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Scaling properties of Navier-Stokes turbulence*

Zhao-cun LIU

(*School of Environment and Planning, Henan University, Kaifeng 475001, China*)

E-mail: liuzc17@yahoo.com.cn

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Abstract: The property of the velocity field and the cascade process of the fluid flow are key problems in turbulence research. This study presents the scaling property of the turbulent velocity field and a mathematical description of the cascade process, using the following methods: (1) a discussion of the general self-similarity and scaling invariance of fluid flow from the viewpoint of the physical mechanism of turbulent flow; (2) the development of the relationship between the scaling indices and the key parameters of the She and Leveque (SL) model in the inertial range; (3) an investigation of the basis of the fractal model and the multi-fractal model of turbulence; (4) a demonstration of the physical meaning of the flowing field scaling that is related to the real flowing vortex. The results illustrate that the SL model could be regarded as an approximate mathematical solution of Navier-Stokes (N-S) equations, and that the phenomena of normal scaling and anomalous scaling is the result of the mutual interactions among the physical factors of nonlinearity, dissipation, and dispersion. Finally, a simple turbulent movement conceptional description model is developed to show the local properties and the instantaneous properties of turbulence.

Key words: Transformation equations, Scaling, Spiral structure, Cascade, Turbulence
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INTRODUCTION

An intriguing aspect of fully developed turbulence is the possible existence of universal scaling behavior for small-scale fluctuations. In fully developed turbulence, large scale fluctuation depends on the specific flowing environment such as boundary conditions, while small scale fluctuation shows much more universal properties. In 1941 Kolmogorov (K41) conjectured the existence of a universal state in fully developed turbulence, called K41 theory. The velocity difference δv_l across a distance l and the energy dissipation ε_l averaged over a ball of size l, have a simple scaling behavior when l is in the inertial range: $\delta v_l^p \sim l^{\varsigma_p}$, $\varepsilon_l^p \sim l^{\tau_p}$, where ς_p and τ_p are called scaling indices (p denotes the pth order). Recently, the turbulent hierarchical structure model developed by She and Leveque (SL) in 1994 (She and Su, 1999) successfully explained the anomalous scaling, from which the scholars developed the hierarchical structure theory of turbulence (Dubrulle, 1994), the SL hierarchy structure model based on the study of turbulent properties including the quantum character showed that the SL hierarchy structure model was also effective in describing the non-linear system (She and Su, 1999). SL scaling and extended self-similarity (ESS) or the generalized extended self-similarity (GESS) have been extensively studied (Frisch, 1995; Liu and Liu, 1993; 1997; 2002). Nevertheless, the following questions should be further studied: (1) What is the relationship between SL scaling and Navier-Stokes (N-S) equations; (2) How to apply knowledge gained from the hierarchical structure theory of turbulence to the concept of turbulence description, especially to velocity distribution widely used in engineering practices.

In recent years attention has been paid to scaling problems (Mitra and Pandit, 2004; Mazzino *et al.*, 2007) and many novel turbulence theories have been initiated as a development of classical mechanics. For example, Chiueh (1998) studied fluid turbulence with

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dynamical quantum chaos, and Marmanis (1998) did the analogy between N-S equations and Maxwell's equations. Some valuable achievements concerning particles and fields in fluid turbulence were reviewed by Falkovich *et al.*(2001). Classical mechanics has developed continuously to give one an opportunity to reconsider these known theories at an early stage and to bring many valuable hints for further studies.

In this paper, from the viewpoint of scaling, some properties of turbulence, such as scaling invariance, SL scaling and the cascade process of fluid flow, were studied.

