Journal of Zhejiang University SCIENCE A
ISSN 1673-565X (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Optimization on bicriterion policies for M/G/1 system with second optional service

Jau-chuan KE[†], Yunn-kuang CHU

(*Department of Applied Statistics, National Taichung Institute of Technology, Taichung 404, Taiwan, China*)

[†]E-mail: jauchuan@ntit.edu.tw

Received Feb. 2, 2008; revision accepted May 14, 2008

Abstract: We compare the optimal operating cost of the two bicriterion policies, <*p*,*T*> and <*p*,*N*>, for an M/G/1 queueing system with second optional service, in which the length of the vacation period is randomly controlled either by the number of arrivals during the idle period or by a timer. After all the customers are served in the queue exhaustively, the server immediately takes a vacation and may operate <*p*,*T*> policy or <*p*,*N*> policy. For the two bicriterion policies, the total average cost function per unit time is developed to search the optimal stationary operating policies at a minimum cost. Based upon the optimal cost the explicit forms for joint optimum threshold values of (p,T) and (p,N) are obtained.

Key words: Average operating cost, Bicriterion policy, Optimization comparisons, Optional service, Optimal threshold values
doi:10.1631/jzus.A0820103 **Document code:** A **CLC number:** O22

INTRODUCTION

Queueing systems with vacations effectively arise in the stochastic modeling of many computers and communication systems, manufacturing/production, and inventory systems. Several excellent surveys on vacation models have been done by Doshi (1986). In general, in order to control the length of the vacation period we either use a timer T [known as T policy by Heyman (1977)] or a queue length N [known as N policy by Yadin and Naor (1963)]. Considerable efforts have been devoted to studying the two types of controllable queueing models by (Teghem, 1987; Gakis *et al*., 1995; etc.). This has also given rise to a vast and rich literature, from which we can review the survey by Tadj and Choudhury (2005). One particular interest is a randomized control over the length of vacation period, which can be achieved by considering T policy or N policy. In this paper we examine the randomized control of T policy and N policy, and carry out comparison on the minimum cost criterion for two bicriterion policies.

The server deactivates and leaves for a vacation with fixed length of time T whenever the system is empty. After a vacation period of length T, the server returns to the system. It begins to serve if there is at least one customer on the waiting line; otherwise, the server waits another period of length T and so on until at least one customer is present. This kind of control policy is called 'T policy' and the T policy for the M/G/1 queueing system was investigated by (Levy and Yechiali, 1975; Heyman, 1977; Gakis *et al*., 1995). Recently, Tadj (2003) has studied an M/G/1 quorum queueing system under T policy with a reliable server where 'quorum' is a bulk service that the server waits until the number of waiting customers reaches a fixed accumulation level γ ($\gamma\geq 1$). Ke (2005) studied a variant T policy for the M/G/1 queueing system with an unreliable server and startup where a single server may take at most J vacations repeatedly until at least one customer appears in the queue upon returning from a vacation, and the server needs a startup time before starting each of his service periods. The optimization work of a variant T policy for the

$M^{[x]}/G/1$ queueing system with at most J vacations and startup/closedowns was examined by Ke (2008). Our vacation policy in this paper differs from the T policy discussed by the authors mentioned above. We consider the $<p,T>$ policy for an M/G/1 queueing system with second optional service where *a single server randomly reactivates when at least one customer appears in the queue upon returning from a vacation*.

Yadin and Naor (1963) first introduced the concept of an N policy, which turns the server on whenever N ($N\geq 1$) or more customers are present, while turns the server off only when none is present. For a reliable server, the N policy M/G/1 queueing system was first studied by Heyman (1968) and was developed by several researchers (Tijms, 1986; Teghem, 1987; Takagi, 1991; Lee *et al*., 1994a; Gakis *et al*., 1995; Wang and Ke, 2000). Analytic steady state solutions of the N policy $M/H_k/1$ queueing system were first obtained by Wang and Yen (2003). For an unreliable server, Wang (1995; 1997) and Wang *et al*.(1999) derived analytic steady state solutions of the N policy M/M/1, the N policy $M/E_k/1$, and the N policy $M/H_2/1$ queueing systems, respectively. Wang *et al*.(2004) extended Wang and Yen (2003)'s system to an unreliable server case. As for N policy queueing systems with vacations/startup, the readers may refer to (Kella, 1989; Lee and Srinivasan, 1989; Takagi, 1991; Medhi and Templeton, 1992; Lee *et al*., 1994b; 1995; Lee and Park, 1997; Zhang *et al*., 1997; Hur and Paik, 1999; Ke, 2003), and so on. Our N policy in this paper differs from the N policy discussed by the authors mentioned above. We consider the $<p,N>$ policy for an M/G/1 queueing system with second optional service where *a single server randomly reactivates when queue length reaches a predetermined threshold since the end of the busy period*.

For the optimization work of T policy and N policy, Table 1 summarizes some important results from the past literature.

