x}\boldsymbol{\psi}_{-i}) \\ &= \sum_{i=0}^{1} m_{0\mathrm{s}}^{(i)}\boldsymbol{v}_{\mathrm{s}}^{(i)} + \sum_{i=0}^{3} m_{0\mathrm{a}}^{(i)}\boldsymbol{v}_{\mathrm{a}}^{(i)} + \sum_{i=1}^{\infty}(m_i\boldsymbol{v}_i + m'_{-i}\,\mathrm{e}^{-\mu_i (x-L)}\boldsymbol{\psi}_{-i}) \\ &= \sum_{i=0}^{1} m_{0\mathrm{s}}^{(i)}\boldsymbol{v}_{\mathrm{s}}^{(i)} + \sum_{i=0}^{3} m_{0\mathrm{a}}^{(i)}\boldsymbol{v}_{\mathrm{a}}^{(i)} + \sum_{i=1}^{\infty}(m_i\boldsymbol{v}_i + m'_{-i}\boldsymbol{v}'_{-i}),\end{aligned} \tag{17}$$

where $m'_{-i} = m_{-i}\,\mathrm{e}^{-\mu_i L}$ and $\boldsymbol{v}'_{-i} = \mathrm{e}^{-\mu_i(x-L)}\boldsymbol{\psi}_{-i}$. Since $x\in[0,L]$, the factor $\mathrm{e}^{-\mu_i(x-L)}$ is always less than 1, i.e. no big number will appear in the new expansion formula, and hence numerical stability can be achieved.

For illustrating Eq.(17), we consider, for example, a clamped-free isotropic beam of span-to-thickness ratio $L/(2h)=5$, subject to a prescribed shear stress $\tau_0=1$ N/m at the free end. Young's modulus is taken to be $E=2.0\times10^{11}$ N/m^2, Poisson's ratio $\nu=0.29$ and the height of beam $2h=1$ m.

Fig.2 displays the distributions of displacements and stresses at the clamped end by virtue of Eq.(17). The results obtained by discarding the β-set eigensolutions are given in Fig.3 for the sake of comparison. It can be seen that although the inclusion of β-set does not affect much the stress distributions, it does have an obvious effect on the displacement distributions. Both the axial and transverse displacements at the clamped end in Fig.2, except at the corners, fluctuate around zero, which accord with the clamped boundary conditions. On the contrary, when the β-set is discarded, the predicted axial displacement varies almost linearly along the depth, while the transverse one is identically less than zero, as shown in Fig.3. It is obvious that the results obtained without β-set eigensolutions are somewhat unreasonable, although both of the displacements are small and approach zero. Note that no results of displacements are presented by Leung and Zheng (2007).