GENERAL SELF-SIMILARITY AND SCALING INVARIANCE OF FLUID FLOW

Scaling invariance of some physical variables of fluid flow

Barotropic, incompressible viscous fluid flow with potential mass force is considered in this study. However, the results could be extrapolated to general cases without any essential difficulty. The motion equations of incompressible Newtonian fluid are deduced by N-S equations as follows:

$$\frac{\partial \boldsymbol{V}}{\partial t}+(\boldsymbol{V}\cdot\nabla)\boldsymbol{V}=-\frac{1}{\rho}\nabla p+\upsilon\nabla^2\boldsymbol{V}, \tag{1}$$

where ∇ is the Hamilton operator, $\boldsymbol{V}$ is velocity, t is time, p is pressure (for convenience, the pressure after absorbing the potential mass force is also called pressure), ρ is density, and υ is the kinematic viscosity coefficient. Eq.(1) could be transformed by the following formulae:

$$\begin{cases}\hat{\boldsymbol{x}}=\lambda^{\beta_1}\boldsymbol{x},\\ \hat{t}=\lambda^{\beta_2}t,\\ \hat{\boldsymbol{v}}=\lambda^{\beta_1-\beta_2}\boldsymbol{v},\\ \hat{p}^*=\lambda^{2(\beta_1-\beta_2)}p^*,\\ \hat{\upsilon}=\lambda^{2\beta_1-\beta_2}\upsilon,\end{cases} \tag{2}$$

where $p^*=p/\rho$, $\boldsymbol{x}$ means the variable of spatial length. Then the form of Eq.(1) remains invariant, moreover, the dimensionless parameters of the Reynolds number of $Re=(vl)/\upsilon$ and Froude's number of $Fr=v^2/(gh)$ remain invariant. The parameters of β_1, β_2, in Eq.(2) are called scaling indices.

After the variables were transformed according to Eq.(2), the relations between these variables, such as the turbulent kinematic energy dissipation rate $\varepsilon\equiv-\frac{\partial}{\partial t}\left(\frac{1}{2}v^2\right)$, vorticity $\boldsymbol{\omega}\equiv\nabla\times\boldsymbol{v}$, enstrophy $Q\equiv|\boldsymbol{\omega}|^2/2$, helicity $h\equiv\boldsymbol{v}\cdot\boldsymbol{\omega}$, and Lamb vector $\boldsymbol{L}=\boldsymbol{\omega}\times\boldsymbol{v}$, are shown as follows:

$$\begin{cases}\hat{\varepsilon}=\lambda^{2\beta_1-3\beta_2}\varepsilon,\\ \hat{\boldsymbol{\omega}}=\lambda^{-\beta_2}\boldsymbol{\omega},\\ \hat{Q}=\lambda^{-2\beta_2}Q,\\ \hat{h}=\lambda^{\beta_1-2\beta_2}h,\\ \hat{\boldsymbol{L}}=\lambda^{\beta_1-2\beta_2}\boldsymbol{L}.\end{cases} \tag{3}$$

Relations between the scaling invariance and SL scaling

The relationship between SL scaling and GESS scaling was discussed by Dubrulle (1994) and She and Su (1999), and the relationship between self-similarity and scaling was discussed by Liu and Liu (2002). It is known that the power spectrum of the physical variable with no characteristic scale satisfies the following formula:

$$S(k)\sim k^{-\mu}, \tag{4}$$

where k is the wave number, and μ is the index of power spectrum that has the following relation to the velocity scaling:

$$\mu=2(\beta_1-\beta_2)+1. \tag{5}$$

If the scaling index of fluctuating velocity of the pth order is ς_p, like that in the K41 theory, i.e., $S_p(\ell)=\left\langle|\boldsymbol{v}(x+\ell)-\boldsymbol{v}(x)|^p\right\rangle=\left\langle|\delta\boldsymbol{v}_\ell|^p\right\rangle\propto\ell^{\varsigma_p}$, the following relations can consequently be developed as in the SL hierarchy model:

$$\beta_1-\beta_2=\frac{\mathrm{d}\varsigma_p}{\mathrm{d}p}=\gamma p+c(1-\beta^p). \tag{6}$$

The physical meaning of β_1, β_2 could be explained subsequently. Let

$$\beta_1=\gamma p+c,\ \beta_2=c\beta^p, \tag{7}$$

like those in the SL hierarchy model, where γ is the index of the most excited state of the energy dissipation rate, and c is the co-dimension of space of the most excited structure covering. Thus β_1 shows the properties of the most excited state, and β_2 shows the co-dimension space of the pth order coherent structure covering. The physical meaning of the constant β of the SL model associates with the parameter β_1, which will be discussed in the following text.