Recently Madan (2000) has investigated an M/G/1 queueing system with a second optional service, in which some of arrivals may require a second optional service immediately after completion of the first essential service. In (Madan, 2000), the service times of the first essential service were assumed to be a general distribution and those of the second optional service were exponential. Also he cited some important applications in daily life situations. Madan (2000)'s work was extended to 'optional service' with general distribution case by Medhi (2002).

In this paper, we examine the finding of the two optimal bicriterion policies that yield the minimum average cost for an M/G/1 system with holding cost and setup cost, where a holding cost of C_h per unit time is for each customer in the system and a fixed charge of C_s is to activate the server. It is assumed that customers arrive according to a Poisson process with rate λ. A more general case for a customer service is considered. All arriving customers require the essential service. However, some customers may further demand a second optional service. On completion of the essential service, the customer may leave the system with probability $1-p$ or may opt for the second optional service with probability p. The times of the essential service and the optional service are assumed to be random variables S_1 and S_2, respectively. Arriving customers form a single waiting line based on the order of their arrivals; that is, they are queued according to the first-come, first-served (FCFS) discipline. The server can only provide either the essential or the optional service for one customer at a time. A customer who arrives and finds the server being busy or on vacation must wait in the queue until he/she is available. After all the customers are served in the queue exhaustively, the server immediately takes a vacation and adopts two bicriterion policies:

(1) If customers are found in the queue after T time units have elapsed since the end of the busy period, the server serves the customers (i.e., starts his busy period) with probability p or leaves for a vacation of the same length T with a probability $1-p$. When a busy period begins, the server is kept in the active state until the system is empty. If no customers are found we say a busy period of zero occurs. In either case, the server takes another vacation with the same length after the end of a busy period. This is so-called $<p,T>$ policy.

(2) If the number of arrivals reaches N, the server serves the waiting customers with probability p or keeps going on vacation (idle) with probability $1-p$. This is so-called $<p,N>$ policy.

Table 1 Some results of optimization work for *T* policy and *N* policy systems

Control policy and queueing types	References	Solutions of the optimal policy	Techniques for optimization	Remarks
Case 1: *T* policy for M/G/1 queueing type	Heyman, 1977	• Show that the optimum solution exists • Explicit closed-form of the optimum solution is obtained	• The optimum solution is obtained using the 'first derivative test' (FDT) and the 'second derivative test' (SDT) from a Calculus handbook (Stewart, 1995, p.197, 203)	• Single decision variable • Show that the *T* policy is not superior to *N* policy based on the optimum cost criterion
Case 2: *N* policy for M/M/1, $M/E_k/1$, $M/H_2/1$, $M/H_k/1$, M/G/1 queueing systems	Wang, 1995; 1997; Wang *et al*., 1999; 2004; Wang and Ke, 2000; Wang and Yen, 2003	• Show that the optimum solution exists but find the approximate • The approximate optimum solution is obtained and this existing optimum solution is not an explicit form	• The approximate optimum solution is obtained using inequalities and the concept of the FDT from a Calculus handbook (Stewart, 1995, p.197)	• Single decision variable • The proof for the existence of optimum solution is not severe and completed
Case 3: *N* policy for M/G/1 queueing type with vacation/setup	Kella, 1989; Lee and Srinivasan, 1989; Lee *et al*., 1994a; 1994b; 1995	• Show that the optimum solution exists • One cannot obtain the explicit closed-form of the optimum solution	• A procedure (algorithm) is developed to find the optimum solution	• Single decision variable
Case 4: *N* policy for M/G/1 queueing type with vacation/setup	Hur and Paik, 1999; Ke, 2003	• One cannot show that the optimum solution exists • One cannot obtain the explicit closed-form of the optimum solution	• A procedure (algorithm) is developed to find the possible optimum solution	• Single decision variable
Case 5: *N* policy for M/G/1 queueing type with vacation/setup	Lee and Park, 1997; Zhang *et al*., 1997; Ke, 2008	• One cannot show that the optimum solution exists • Possibly find some properties of the total average cost • One cannot obtain the explicit closed-form of the optimum solution	• A procedure (algorithm) is developed to find the possible optimum solution	• Two decision variables*
Case 6: <*p*,*T*> and <*p*,*N*> policies for M/G/1 queueing system	This paper	• Show that the optimum solution exists for the <*p*,*T*> and <*p*,*N*> policies, respectively • Explicit closed-forms of the two optimum solutions are obtained • Provide an analytical and numerical comparison on optimum cost between the <*p*,*T*> policy and the <*p*,*N*> policy	• The optimum solutions of two random control policies are derived merely using the FDT and the 'test for monotonic functions' (see Sections 5 and 6), rather than using the SDT for function of two variables (Stewart, 1995, p.196, 814) • For the optimization of two variables considered in this paper, the FDT and the 'test for monotonic functions' are easily implemented with no constraints	• The two bicriterion policies considered in this paper are new • Note that the exact optimal solution of Case 2 can be deduced as a special case of the results developed in this paper • Two decision variables* • We analytically show that the <*p*,*N*> policy is superior to the <*p*,*T*> policy based on the optimum cost criterion • We perform an analytical and numerical comparison for two bicriterion policies • The optimum solutions of two random control policies are easily computed for practitioners and managers

