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A minimax optimal control strategy for uncertain quasi-Hamiltonian systems*

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Abstract: A minimax optimal control strategy for quasi-Hamiltonian systems with bounded parametric and/or external disturbances is proposed based on the stochastic averaging method and stochastic differential game. To conduct the system energy control, the partially averaged Itô stochastic differential equations for the energy processes are first derived by using the stochastic averaging method for quasi-Hamiltonian systems. Combining the above equations with an appropriate performance index, the proposed strategy is searching for an optimal worst-case controller by solving a stochastic differential game problem. The worst-case disturbances and the optimal controls are obtained by solving a Hamilton-Jacobi-Isaacs (HJI) equation. Numerical results for a controlled and stochastically excited Duffing oscillator with uncertain disturbances exhibit the efficacy of the proposed control strategy.

Key words: Nonlinear quasi-Hamiltonian system, Minimax optimal control, Stochastic excitation, Uncertain disturbance, Stochastic averaging, Stochastic differential game

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INTRODUCTION

The mathematical theory of stochastic optimal control has been quite well developed (Fleming and Soner, 1993; Yong and Zhou, 1999). However, for several decades, only the linear quadratic Gaussian (LQG) and bang-bang control strategies have been used mostly in structural engineering. In recent years, a nonlinear stochastic optimal control strategy has been proposed for quasi-Hamiltonian systems with external and/or parametric stochastic excitations based on the stochastic averaging method and stochastic dynamical programming principle (Zhu and Ying, 1999; Zhu *et al*., 2001). The strategy has been extended to the stochastic optimal control of partially observable systems (Zhu and Ying, 2002; Ying and Zhu, 2008), the stochastic optimal semi-active control (Ying *et al*., 2003; Cheng *et al*., 2006), the stochastic optimal bounded control (Zhu and Deng, 2004; Ying and Zhu, 2006), the stochastic stabilization (Zhu and Huang, 2003; Zhu, 2004), and applied to hysteretic systems (Zhu *et al*., 2000; Ying and Zhu, 2003). The proposed control strategy has several advantages over the LQG controller and deserves further development.

In the above investigation, uncertain disturbances are not under consideration. In practice, however, parametric and external disturbances cannot be avoided, and that will degenerate the performance of the controller which is established based on the nominal system. In the past several decades, the robust control of the deterministic linear and nonlinear systems with uncertain disturbances has been studied extensively (Zhou *et al*., 1996). The robust control of linear stochastic uncertain systems has also been investigated by minimizing worst-case performance

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under certain uncertainty constraints (Petersen *et al*., 2000). The multiplicative and additive noise cases are discussed respectively in (Petersen and James, 1996; Ugrinovskii and Petersen, 1997), and for the latter case, the LQG stochastic optimal control method is extended into a minimax stochastic optimal control method.

In this paper, a minimax optimal control strategy for uncertain nonlinear quasi-Hamiltonian systems is proposed. The proposed strategy is searching for an optimal worst-case controller by solving a stochastic differential game problem, and the worst-case disturbances and the optimal controls are obtained by solving a Hamilton-Jacobi-Isaacs (HJI) equation.

Stochastic averaging

Considering the following controlled, stochastically excited and dissipated Hamiltonian system with parametric and/or external disturbances:

$$\dot{Q}_i = \frac{\partial H'(\boldsymbol{Q},\boldsymbol{P},\bar{\boldsymbol{s}})}{\partial P_i},$$
$$\dot{P}_i = -\frac{\partial H'(\boldsymbol{Q},\boldsymbol{P},\bar{\boldsymbol{s}})}{\partial Q_i} - [\bar{c}_{ij}(\boldsymbol{Q},\boldsymbol{P}) + \tilde{c}_{ij}(t)]\frac{\partial H'(\boldsymbol{Q},\boldsymbol{P},\bar{\boldsymbol{s}})}{\partial P_j}$$
$$+ u_i(\boldsymbol{Q},\boldsymbol{P}) + f_{ik}(\boldsymbol{Q},\boldsymbol{P})\xi_k(t) - \tilde{s}_l(t)g_{il}(\boldsymbol{Q}) + \tilde{w}_i(t),$$
$$i,j=1,2,\cdots,n;\quad k=1,2,\cdots,m;\quad l=1,2,\cdots,p, \qquad (1)$$

