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Optimal approximate merging of a pair of Bézier curves with G^2-continuity*

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Abstract: We present a novel approach for dealing with optimal approximate merging of two adjacent Bézier curves with G^2-continuity. Instead of moving the control points, we minimize the distance between the original curves and the merged curve by taking advantage of matrix representation of Bézier curve's discrete structure, where the approximation error is measured by L_2-norm. We use geometric information about the curves to generate the merged curve, and the approximation error is smaller. We can obtain control points of the merged curve regardless of the degrees of the two original curves. We also discuss the merged curve with point constraints. Numerical examples are provided to demonstrate the effectiveness of our algorithms.

Key words: Approximate merging, G^1-continuity, G^2-continuity, Discrete subdivision, Point constraints
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INTRODUCTION

A fast growing variety of geometric modeling data have been transferred and exchanged frequently between various CAD systems along with the development of network technology. This demands that the geometric model should ensure data high accuracy and reduce data communication as much as possible when the error is given. To achieve this goal, Hoschek (1987) proposed approximate conversion including degree reduction and approximate merging.

In recent years, degree reduction methods of Bézier curves and surfaces have been investigated by many researchers (Zheng and Wang, 2003; Ahn *et al.*, 2004; Chen and Wu, 2004; Sunwoo, 2005; Hu and Wang, 2008). However, there has been far less research concerning an approximate merging of Bézier curves. Hu *et al.*(2001) dealt with the problem of approximate merging by finding conditions for precise merging of Bézier curves. Then Wu and Chen (2002) proposed a merging method of interval Bézier curves.

Hu *et al.*(2001) is one classic paper on approximate merging for polynomial Bézier curves. They proposed the conditions for precise merging and made use of control points perturbation with two different optimization criteria to obtain control points of the merged Bézier curve. They also considered the conditions for matching original endpoints and derivatives and merging with point constraints. Meanwhile they pointed out that degree elevation of the original Bézier curves could reduce the approximation error.

In this paper we consider approximate merging of two adjacent Bézier curves with geometric continuity by minimizing the objective function based on L_2-error between the original curves and the merged curve. G^2-continuity provides four additional parameters. By using these parameters, we can optimize the approximation and obtain a smaller error in L_2-norm. Compared to (Hu *et al.*, 2001), our method uses the matrix representation of a discrete Bézier curve and can directly obtain control points of the merged curve, regardless of the degrees of the original

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curves. The approximation error of the merged curve is smaller than that in (Hu *et al*., 2001). Furthermore, we obtain a higher degree merged curve through raising the merged Bézier curve's degree instead of degree elevation of the original Bézier curves.

The outline of this paper is as follows. We first propose the problem of approximate merging of Bézier curves with G^α-continuity in Section 2. Then we consider G^2-constrained approximate merging in Section 3. In Section 4, approximate merging with G^2-continuity and point constraints is discussed. Numerical examples are given in Sections 3 and 4 to demonstrate our algorithm's advantage. Finally, we summarize this paper and indicate future work in Section 5.

PRELIMINARIES

Definitions and notations

A Bézier curve of degree n is defined by the control points $\{\boldsymbol{p}_i\}_{i=0}^n$ in the form

$$\boldsymbol{P}(t)=\sum_{i=0}^{n}B_i^n(t)\boldsymbol{p}_i,\ \ 0\le t\le 1,$$

where $B_i^n(t)=\binom{n}{i}(1-t)^{n-i}t^i$ is the Bernstein basis function. We rewrite the Bézier curve in the matrix form:

$$\boldsymbol{P}(t)=\boldsymbol{B}_n\boldsymbol{P}_n, \tag{1}$$

where

$$\boldsymbol{B}_n=[B_0^n(t), B_1^n(t), ..., B_n^n(t)],\ \ \boldsymbol{P}_n=[\boldsymbol{p}_0, \boldsymbol{p}_1, ..., \boldsymbol{p}_n]^{\mathrm{T}}.$$

We rewrite the discrete construction theorem (Wang *et al*., 2001) of the Bézier curve in the matrix form:

Theorem 1 For an n degree Bézier curve $\boldsymbol{P}(t)$ ($0\le t\le 1$) and any point $\lambda\in[0, 1]$, $\boldsymbol{P}(t)$ can be scattered into two sub-Bézier curves, where $t=\lambda$, with the same degree. That is,

$$\boldsymbol{P}(t)=\begin{cases}\boldsymbol{P}_{\mathrm{l}}(t)=\sum_{i=0}^{n}B_i^n(t)\boldsymbol{p}_i^i(\lambda)=\boldsymbol{B}_n\boldsymbol{P}_{n\mathrm{l}}, \\ \boldsymbol{P}_{\mathrm{r}}(t)=\sum_{i=0}^{n}B_i^n(t)\boldsymbol{p}_n^{n-i}(\lambda)=\boldsymbol{B}_n\boldsymbol{P}_{n\mathrm{r}},\end{cases}\ \ 0\le t\le 1, \tag{2}$$