* It is very difficult to obtain the analytical optimum solutions of two thresholds for the controllable queueing systems including combined *T* policy and *N* policy (Lee and Park, 1997; Zhang *et al*., 1997; Hur *et al*., 2003; Tadj and Ke, 2005)

The organization of this paper is as follows. First, we present some results of the classical control policies for M/G/1 system with second optional service. Second, we develop some important system characteristics for the <p,T> policy and <p,N> policy M/G/1 system with second optional service, which is such as the expected length of the idle period, busy period, busy cycle, and the expected number of customers in the system. Third, we construct a total average cost function per customer per unit time for the two bicriterion policies and then obtain the explicit forms for the joint optimum values of (p,T) and (p,N) under the minimum cost criterion. Finally, we perform analytical comparison on the optimum cost criterion for two bicriterion policies.

CLASSICAL CONTROL POLICIES FOR M/G/1 SYSTEM WITH SECOND OPTIONAL SERVICE

In this section, we respectively develop the system characteristics for the T and N policies M/G/1 system with second optional service by means of results of (Heyman, 1977; Wang and Ke, 2000; Medhi, 2002).

Some results for the *T* policy

We denote by I_T and B_T the idle and busy periods, respectively, for the T policy M/G/1 system with second optional service. Let C_T denote a busy cycle, which is a sum of consecutive idle and busy periods. From (Heyman, 1977; Medhi, 2002), we have the following formulas:

$$E[I_T] = T, \tag{1}$$

$$E[B_T] = \rho T/(1-\rho), \tag{2}$$

$$E[C_T] = T/(1-\rho), \tag{3}$$

where $\rho = \lambda(E[S_1] + rE[S_2])$.

Moreover, let L_T represent the expected number of customers in the T policy M/G/1 system with second optional service. Utilizing the results by Heyman (1977) and Medhi (2002) again, we obtain

$$L_T = \frac{\lambda T}{2} + \rho + \frac{\rho^2(c_V^2+1)}{2(1-\rho)} := \frac{\lambda T}{2} + L\ , \tag{4}$$

where

$$c_V^2 = \frac{E[S_1^2] + 2rE[S_1]E[S_2] + E[S_2^2]}{(E[S_1] + rE[S_2])^2} - 1,$$

$$L = \rho + \frac{\rho^2(c_V^2+1)}{2(1-\rho)}.$$

Some results for the *N* policy

We develop important system characteristics for the N policy M/G/1 system with second optional service. Let us define the following notations: I_N the idle period random variable, B_N the busy period random variable, C_N the busy cycle random variable (idle period + busy period), and I_N the expected number of customers in the N policy M/G/1 system with second optional service.

Similar to analysis in previous section and using the arguments by Wang and Ke (2000), we obtain the following results:

$$E[I_N] = N/\lambda\ , \tag{5}$$

$$E[B_N] = \rho N/[\lambda(1-\rho)]\ , \tag{6}$$

$$E[C_N] = N/[\lambda(1-\rho)]\ , \tag{7}$$

$$L_N = (N-1)/2 + L, \tag{8}$$

where ρ, c_V^2 and L are given in the previous subsection.

<p,T> POLICY FOR M/G/1 SYSTEM WITH SECOND OPTIONAL SERVICE

We denote by (I_{2T}, B_{2T}) and ($I_{p,T}$, $B_{p,T}$) the idle and busy periods, for the $2T$ policy and <p,T> policy M/G/1 systems with second optional service, respectively. Let C_{2T} and $C_{p,T}$ be a busy cycle period for $2T$ policy and <p,T> policy M/G/1 systems with second optional service, respectively. It should be noted from (Feinberg and Kim, 1996) that the system characteristics of a <p,T> policy are a convex combination of the system characteristics of T policy and $2T$ policy systems. From Eqs.(1)~(3), we obtain

$$E[I_{p,T}] = pE[I_T] + (1-p)E[I_{2T}] = (2-p)T, \tag{9}$$

$$\begin{aligned} E[B_{p,T}] &= pE[B_T] + (1-p)E[B_{2T}] \\ &= (2-p)\rho T/(1-\rho), \end{aligned} \tag{10}$$

$$\begin{aligned} E[C_{p,T}] &= pE[C_T] + (1-p)E[C_{2T}] \\ &= (2-p)T/(1-\rho). \end{aligned} \tag{11}$$