From the viewpoint of mathematics, the essential difficulty to be examined in the cascade-down of turbulent energy is the function of the non-linear term that plays an important role in the energy transform process. As $\hat{\varepsilon} = \lambda^{2\beta_1 - 3\beta_2}\varepsilon$, if we take $\hat{\varepsilon} = \varepsilon$, then $\beta_2 = 2\beta_1/3$. Let $S_p(x) \equiv \langle v \rangle^p$. Because $\hat{\boldsymbol{x}} = \lambda^{\beta_1}\boldsymbol{x}$ and $\hat{\boldsymbol{v}} = \lambda^{\beta_1 - \beta_2}\boldsymbol{v}$, , if we have known $S_p(x) \propto \ell^{\tilde{p}}$, then there should be $\tilde{p} = p/3$. The result is accordant with the following formulae developed from the hypothesis of K41 Theory (Frisch, 1995):

$$S_p(x) \propto \ell^{p/3}, \tag{8}$$

$$S_p(x) = C_p \varepsilon^{p/3} \ell^{p/3}. \tag{9}$$

Relationships between the scaling and the β model, the fractal turbulence model and the multi-fractal turbulence model

The basic hypotheses of the bi-fractal turbulence model or the multi-fractal turbulence model, e.g., the natural generalized results of β model developed by Frisch *et al.*(1978) (here the bi-fractal model is taken as an example; the physical hypotheses of the bi-fractal model and the multi-fractal model are essentially the same), which have two sets, namely $\wp_1$ and $\wp_2$, both of which are imbedded in the physical space of the fluid flow. The velocity near $\wp_1$ has the scaling exponent h_1 and the velocity near $\wp_2$ has the scaling exponent h_2, and they possess the following properties:

$$\frac{\delta v_\ell(x)}{v_0} = \begin{cases} \left(\dfrac{\ell}{\ell_0}\right)^{h_1}, & x \in \wp_1, \ \dim \wp_1 = D_1, \\ \left(\dfrac{\ell}{\ell_0}\right)^{h_2}, & x \in \wp_2, \ \dim \wp_2 = D_2. \end{cases} \tag{10}$$

The essential hypothesis is that ζ_p could be given with different values, i.e., ζ_p takes two different values in two different cases: $\zeta_p = h_1$ in $\wp_1$ and $\zeta_p = h_2$ in $\wp_2$, respectively. Let $\beta_2 = h_2$ and $\beta_1 = h_1 + h_2$; β_1 means the fractal of space filled by the fluid flow. The multi-fractal models could be treated in the same manner.

The mathematical physics basis of the above models, such as β model, fractal model, and multi-fractal model of turbulence, could be regarded as the scaling transform relations of N-S equations. Moreover, in the practical fluid flow, the basis is fulfilled by the vortex. By the way, SL scaling can be developed from the bi-fractal model and log-Poisson distribution law as described by Dubrulle (1994).

Although the scaling of the flow field is invariant as mentioned above, it is considered that no universality exists in the probability distribution function (PDF) of physical variables in all kinds of fluid movements, because the PDF relates to the concrete mechanism of turbulence. On the other hand, it is also considered that there is no universality for the tail function deviating from the selected PDF, e.g., the Gaussian normal distribution law of the physical variable. Both phenomena imply that the specific PDF is related to the evolution style, e.g., symmetry breaking and intermittence, which can be described as the mutual relationship of β_1 and β_2, to some extent.