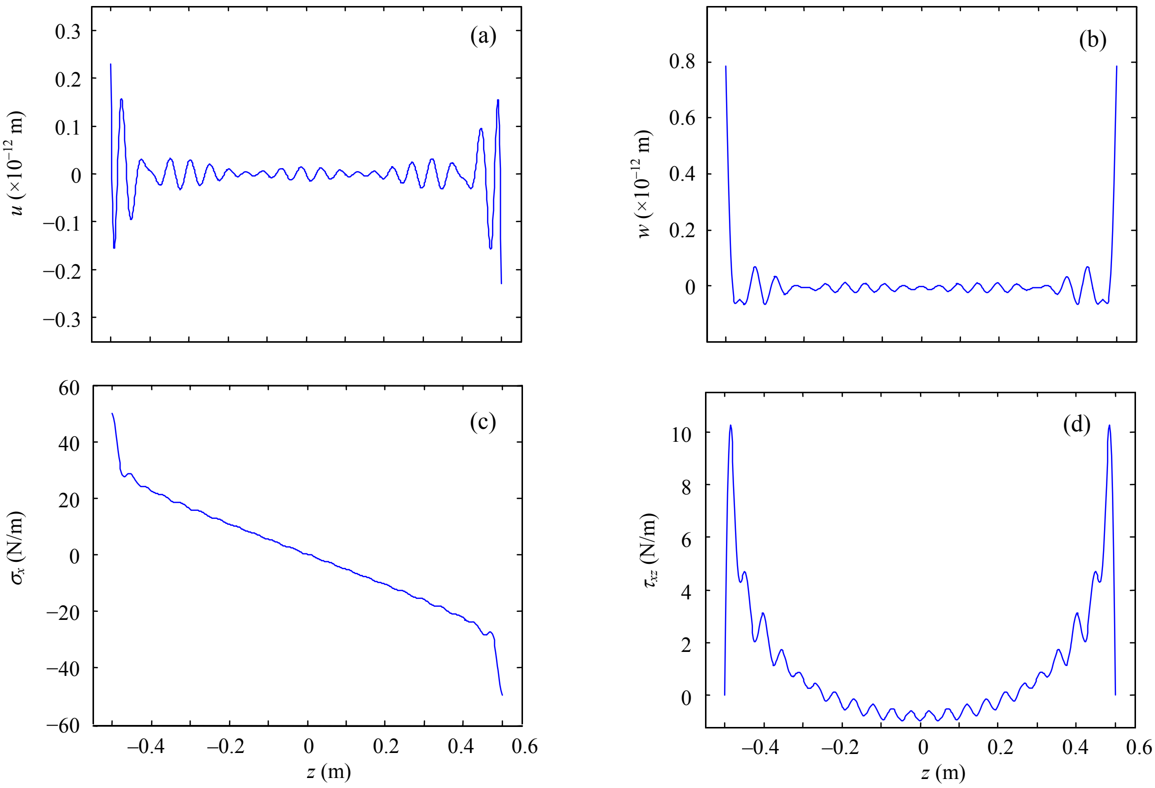

Fig.2 Stresses and displacements at the clamped end using Eq.(17). (a) u; (b) w; (c) σ_x; (d) τ_{xz}

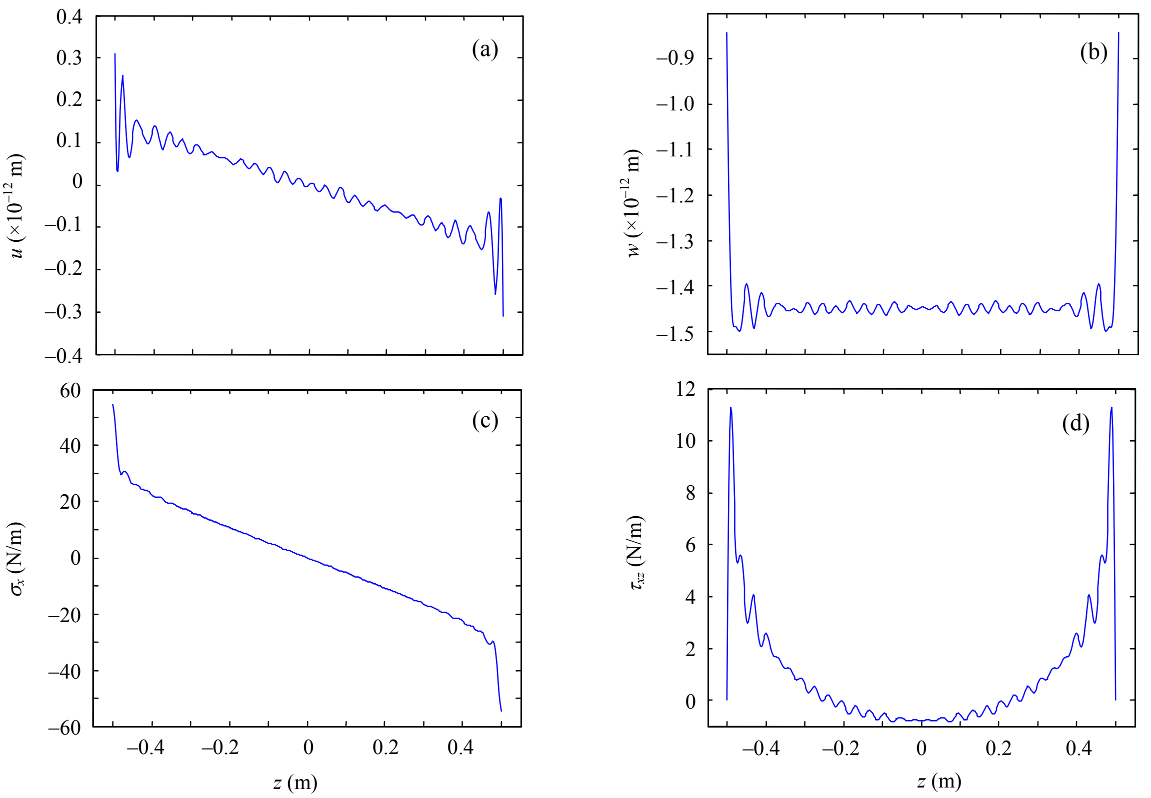

Fig.3 Stresses and displacements at the clamped end by discarding the β-set in Eq.(15). (a) u; (b) w; (c) σ_x; (d) τ_{xz}

CONCLUSION

There is no requirement of experience in the symplectic approach for solving a concrete elasticity problem since it is rational and bears a unified mathematical treatment. Moreover, the symplectic approach provides physical interpretations of the problem that cannot be attained by the traditional methods (such as the stress function method). But the symplectic approach involves the expansion in terms of eigensolutions, by which the boundary conditions could not be strictly satisfied in many cases. A linear system of algebraic equations is then derived to solve approximately the unknown constants in the expansion formula. Matrix singularity usually arises because of the distinct orders of magnitude of coefficients, leading to the numerical difficulty. This paper rewrites the symplectic expansion formula to exclude large numbers. Numerical example indicates that the new expansion formula can effectively avoid the numerical instability and hence give more reasonable results (especially the displacement distributions) at the clamped end than those using the conventional formula.

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