Thus we have the number of busy cycles per unit time

$$1/E[C_{p,T}]=(1-\rho)/[(2-p)T]. \quad (12)$$

Let $\bar{\Psi}_T$, $\bar{\Psi}_{2T}$ and $\bar{\Psi}_{p,T}$ represent the cumulative amount of time that all customers spent in the system during a busy cycle for T policy, $2T$ policy and <p,T> policy M/G/1 systems with second optional service, respectively. Employing the results of (Feinberg and Kim, 1996), we obtain

$$E[\bar{\Psi}_T]=L_T E[C_T]=\left(L+\frac{\lambda T}{2}\right)\frac{T}{1-\rho}. \quad (13)$$

It follows that

$$\begin{aligned}E[\bar{\Psi}_{p,T}]&=pE[\bar{\Psi}_T]+(1-p)E[\bar{\Psi}_{2T}]\\&=\frac{T}{1-\rho}\left(L(2-p)+\lambda T\left(2-\frac{3}{2}p\right)\right).\end{aligned} \quad (14)$$

By the similar arguments of (Feinberg and Kim, 1996), the expected number of customers in the <p,T> policy M/G/1 system with second optional service is given by

$$L_{p,T}=\frac{E[\bar{\Psi}_{p,T}]}{E[C_{p,T}]}=L+\frac{\lambda T(2-3p/2)}{2-p}. \quad (15)$$

Eq.(15) can be expressed as a convex combination of characteristics of T policy and $2T$ policy systems. That is,

$$L_{p,T}=\left(\frac{p}{2-p}\right)L_T+\left(1-\frac{p}{2-p}\right)L_{2T}, \quad (16)$$

which confirms the results of (Feinberg and Kim, 1996).

<p,N> POLICY FOR M/G/1 SYSTEM WITH SECOND OPTIONAL SERVICE

In this section, we develop important system characteristics for the <p,N> policy M/G/1 system with second optional service. Let us define the following notations: $I_{p,N}$ the idle period random variable, $B_{p,N}$ the busy period random variable, $C_{p,N}$ the busy cycle random variable, and $L_{p,N}$ the expected number of customers in the <p,N> policy M/G/1 system with second optional service.

Following the analysis in Section 3 and using the arguments by Wang and Ke (2000), we have that

$$E[I_{p,N}]=(N+1-p)/\lambda, \quad (17)$$

$$E[B_{p,N}]=\rho(N+1-p)/[\lambda(1-\rho)], \quad (18)$$

$$E[C_{p,N}]=(N+1-p)/[\lambda(1-\rho)], \quad (19)$$

and

$$L_{p,N}=\frac{N(N+1-2p)}{2(N+1-p)}+L. \quad (20)$$

Eq.(20) can be expressed as a convex combination of characteristics of N policy and N+1 policy systems. That is,

$$L_{p,N}=\left(\frac{pN}{N+1-p}\right)L_N+\left(1-\frac{pN}{N+1-p}\right)L_{N+1},$$

which confirms the results of (Feinberg and Kim, 1996).

OPTIMAL <p,T> POLICY

In this section, we construct a total average cost function per customer per unit time for the <p,T> policy M/G/1 system with second optional service, in which p and T are decision variables. Our objective is to determine the joint optimal thresholds, say (p^*,T^*), to minimize this cost function. Since $E[B_{p,T}]/E[C_{p,T}]=\rho$ and $E[I_{p,T}]/E[C_{p,T}]=1-\rho$ are independent of p and T, we consider the following cost elements: C_s the setup cost for per busy cycle, and C_h the holding cost per unit time for each customer present in the system.

Employing the definition of each cost element and its corresponding system performance, the total average cost function with (p,T) is given by

$$\begin{aligned}F_1(p,T)&=C_h L_{p,T}+C_s/E[C_{p,T}]\\&=C_h L+C_h\frac{\lambda T(2-3p/2)}{2-p}+C_s\frac{1-\rho}{T(2-p)}.\end{aligned} \quad (21)$$

To find the joint optimal threshold values (p^*,T^*), first we let $T_0=\sqrt{C_s(1-\rho)/(\lambda C_h)}$ and $t=T/T_0$. Then Eq.(21) can be re-expressed as

$$F_1(p,T)=C_\mathrm{h}L+\sqrt{C_\mathrm{h}C_\mathrm{s}\lambda(1-\rho)}\left(\frac{3}{2}t+\frac{1}{2-p}\left(\frac{1}{t}-t\right)\right). \tag{22}$$

Moreover, we define

$$f(p,t)=\frac{3}{2}t+\frac{1}{2-p}\left(\frac{1}{t}-t\right). \tag{23}$$

Obviously, the following results can be obtained:

(1) As $0<t<1$, $f(p,t)$ is an increasing function in $p\in[0,1]$. It implies

$$\min_{0\le p\le 1} f(p,t)=f(0,t)=t+\frac{1}{2t},\ 0<t<1. \tag{24}$$

(2) As $t=1$, $f(p,t)=f(p,1)=3/2$, for $p\in[0,1]$.