Analysis of velocity field

As Eq.(1) is the governing equation for fluid flow, the property of general similarity is attached to the solutions corresponding to different scales of the fluid flow. For example, the velocity is

$$\boldsymbol{V}(\lambda^{\beta_1}x, \lambda^{\beta_2}t) = \lambda^{\beta_1 - \beta_2}\boldsymbol{V}(x,t). \tag{11}$$

In practical engineering, if the velocity distribution was analytically expressed as the function of spatial and time variables, then why does Eq.(2) require concrete expression for the forms of velocity? For the functional equations, the solution of $f(\lambda^{\beta_1}x) = \lambda^{\beta_1} f(x)$ is $f(x)=cx$, and the solution of $s(\lambda^{\beta_2}t) = \lambda^{\beta_2} s(t)$ is $s(t)=ct$, where c denotes constant. The solutions show that if any one of the spatial variables or the temporal variables was taken as the reference parameter, there would be no specific requirement for another in velocity distribution. In Eq.(2), let $\beta_1-\beta_2=1$, and then

$$V(\lambda^{\beta_2+1}x,\lambda^{\beta_2}t)=\lambda V(x,t). \qquad (12)$$

According to the renormalization group, the solution of $f(\lambda^{\beta_2+1}x,\lambda^{\beta_2}t)=\lambda f(x,t)$ is

$$f(x,t)=t^{1/\beta_2}F_1\left(\frac{x}{t^{(\beta_2+1)/\beta_2}}\right), \text{ or}$$
$$f(x,t)=x^{1/(\beta_2+1)}F_2\left(\frac{t}{x^{\beta_2/(\beta_2+1)}}\right), \qquad (13)$$

where $F_1(\cdot)$ and $F_2(\cdot)$ are arbitrary functions.

$C_i(r)=\overline{v(x)v(x+r)}$ represents an auto-correlation function of turbulent velocity with the distance of r, which is the exponent function for each scale and the power function for the whole turbulence, according to Liu and Liu (2002)'s study. After calculation, this study presents the auto-correlation function of the model as follows:

$$C(r)=ar^{2\beta_3}+\frac{\lambda^{2\beta_3}C_0(r)}{\lambda^{2\beta_3}-1}, \qquad (14)$$

where $\beta_3=1-\beta_2$. Although it is difficult to find the concrete analytic solution of Eq.(14), practical engineering shows that there are two types of velocity distributions widely used for shear flow, e.g., open channel flow, namely power type and a logarithmic one:

$$v=Ax^{\alpha}+B \text{ and } v=A_1\ln(x)+B_1, \qquad (15)$$

where v and x could be the dimensional physical variables or the dimensionless physical variables. The two types of velocity distribution formulae should be the approximate solution of the N-S equation under the particular conditions.

It is self-similarity and self-preservation that result in the power type and the logarithmic type of velocity distribution. Different phases of flow self-similarity result in the problem of deciding which one will be the most appropriate type to describe the distribution of velocities (Hunt, 2001).

Physical meaning of flow field scaling

Description of the cascade process seems solid in virtue of the scaling transform relations of the flow field. The turbulence scaling connects with the cascade process of the fluid flow. The cascade process of turbulent energy is mainly determined by the stretch of vortexes (Wu *et al.*, 2006). The mutual actions among many vortexes make vortexes themselves become slimmer and slimmer. Thus the turbulent energy is transferred from the large vortexes to the small ones, and the cascade process of turbulent energy is formed. As the vortex becomes slimmer and slimmer, the turbulent energy will finally dissipate. This dissipation process belongs to the entropy increasing process. At the same time, a turbulent field is a dissipative structure system according to the theory of Prigogine. Because of the self-organization, such as the instabilities of the fluid flow, the vortexes rolling up, and the vortexes pairing, the energy of the small vortexes may be transferred to that of the large ones to make the flow field become ordered. The above fact, a typical phenomenon of Ranque-Hilsh, is called the entropy decreasing process. In this process, the commutating mechanism plays an important role, which causes the small vortexes to become large ones because of the Coriolis force. But from the viewpoint of the mechanism of phenomenology, the infinite hierarchy structures of flow are only a deterministic process in which the energy of mean flow transfers to the energy of fluctuating flow because of the available viscosity and finally turns into molecule heat.

The scaling transform relations of the above flow field strongly support the cascade process.