where Q_i and P_i are generalized displacements and momenta, respectively; $\boldsymbol{Q}=[Q_1, Q_2, \ldots, Q_n]^{\mathrm{T}}$, $\boldsymbol{P}=[P_1, P_2, \ldots, P_n]^{\mathrm{T}}$; $H'(\boldsymbol{Q},\boldsymbol{P},\bar{\boldsymbol{s}})$ is the Hamiltonian generally representing total system energy; $\bar{\boldsymbol{s}}$ and $\bar{c}_{ij}(\boldsymbol{Q},\boldsymbol{P})$ are the nominal values of the stiffness and damping coefficients, respectively; $f_{ik}(\boldsymbol{Q}, \boldsymbol{P})$ are excitation amplitudes; $\xi_k(t)$ are Gaussian white noises with zero mean and correlation function $2D_{kl}\delta(\tau)$; $u_i(\boldsymbol{Q}, \boldsymbol{P})$ represent feedback controls; $\tilde{s}_l(t)$ and $\tilde{c}_{ij}(t)$ are the parametric disturbances; and $\tilde{w}_i(t)$ are the external disturbances. $g_{il}(\boldsymbol{Q})$ are determined by the special system. Assumed that $\bar{c}_{ij}=O(\varepsilon)$, $f_{ik}=O(\varepsilon^{1/2})$, $u_i=O(\varepsilon)$, $\tilde{c}_{ij}(t)=O(\varepsilon)$, $\tilde{s}_l(t)=O(\varepsilon)$, $\tilde{w}_i(t)=O(\varepsilon)$, where ε is a small parameter, and that $\tilde{c}_{ij}(t)$, $\tilde{s}_l(t)$ and $\tilde{w}_i(t)$ are all bounded, i.e., $\tilde{c}_{ij}(t)\in[-c_{ij}^0, c_{ij}^0]$, $\tilde{s}_l(t)\in[-s_l^0, s_l^0]$ and $\tilde{w}_i(t)\in[-w_i^0, w_i^0]$.

The Itô stochastic differential equations of system Eq.(1) are

$$\mathrm{d}Q_i = \frac{\partial H(\boldsymbol{Q},\boldsymbol{P},\bar{\boldsymbol{s}})}{\partial P_i}\mathrm{d}t,$$
$$\mathrm{d}P_i = -\left[\frac{\partial H(\boldsymbol{Q},\boldsymbol{P},\bar{\boldsymbol{s}})}{\partial Q_i} + (m_{ij}(\boldsymbol{Q},\boldsymbol{P}) + \tilde{c}_{ij}(t))\frac{\partial H(\boldsymbol{Q},\boldsymbol{P},\bar{\boldsymbol{s}})}{\partial P_j}\right.$$
$$\left. - u_i(\boldsymbol{Q},\boldsymbol{P}) + \tilde{s}_l(t)g_{il}(\boldsymbol{Q}) - \tilde{w}_i(t)\right]\mathrm{d}t + \sigma_{ik}(\boldsymbol{Q},\boldsymbol{P})\mathrm{d}B_k(t),$$
$$i,j=1,2,\cdots,n;\quad k=1,2,\cdots,m;\quad l=1,2,\cdots,p, \qquad (2)$$

where $H(\boldsymbol{Q},\boldsymbol{P},\bar{\boldsymbol{s}})$ and $m_{ij}(\boldsymbol{Q},\boldsymbol{P})$ are respectively the new Hamiltonian and damping coefficients possibly modified by the Wong-Zakai correction terms; $\sigma_{ik}(\boldsymbol{Q}, \boldsymbol{P})$ are the elements of matrix $\boldsymbol{\sigma}$ with $\boldsymbol{\sigma}\boldsymbol{\sigma}^{\mathrm{T}}=2\boldsymbol{f}\boldsymbol{D}\boldsymbol{f}^{\mathrm{T}}$; $B_k(t)$ are the standard Wiener processes.