(3) As $t>1$, $f(p,t)$ is an decreasing function in $p\in[0,1]$. It yields

$$\min_{0\le p\le 1} f(p,t)=f(1,t)=\frac{t}{2}+\frac{1}{t},\ t>1. \tag{25}$$

Set $g(t)=t+1/(2t),\ \ 0<t<1.$ Then

$$g'(t)=1-\frac{1}{2t^2}\begin{cases}<0, & 0<t<1/\sqrt{2},\\ =0, & t=1/\sqrt{2},\\ >0, & 1/\sqrt{2}<t<1.\end{cases}$$

According to the 'first derivative test' (FDT), we have

$$\min g=g(1/\sqrt{2})=1/\sqrt{2}+\sqrt{2}/2=\sqrt{2}. \tag{26}$$

Set $h(t)=\dfrac{t}{2}+\dfrac{1}{t},\ t>1.$ Then

$$h'(t)=\frac{1}{2}-\frac{1}{t^2}\begin{cases}<0, & 1<t<\sqrt{2},\\ =0, & t=\sqrt{2},\\ >0, & t>\sqrt{2}.\end{cases}$$

Using the FDT again, we have

$$\min h=h(\sqrt{2})=\sqrt{2}/2+1/\sqrt{2}=\sqrt{2}. \tag{27}$$

Subsequently based on Eqs.(22)~(27), we deduce that

$$\begin{aligned}\min F_1&=C_\mathrm{h}L+\sqrt{C_\mathrm{h}C_\mathrm{s}\lambda(1-\rho)}\min f\\ &=C_\mathrm{h}L+\sqrt{2C_\mathrm{h}C_\mathrm{s}\lambda(1-\rho)}\\ &=F(0,T_0/\sqrt{2})=F(1,\sqrt{2}T_0).\end{aligned} \tag{28}$$

Summarizing the above results, we have the following theorem:

Theorem 1 Let (p^*,T^*) be the joint optimal threshold values that minimize the total average cost Eq.(21). Then $(p^*,T^*)=(0,T_0/\sqrt{2})$ or $(1,\sqrt{2}T_0)$. That is,

$$(p^*,T^*)=\left(0,\sqrt{C_\mathrm{s}(1-\rho)/(2\lambda C_\mathrm{h})}\right)$$

or

$$(p^*,T^*)=\left(1,\sqrt{2C_\mathrm{s}(1-\rho)/(\lambda C_\mathrm{h})}\right).$$

Note that, when $(p^*,T^*)=\left(0,\sqrt{C_\mathrm{s}(1-\rho)/(2\lambda C_\mathrm{h})}\right)$, it is exactly the same as the result of the classical T policy for M/G/1 queue (Heyman, 1977).

OPTIMAL <p,N> POLICY

In this section we find the joint optimal thresholds (p^*,N^*) that minimize the total average cost function for the $<p,N>$ policy M/G/1 system with second optional service. As before, C_s and C_h represent setup cost and holding cost, respectively. Then the total average cost function with two thresholds (p,N) becomes

$$\begin{aligned}F_2(p,N)&=C_\mathrm{h}L_{p,N}+C_\mathrm{s}/E[C_{p,N}]\\ &=C_\mathrm{h}L+C_\mathrm{h}\frac{N(N+1-2p)}{2(N+1-p)}+C_\mathrm{s}\frac{\lambda(1-\rho)}{N+1-p}.\end{aligned} \tag{29}$$

To find the joint optimal threshold values (p^*,N^*), we set $k_0=2\lambda C_\mathrm{s}(1-\rho)/C_\mathrm{h}$ and $x_0=-0.5+\sqrt{0.25+k_0}$. Then taking partial differentiation on Eq.(29) with respect to p, we have

$$\frac{\partial F_2(p,N)}{\partial p}=\frac{C_\mathrm{h}}{2(N+1-p)^2}(k_0-N(N+1)). \tag{30}$$

This implies that for any p in (0,1),

$$\frac{\partial F_2(p,N)}{\partial p}\begin{cases}>0, & N(N+1)<k_0,\\ =0, & N(N+1)=k_0,\\ <0, & N(N+1)>k_0.\end{cases}$$

Apparently, the following outcomes can be deduced:

(1) As $N(N+1)<k_0$, $F_2(p,N)$ is an increasing function in $p\in[0,1]$. Based upon the 'test for monotonic functions' (Stewart, 1995), it asserts

$$\min_{0\le p\le 1} F_2(p,N)=F_2(0,N)=C_{\rm h}L+\frac{C_{\rm h}}{2}N+\frac{\lambda C_{\rm s}(1-\rho)}{N+1}. \quad (31)$$

(2) As $N(N+1)>k_0$, $F_2(p,N)$ is a decreasing function in $p\in[0,1]$. From the 'test for monotonic functions', it ascertains

$$\begin{aligned}\min_{0\le p\le 1} F_2(p,N)&=F_2(1,N)\\&=C_{\rm h}L+\frac{C_{\rm h}}{2}(N-1)+\frac{\lambda C_{\rm s}(1-\rho)}{N}.\end{aligned} \quad (32)$$