From the physics viewpoint, turbulence could be regarded as an excited medium. The fourth effect of turbulence, dispersion, recently proposed, connects with the excited medium's following properties: (1) the local nature of the excitation; (2) the hysteretic nature of the excitation propagation. She and Su (1999) figured out that the most important hypothesis of H4 in the SL model originated from the hidden symmetry of the N-S equation. Furthermore, the symmetry could be described with the general co-variance (Dubrulle, 1994). Comparing the above hypothesis of H4 with Eq.(3), it could be concluded that the SL model basically illustrated the mathematical physics properties of the N-S equation with a different Reynolds number. The scaling that displays the normal one and anomalous one comes from the different behavior of the intrinsic properties of the N-S equation (Qian, 2001).

Then two problems arise: What is the physical mechanism of the scaling? How to explain the different mechanisms of the scaling from the viewpoint of turbulent evolution, for normal scaling and anomalous scaling corresponding to a different Reynolds number with different mechanisms.

Different values of the Reynolds number essentially indicate the different cooperative evolutional results of wave behavior and particle behavior, and result in different excited levels of mutual action mechanism on dissipation and dispersion with normal scaling and anomalous scaling. Normal scaling corresponds to the cooperative evolutional result of particle behavior and wave one, and anomalous scaling corresponds to dominating particle behavior. In other words, normal scaling and anomalous scaling are different properties of fluid flow with a deterministic evolutional mechanism. Compared with experimental results (Lee and Lee, 2001), normal scaling and anomalous scaling represent the different structural behavior of the fluid flow in different evolutionary stages. Traditionally, the flow field can be regarded as the stochastic field with non-white noise. When the Reynolds number is large enough, small scale eddies are independent of their evolutionary history. It is the independence that leads the fluid to realize the deterministic statistical state. According to the viewpoint of the dissipative and dispersive mechanism of turbulence (Liu and Liu, 1998), different dominating physical factors exist resulting in different structures and different dominating mechanisms at different evolutionary phases.

The wave nature and the particle nature at some evolutionary phases appear due to random fluctuation, and the random nature of small scale eddies arises subsequently. On one hand, the above phenomena demonstrate the reason why no universality for the probability distribution function exists. On the other hand, it may be a possible reason for the fact that the scaling becomes different when the scaling order is infinite and the Reynolds number is infinite (Nelkin, 1995; Zou, 2003).

SIMPLE TURBULENT DESCRIPTION MODEL

It is confirmed by experimental results (Lee and Lee, 2001) that turbulence and transition are different stages with the same physical process, the resonance is their common physical mechanism, and CS-solitons, combined-CS-solitons, and ring vortex are the uniform basic structures of fluid flow.

A simple turbulent conceptional description model is proposed for fully developed turbulence to describe the relationship between mechanics and statistics. For any element in fluid flow, a fiber bundle structure exists which is composed of a direct product of 3D space and 1D time:

$$\boldsymbol{M}^4=\boldsymbol{R}^3\otimes\boldsymbol{R}^1. \tag{16}$$

The style of transform from the base space to the state space is as follows: (1) the spatial shift, (2) the spatial rotation, (3) the spatial distortion, and (4) the time delay. In this study, the base space takes a continuous form, and the state space takes a discrete form of which the value may be definite or countably infinite.

Some indices for measuring the fully developed turbulent flow manifest more local and instantaneous properties. It is suggested that random fluctuation should exist to some extent in the premise of deterministic quantity if some indices were used to quantitatively describe turbulence. Possibility distribution of the variables corresponding to different evolution levels of turbulence should be permitted to fluctuate randomly in some respects besides the ultimate deterministic distribution function. This is one of the effects of symmetry breaking in deterministic chaos.

CONCLUSION

This study presented scaling transformation equations and case studies in the flow field. The scaling invariance and the mathematical basis for the fractal and the multi-fractal model were studied. In the inertial range, relationships between the scaling indices and the key parameters of the SL model suggest that the SL model could be regarded as an approximate mathematical solution for N-S equations.

The results indicated that interactions of non-linearity, dissipation and dispersion may lead to normal scaling and anomalous scaling.

Connections between the scaling transformation equations and the spiral structures widely existing in fluid dynamics were explored to illustrate the cascade process. Finally, a simple turbulent movement conceptional description model was developed to emphasize the discrete properties in space and in time of turbulence.

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