By applying the stochastic averaging method for quasi-Hamiltonian systems (Zhu and Yang, 1997; Zhu *et al*., 1997) to Eq.(2), the partially averaged Itô stochastic differential equations can be obtained. Their dimension and expressions depend on the integrability and resonance of the associated Hamiltonian system. In the integrable and nonresonant case, the partially averaged Itô stochastic differential equations are of the form

$$\mathrm{d}H_r(\boldsymbol{Q},\boldsymbol{P},\bar{\boldsymbol{s}})$$
$$= \left[\bar{m}_r(\boldsymbol{H}(\boldsymbol{Q},\boldsymbol{P},\bar{\boldsymbol{s}})) + \left\langle \frac{\partial H_r(\boldsymbol{Q},\boldsymbol{P},\bar{\boldsymbol{s}})}{\partial P_i}\left(u_i(\boldsymbol{Q},\boldsymbol{P}) \right.\right.\right.$$
$$\left.\left.\left. - \tilde{s}_l(t)g_{il}(\boldsymbol{Q}) - \tilde{c}_{ij}(t)\frac{\partial H(\boldsymbol{Q},\boldsymbol{P},\bar{\boldsymbol{s}})}{\partial P_j} + \tilde{w}_i(t)\right)\right\rangle\right]\mathrm{d}t$$
$$+ \bar{\sigma}_{rk}(\boldsymbol{H}(\boldsymbol{Q},\boldsymbol{P},\bar{\boldsymbol{s}}))\mathrm{d}B_k(t),$$
$$r,i,j=1,2,\cdots,n;\quad k=1,2,\cdots,m;\quad l=1,2,\cdots,p, \quad (3)$$

where H_r are the independent first integrals; $\boldsymbol{H}(\boldsymbol{Q},\boldsymbol{P},\bar{\boldsymbol{s}})=[H_1, H_2, \cdots, H_n]$; $\langle\cdot\rangle$ denotes the averaging operation; $\bar{m}_r(\boldsymbol{H}(\boldsymbol{Q},\boldsymbol{P},\bar{\boldsymbol{s}}))$ and $\bar{\sigma}_{rk}(\boldsymbol{H}(\boldsymbol{Q},\boldsymbol{P},\bar{\boldsymbol{s}}))$ are the drift and diffusion coefficients respectively determined by

$$\bar{m}_r(\boldsymbol{H}) = \frac{1}{T(\boldsymbol{H})}\oint\left[\left(-m_{ij}\frac{\partial H}{\partial p_j}\frac{\partial H_r}{\partial p_i} + D_{kl}f_{ik}f_{jl}\frac{\partial^2 H_r}{\partial p_i \partial p_j}\right)\right.$$

$$\cdot\left(\frac{\partial H_1}{\partial p_1}\frac{\partial H_2}{\partial p_2}\cdots\frac{\partial H_n}{\partial p_n}\right)^{-1}\Bigg]\mathrm{d}q_1\mathrm{d}q_2\cdots\mathrm{d}q_n,$$

$$\bar{\sigma}_{ru}(\boldsymbol{H})\bar{\sigma}_{su}(\boldsymbol{H})=\frac{1}{T(\boldsymbol{H})}\oint\Bigg[\left(2D_{kl}f_{ik}f_{jl}\frac{\partial H_r}{\partial p_i}\frac{\partial H_s}{\partial p_j}\right)$$

$$\cdot\left(\frac{\partial H_1}{\partial p_1}\frac{\partial H_2}{\partial p_2}\cdots\frac{\partial H_n}{\partial p_n}\right)^{-1}\Bigg]\mathrm{d}q_1\mathrm{d}q_2\cdots\mathrm{d}q_n,$$