(3) If there exists an integer $N_0>0$ such that $N(N+1)=k_0$, then

(i) $F_2(p,N_0)=C_{\rm h}L+C_{\rm h}N_0$ for $p\in[0,1]$;

(ii) $N_0=x_0$;

(iii) $$\begin{aligned}\min_{0\le p\le 1} F_2(p,N_0)&=F_2(0,N_0)=F_2(1,N_0)\\&=C_{\rm h}L+C_{\rm h}N_0.\end{aligned} \quad (33)$$

(4) Define $\alpha(N)=C_{\rm h}L+\frac{C_{\rm h}}{2}N+\frac{\lambda C_{\rm s}(1-\rho)}{N+1}$, $N<x_0$. Note that $N(N+1)<k_0$ is equivalent to $N<x_0$. Then

$$\alpha(N)-\alpha(N-1)=\frac{C_{\rm h}}{2}\left(1-\frac{k_0}{N(N+1)}\right)<0,$$

and hence

$$\min\alpha=\alpha\left([x_0]\right)=C_{\rm h}L+\frac{C_{\rm h}}{2}[x_0]+\frac{\lambda C_{\rm s}(1-\rho)}{[x_0]+1}, \quad (34)$$

where $[x_0]$ denotes the greatest integer less than or equal to x_0.

(5) Define $\beta(N)=C_{\rm h}L+\frac{C_{\rm h}}{2}(N-1)+\frac{\lambda C_{\rm s}(1-\rho)}{N}$, $N>x_0$. Then

$$\beta(N+1)-\beta(N)=\frac{C_{\rm h}}{2}\left(1-\frac{k_0}{N(N+1)}\right)>0,$$

and hence

$$\min\beta=\beta\left([x_0]+1\right)=C_{\rm h}L+\frac{C_{\rm h}}{2}[x_0]+\frac{\lambda C_{\rm s}(1-\rho)}{[x_0]+1}. \quad (35)$$

Based upon Eqs.(31)~(35), we derive that

$$\min F_2=\begin{cases}C_{\rm h}L+C_{\rm h}N_0=F_2(0,N_0)=F_2(1,N_0),\\ \quad \text{if } k_0=N_0(N_0+1) \text{ for some integer } N_0;\\ C_{\rm h}L+\frac{C_{\rm h}}{2}[x_0]+\frac{\lambda C_{\rm s}(1-\rho)}{[x_0]+1}\\ =F_2\left(0,[x_0]\right)=F_2\left(1,[x_0]+1\right),\\ \quad \text{if } k_0\ne N(N+1) \text{ for all integers } N.\end{cases} \quad (36)$$

In a practical M/G/1 system, there does not exist an integer N_0 satisfying $k_0=N_0(N_0+1)$. Therefore, we obtain the following theorem:

Theorem 2 Let (p^*,N^*) be the joint optimal threshold values that minimize the total average cost Eq.(29). Then $(p^*,N^*)=(0, [x_0])$ or $(1, [x_0]+1)$. That is,

$$(p^*,N^*)=\left(0,\left[-0.5+\sqrt{0.25+k_0}\right]\right)$$

or

$$(p^*,N^*)=\left(1,\left[-0.5+\sqrt{0.25+k_0}\right]+1\right),$$

where $k_0=2\lambda C_{\rm s}(1-\rho)/C_{\rm h}$.

Note that, when $(p^*,N^*)=\left(0,\left[-0.5+\sqrt{0.25+k_0}\right]\right)$, it is an exact optimum threshold solution for the classical N policy M/G/1 system. The optimum threshold solution previously studied is approximated for the N policy M/G/1 queue (Wang, 1995; 1997; Wang and Ke, 2000; Wang and Yen, 2003; Wang *et al*., 1999; 2004).

COMPARISON ANALYSIS OF OPTIMAL <p,T> POLICY AND OPTIMAL <p,N> POLICY

Fixing the level values of c_V^2, $C_{\rm h}$, $C_{\rm s}$, λ and ρ, we have shown that

(1) In <p,T> policy, the minimum total average cost is

$$m_1=\min F_1=C_{\rm h}L+C_{\rm h}\sqrt{k_0}. \quad (37)$$

(2) In <p,N> policy, the minimum total average cost is

$$m_2=\min F_2=C_{\rm h}L+\frac{C_{\rm h}}{2}\left([x_0]+\frac{k_0}{[x_0]+1}\right). \quad (38)$$

Using

$$[x_0]([x_0]+1) < k_0 < ([x_0]+1)([x_0]+2)$$

yields

$$[x_0]([x_0]+1)+k_0 < 2k_0 < 2\sqrt{k_0}\,([x_0]+1),$$

and

$$\frac{[x_0]}{\sqrt{k_0}}+\frac{\sqrt{x_0}}{[x_0]+1}=\frac{[x_0]([x_0]+1)+k_0}{\sqrt{k_0}\,([x_0]+1)}<2.$$

It finally follows that

$$m_1-m_2=C_{\rm h}\sqrt{k_0}\left(1-\frac{1}{2}\left(\frac{[x_0]}{\sqrt{k_0}}+\frac{\sqrt{k_0}}{[x_0]+1}\right)\right)>0. \quad (39)$$

We therefore establish the theorem below:

Theorem 3 Setting the same level values of c_V^2, $C_{\rm h}$, $C_{\rm s}$, λ and ρ in <*p*,*T*> policy and <*p*,*N*> policy, the minimum total average cost in <*p*,*N*> policy is less than that in <*p*,*T*> policy. That is, $m_2<m_1$.

In order to examine the above theorem numerically, we produce Table 2 for numerical comparison of m_1 and m_2.

Table 2 Numerical comparison of m_1 and m_2

$C_{\rm h}$	$C_{\rm s}$	λ	ρ	L	k_0	$[x_0]$	m_1	m_2
2	1000	2	0.8	10	400	19	60.00	59.00
2	5000	2	0.8	50	2000	44	189.44	188.44
2	10000	2	0.8	90	4000	62	306.49	305.49
2	1000	5	0.8	10	1000	31	83.25	82.25
2	5000	5	0.8	50	5000	70	241.42	240.42
2	10000	5	0.8	90	10000	99	380.00	379.00
8	1000	2	0.9	10	450	20	249.71	245.71
8	5000	2	0.9	50	2250	46	779.47	775.49
8	10000	2	0.9	90	4500	66	1256.66	1252.66
8	1000	5	0.9	10	1125	33	348.33	344.35
8	5000	5	0.9	50	5625	74	1000.00	996.00
8	10000	5	0.9	90	11250	105	1568.53	1564.53

Note that the values of L are obtained from the subsection of 'Some results for the T policy' by choosing appropriate values of c_V^2 $(=2(1-\rho)(L-\rho)/\rho^2-1)$

CONCLUSION

In this paper, we showed that the optimum solution exists for the <*p*,*T*> and <*p*,*N*> policies, respectively. Explicit closed-forms of the two optimum solutions were obtained. We also provided an analytical and numerical comparison of optimum cost between <*p*,*T*> policy and <*p*,*N*> policy. More importantly, we analytically showed that the <*p*,*N*> policy is superior to the <*p*,*T*> policy based on the optimum cost criterion. Past studies merely provided an approximate optimum threshold value for classical *N* policy M/G/1 queueing systems, but their exact optimum threshold value can be deduced as a special case of the results developed in this paper.

References

Doshi, B.T., 1986. Queueing system with vacations—a survey. *Queueing Systems*, **1**(1):29-66. [doi:10.1007/BF01149327]

Feinberg, E.A., Kim, D.J., 1996. Bicriterion optimization of an M/G/1 queue with a removable server. *Prob. Eng. Inf. Sci.*, **10**:57-73.

Gakis, K.G., Rhee, H.K., Sivazlian, B.D., 1995. Distributions and first moments of the busy and idle periods in controllable M/G/1 queueing models with simple and dyadic policies. *Stoch. Anal. Appl.*, **13**(1):47-81. [doi:10.1080/07362999508809382]

Heyman, D.P., 1968. Optimal operating policies for M/G/1 queueing system. *Oper. Res.*, **16**:362-382.

Heyman, D.P., 1977. The *T* policy for the M/G/1 queue. *Manag. Sci.*, **23**:775-778.

Hur, S., Paik, S.J., 1999. The effect of different arrival rates on the *N*-policy of M/G/1 with server setup. *Appl. Math. Model.*, **23**(4):289-299. [doi:10.1016/S0307-904X(98)10088-4]

Hur, S., Kim, J., Kang, C., 2003. An analysis of the M/G/1 system with *N* and *T* policy. *Appl. Math. Model.*, **27**(8):665-675. [doi:10.1016/S0307-904X(03)00074-X]

Ke, J.C., 2003. The optimal control of an M/G/1 queueing system with server vacations, startup and breakdowns. *Comput. Ind. Eng.*, **44**(4):567-579. [doi:10.1016/S0360-8352(02)00235-8]

Ke, J.C., 2005. Modified *T* vacation policy for an M/G/1 queueing system with an un-reliable server and startup. *Math. Comput. Model.*, **41**(11-12):1267-1277. [doi:10.1016/j.mcm.2004.08.009]

Ke, J.C., 2008. Two thresholds of a batch arrival queueing system under modified *T* vacation policy with startup and closedown. *Math. Methods Appl. Sci.*, **31**(2):229-247. [doi:10.1002/mma.905]

Kella, O., 1989. The threshold policy in the M/G/1 queue with server vacations. *Nav. Res. Logist.*, **36**(1):111-123. [doi:10.1002/1520-6750(198902)36:1<111::AID-NAV3220360109>3.0.CO;2-3]

Lee, H.S., Srinivasan, M.M., 1989. Control policies for the $M^{[x]}$/G/1 queueing system. *Manag. Sci.*, **35**:708-721.