$$T(\boldsymbol{H})=\oint\left(\frac{\partial H_1}{\partial p_1}\frac{\partial H_2}{\partial p_2}\cdots\frac{\partial H_n}{\partial p_n}\right)^{-1}\mathrm{d}q_1\mathrm{d}q_2\cdots\mathrm{d}q_n,$$

$$\langle\cdot\rangle=\frac{1}{T(\boldsymbol{H})}\oint\left[\cdot\Big/\left(\frac{\partial H_1}{\partial p_1}\frac{\partial H_2}{\partial p_2}\cdots\frac{\partial H_n}{\partial p_n}\right)\right]\mathrm{d}q_1\mathrm{d}q_2\cdots\mathrm{d}q_n. \quad (4)$$

Note that the dimension of the controlled system with uncertain disturbances is reduced from $2n$ to n.

Minimax optimal control

A performance index is introduced for the semi-infinite time-interval ergodic control as

$$J(\tilde{s}_l,\tilde{c}_{ij},\tilde{w}_i,u_i)=\lim_{t_f\to\infty}\frac{1}{t_f}\int_0^{t_f}L(\boldsymbol{H}(t),\langle\boldsymbol{u}(t)\rangle)\mathrm{d}t, \quad (5)$$

where $L(\boldsymbol{H},\langle\boldsymbol{u}\rangle)$ is a continuous differential convex cost function; $\boldsymbol{u}=[u_1, u_2, \ldots, u_n]^{\mathrm{T}}$. Let the cost function be quadratic in control, i.e.,

$$L(\boldsymbol{H},\ \langle\boldsymbol{u}\rangle)=f(\boldsymbol{H})+\langle\boldsymbol{u}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{u}\rangle, \quad (6)$$

where $\boldsymbol{R}$ is a positive-definite symmetric matrix and $f(\boldsymbol{H})>0$.

The proposed strategy is searching for an optimal worst-case controller by solving the following stochastic differential game problem:

$$\inf_{u_i}\sup_{\tilde{s}_l,\tilde{c}_{ij},\tilde{w}_i}J(\tilde{s}_l,\tilde{c}_{ij},\tilde{w}_i,u_i). \quad (7)$$

The HJI equation is obtained by using the principle of optimality (Yong and Zhou, 1999) to system Eq.(3) and the performance index Eq.(5) as follows:

$$\gamma=\inf_{u_i}\sup_{\tilde{s}_l,\tilde{c}_{ij},\tilde{w}_i}\Bigg\{L(\boldsymbol{H},\ \langle\boldsymbol{u}\rangle)+\Bigg[\bar{m}_r(\boldsymbol{H})$$

$$+\left\langle\frac{\partial H_r}{\partial P_i}\left(u_i-\tilde{s}_l(t)g_{il}(\boldsymbol{Q})-\tilde{c}_{ij}(t)\frac{\partial H}{\partial P_j}+\tilde{w}_i(t)\right)\right\rangle\Bigg]\frac{\partial V}{\partial H_r}$$

$$+\frac{1}{2}\bar{\sigma}_{rk}(\boldsymbol{H})\bar{\sigma}_{sk}(\boldsymbol{H})\frac{\partial^2 V}{\partial H_r\partial H_s}\Bigg\}, \quad (8)$$

where $V=V(\boldsymbol{H}, t)$ is the value function; γ is the optimal average cost,

$$\gamma=\lim_{t_f\to\infty}\frac{1}{t_f}\int_0^{t_f}L(\boldsymbol{H}(t),\ \langle\boldsymbol{u}^*(t)\rangle)\mathrm{d}t. \quad (9)$$