Lee, H.W., Park, J.O., 1997. Optimal strategy in *N*-policy production system with early set-up. *J. Oper. Res. Soc.*, **48**:306-313. [doi:10.2307/3010429]

Lee, H.W., Lee, S.S., Chae, K.C., 1994a. Operating characteristics of $M^{[x]}/G/1$ queue with *N* policy. *Queueing Systems*, **15**(1-4):387-399. [doi:10.1007/BF01189247]

Lee, H.W., Lee, S.S., Park, J.O., Chae, K.C., 1994b. Analysis of $M^{[x]}/G/1$ queue with *N* policy and multiple vacations. *J. Appl. Prob.*, **31**(2):467-496. [doi:10.2307/3215040]

Lee, S.S., Lee, H.W., Yoon, S.H., Chae, K.C., 1995. Batch arrival queue with *N* policy and single vacation. *Comput. Oper. Res.*, **22**(2):173-189. [doi:10.1016/0305-0548(94)E0015-Y]

Levy, Y., Yechiali, U., 1975. Utilization of idle time in an M/G/1 queueing system. *Manag. Sci.*, **22**:202-211.

Madan, K.C., 2000. An M/G/1 queue with second optional service. *Queueing Systems*, **34**(1/4):37-46. [doi:10.1023/A:1019144716929]

Medhi, J., 2002. A single server Poisson input queue with a second optional channel. *Queueing Systems*, **42**(3):239-242. [doi:10.1023/A:1020519830116]

Medhi, J., Templeton, J.G.C., 1992. A Poisson input queue under *N*-policy and with a general start up time. *Comput. Oper. Res.*, **19**(1):35-41. [doi:10.1016/0305-0548(92)90057-C]

Stewart, J., 1995. Calculus (3rd Ed.). Brooks/Cole Publishing Company, Pacific Grove.

Tadj, L., 2003. A quorum queueing system under *T*-policy. *J. Oper. Res. Soc.*, **54**(5):466-471. [doi:10.1057/palgrave.jors.2601536]

Tadj, L., Choudhury, G., 2005. Optimal design and control of queues. *Top*, **13**(2):359-414. [doi:10.1007/BF02579061]

Tadj, L., Ke, J.C., 2005. Control policy of a hysteretic bulk queueing system. *Math. Comput. Model.*, **41**(4-5):571-579. [doi:10.1016/j.mcm.2003.07.017]

Takagi, H., 1991. Queueing Analysis: A Foundation of Performance Evaluation. North-Holland, Amsterdam.

Teghem, J.Jr., 1987. Optimal control of a removable server in an M/G/1 queue with finite capacity. *Eur. J. Oper. Res.*, **31**(3):358-367. [doi:10.1016/0377-2217(87)90045-2]

Tijms, H.C., 1986. Stochastic Modelling and Analysis. Wiley, New York.

Wang, K.H., 1995. Optimal control of a Markovian queueing system with a removable and non-reliable server. *Microelectron. Rel.*, **35**(8):1131-1136. [doi:10.1016/0026-2714(94)00164-J]

Wang, K.H., 1997. Optimal control of an $M/E_k/1$ queueing system with removable service station subject to breakdowns. *J. Oper. Res. Soc.*, **48**(9):936-942. [doi:10.1057/palgrave.jors.2600419]

Wang, K.H., Ke, J.C., 2000. A recursive method to the optimal control of an M/G/1 queueing system with finite capacity and infinite capacity. *Appl. Math. Model.*, **24**(12):899-914. [doi:10.1016/S0307-904X(00)00024-X]

Wang, K.H., Yen, K.L., 2003. Optimal control of an $M/H_k/1$ queueing system with a removable server. *Math. Methods Oper. Res.*, **57**(2):255-262. [doi:10.1007/s001860200263]

Wang, K.H., Chang, K.W., Sivazlian, B.D., 1999. Optimal control of a removable and non-reliable server in an infinite and a finite $M/H_2/1$ queueing system. *Appl. Math. Model.*, **23**(8):651-666. [doi:10.1016/S0307-904X(99)00002-5]

Wang, K.H., Kao, H.T., Chen, G., 2004. Optimal management of a removable and non-reliable server in an infinite and a finite $M/H_k/1$ queueing system. *Qual. Technol. Quant. Manag.*, **1**(2):325-339.

Yadin, M., Naor, P., 1963. Queueing systems with a removable service station. *Oper. Res. Quart.*, **14**:393-405.

Zhang, Z.G., Vickson, R.G., van Eenige, M.J.A., 1997. Optimal two-threshold policies in an M/G/1 queue with two vacation types. *Perform. Eval.*, **29**(1):63-80. [doi:10.1016/S0166-5316(96)00005-3]