The worst-case disturbances can be determined by maximizing the right-hand side of Eq.(8) with respect to $\tilde{c}_{ij}(t)$, $\tilde{s}_l(t)$ and $\tilde{w}_i(t)$, respectively, while the optimal controls determined by minimizing it with respect to u_i. Because the disturbances are bounded, the worst-case disturbances are

$$\tilde{s}_l^*(t)=-s_l^0\,\mathrm{sgn}\left[g_{il}(\boldsymbol{Q})\frac{\partial H_r}{\partial p_i}\frac{\partial V}{\partial H_r}\right],$$

$$\tilde{c}_{ij}^*(t)=-c_{ij}^0\,\mathrm{sgn}\left[\frac{\partial H}{\partial p_j}\frac{\partial H_r}{\partial p_i}\frac{\partial V}{\partial H_r}\right], \quad (10)$$

$$\tilde{w}_i^*(t)=w_i^0\,\mathrm{sgn}\left[\frac{\partial H_r}{\partial p_i}\frac{\partial V}{\partial H_r}\right],$$

no summation with respect to l, i, j,

where sgn[·] denotes the sign function. The optimal controls are expressed as

$$u_i^*=-\frac{1}{2}(\boldsymbol{R}^{-1})_{ij}\frac{\partial H_r}{\partial P_j}\frac{\partial V}{\partial H_r},\quad i=1,2,\cdots,n. \quad (11)$$

Substituting the worst-case disturbances $\tilde{s}_l^*,\tilde{c}_{ij}^*,\tilde{w}_i^*$ and the optimal controls u_i^* into Eq.(8) and completing the averaging, we obtain the final HJI equation

$$\gamma=f(\boldsymbol{H})+\bar{m}_r(\boldsymbol{H})\frac{\partial V}{\partial H_r}$$

$$-\frac{1}{4}(\boldsymbol{R}^{-1})_{ij}\left\langle\frac{\partial H_r}{\partial P_i}\frac{\partial H_s}{\partial P_j}\right\rangle\frac{\partial V}{\partial H_s}\frac{\partial V}{\partial H_r}$$

$$+\left[s_l^0\left\langle\left|g_{il}(\boldsymbol{Q})\frac{\partial H_r}{\partial P_i}\right|\right\rangle+c_{ij}^0\left\langle\left|\frac{\partial H}{\partial P_j}\frac{\partial H_r}{\partial P_i}\right|\right\rangle\right.$$
$$\left.+w_i^0\left\langle\left|\frac{\partial H_r}{\partial P_i}\right|\right\rangle\right]\left|\frac{\partial V}{\partial H_r}\right|+\frac{1}{2}\bar{\sigma}_{rk}(\boldsymbol{H})\bar{\sigma}_{sk}(\boldsymbol{H})\frac{\partial^2 V}{\partial H_r\partial H_s}, \quad (12)$$

where |·| denotes absolute value. Eq.(12) can be solved to yield $\partial V/\partial H_r$. The worst-case disturbances $\tilde{s}_l^*, \tilde{c}_{ij}^*, \tilde{w}_i^*$ and the optimal controls u_i^* are then obtained by substituting $\partial V/\partial H_r$ into Eqs.(10) and (11).

The response of controlled worst-case can be predicted by solving the Fokker-Planck-Kolmogorov equation associated with fully averaged Itô Eq.(3) with worst-case disturbances and optimal controls, then control effectiveness and control efficiency can be evaluated.

Example

To illustrate the application and efficacy of the proposed minimax optimal control strategy, consider the following controlled and stochastically excited Duffing oscillator with parametric and external disturbances:

$$\begin{aligned}&\dot{Q}=P,\\&\dot{P}=-[\bar{a}+\tilde{a}(t)]Q-[\bar{b}+\tilde{b}(t)]Q^3\\&\quad-[\bar{c}+\tilde{c}(t)]P+u+e\xi(t)+\tilde{w}(t),\end{aligned} \quad (13)$$

where $\bar{a}$, $\bar{b}$ and $\bar{c}$ are the nominal values of linear stiffness, nonlinear stiffness and damping coefficient, respectively; e is excitation amplitude; $\xi(t)$ is Gaussian white noise with intensity $2D$; u is control. $\tilde{a}(t), \tilde{b}(t), \tilde{c}(t)$ are parameter disturbances, and $\tilde{w}(t)$ is external disturbance. They are bounded, i.e., $\tilde{a}(t)\in[-a^0, a^0]$, $\tilde{b}(t)\in[-b^0, b^0]$, $\tilde{c}(t)\in[-c^0, c^0]$ and $\tilde{w}(t)\in[-w^0, w^0]$.

The proposed procedure yields the following worst-case disturbances and the optimal control

$$\tilde{a}^*=-a^0\operatorname{sgn}[Q\dot{Q}], \quad \tilde{b}^*=-b^0\operatorname{sgn}[Q\dot{Q}],$$
$$\tilde{c}^*=-c^0, \quad \tilde{w}^*=w^0\operatorname{sgn}[\dot{Q}], \quad (14)$$

$$u^*=-\frac{1}{2R}\frac{\mathrm{d}V}{\mathrm{d}H}\frac{\partial H}{\partial P}, \quad (15)$$

where dV/dH are determined by the final HJI equation.

The control effectiveness and efficiency are respectively defined as follows

$$K=[(\sigma_q)_u-(\sigma_q)_c]/(\sigma_q)_u, \quad (16a)$$
$$\mu=K/(\sigma_{u^*}/\sqrt{2D}), \quad (16b)$$

which are used to evaluate the control performance.

Numerical results have been obtained for system Eq.(13) with parameter values: a=1.0, b=0.2, c=0.1, e=1.0, D=0.5, a^0=0.04, b^0=0.005, c^0=0.01, w^0=0.01, unless otherwise mentioned, and given in Figs.1 and 2. For the worst-case, the effect of excitation intensity D on the control effectiveness and efficiency are shown in Fig.1. Also, the effect of the bound of uncertain disturbance $\tilde{a}(t)$ on the control effectiveness and efficiency are shown in Fig.2. It is seen from these figures that the proposed control strategy is quite effective. Fig.1 shows that the control effectiveness decreases slightly and the control efficiency increases correspondingly as excitation intensity D increases. Fig.2 shows that both the control effectiveness and efficiency increase as the bound of linear stiffness disturbance increases.

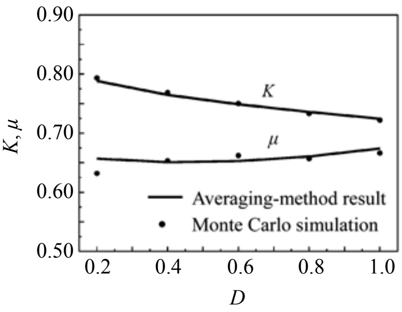

Fig.1 Effectiveness (K) and efficiency (μ) of the minimax optimal control vs excitation intensity D

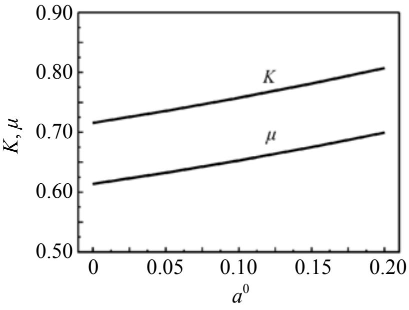

Fig.2 Effectiveness (K) and efficiency (μ) of the minimax optimal control vs the bound of linear stiffness disturbance a^0

CONCLUSION

A minimax optimal control strategy for quasi-Hamiltonian systems with bounded parametric and/or external disturbances has been proposed based on the stochastic averaging method and stochastic differential game. The stochastic averaging method for quasi-Hamiltonian systems was first applied to yield the partially averaged Itô stochastic differential equations. An optimal worst-case controller was then obtained by solving the stochastic differential game problem. Numerical results exhibit that the proposed control strategy is quite effective.

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