where $\boldsymbol{P}_{n\mathrm{l}}=\boldsymbol{A}_1(\lambda)\boldsymbol{P}_n$, $\boldsymbol{P}_{n\mathrm{r}}=\boldsymbol{A}_2(\lambda)\boldsymbol{P}_n$, and

$$\boldsymbol{A}_1(\lambda)=\begin{bmatrix}1 & 0 & ... & 0 & 0\\ B_0^1(\lambda) & B_1^1(\lambda) & ... & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ B_0^{n-1}(\lambda) & B_1^{n-1}(\lambda) & ... & B_{n-1}^{n-1}(\lambda) & 0\\ B_0^n(\lambda) & B_1^n(\lambda) & ... & B_{n-1}^n(\lambda) & B_n^n(\lambda)\end{bmatrix},$$

$$\boldsymbol{A}_2(\lambda)=\begin{bmatrix}B_0^n(\lambda) & B_1^n(\lambda) & \cdots & B_{n-1}^n(\lambda) & B_n^n(\lambda)\\ 0 & B_0^{n-1}(\lambda) & \cdots & B_{n-2}^{n-1}(\lambda) & B_{n-1}^{n-1}(\lambda)\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & B_0^1(\lambda) & B_1^1(\lambda)\\ 0 & 0 & \cdots & 0 & 1\end{bmatrix}.$$

Proof According to the discrete construction theorem,

$$\boldsymbol{P}(t)=\begin{cases}\sum_{i=0}^{n}B_i^n\left(\dfrac{t}{\lambda}\right)\boldsymbol{p}_i^i(\lambda),\ \ 0\le t\le\lambda,\\ \sum_{i=0}^{n}B_i^n\left(\dfrac{t-\lambda}{1-\lambda}\right)\boldsymbol{p}_n^{n-i}(\lambda),\ \ \lambda\le t\le 1.\end{cases}$$

By introducing the parametric transformation, Eq.(2) is obtained. Control points sequences $\boldsymbol{P}_{n\mathrm{l}}$, $\boldsymbol{P}_{n\mathrm{r}}$ of $\boldsymbol{P}_{\mathrm{l}}(t)$ and $\boldsymbol{P}_{\mathrm{r}}(t)$ can be obtained by the corner cutting algorithm respectively. It is easy to deduce $\boldsymbol{A}_1(\lambda)$ and $\boldsymbol{A}_2(\lambda)$.

Theorem 2 If the elements of an $m\times m$ matrix $\boldsymbol{M}=(m_{ij})$ satisfy $m_{ij}=\int_0^1 B_i^m(t)B_j^m(t)\mathrm{d}t$, then $\boldsymbol{M}$ is a real symmetric positive definite matrix (Hu *et al*., 2001).

Problem statement

Problem 1 For two adjacent Bézier curves $\boldsymbol{P}(u)=\sum_{i=0}^{n_1}B_i^{n_1}(u)\boldsymbol{p}_i$ and $\boldsymbol{Q}(v)=\sum_{i=0}^{n_2}B_i^{n_2}(v)\boldsymbol{q}_i$ ($0\le u\le 1$, $0\le v\le 1$, $\boldsymbol{p}_{n_1}=\boldsymbol{q}_0$), their control points are $\boldsymbol{p}_i$ (i=0, 1, …, n_1) and $\boldsymbol{q}_i$ (i=0, 1, …, n_2) respectively. Merging of $\boldsymbol{P}(u)$ and $\boldsymbol{Q}(v)$ is a process that amounts to finding an n ($\ge\max(n_1, n_2)$) degree Bézier curve $\boldsymbol{R}(t)$ with control points $\boldsymbol{r}_i$ (i=0, 1, …, n), such that a suitable distance function $d(\boldsymbol{R}, \overline{\boldsymbol{R}})$ between $\boldsymbol{R}(t)$ and $\overline{\boldsymbol{R}}(t)$ is minimized on the interval [0, 1], where

$$\bar{\boldsymbol{R}}(t)=\begin{cases}\sum_{i=0}^{n_1}B_i^{n_1}\left(\dfrac{t}{\lambda}\right)\boldsymbol{p}_i, & 0\le t\le\lambda,\\ \sum_{i=0}^{n_2}B_i^{n_2}\left(\dfrac{t-\lambda}{1-\lambda}\right)\boldsymbol{q}_i, & \lambda\le t\le 1,\end{cases}$$

and λ is a subdivision parameter. In this paper, λ is defined as

$$\lambda=\frac{\int_0^1|\boldsymbol{P}'(u)|\,\mathrm{d}u}{\int_0^1|\boldsymbol{P}'(u)|\,\mathrm{d}u+\int_0^1|\boldsymbol{Q}'(v)|\,\mathrm{d}v},$$

and $d(\boldsymbol{R},\bar{\boldsymbol{R}})$ is measured by the squared L_2-norm:

$$d=\int_0^1\|\boldsymbol{R}(t)-\bar{\boldsymbol{R}}(t)\|^2\,\mathrm{d}t. \tag{3}$$

In addition to this, $\boldsymbol{R}(t)$ should be G^α-continuous (i) with $\boldsymbol{P}(u)$ at the left endpoint and (ii) with $\boldsymbol{Q}(v)$ at the right endpoint.

$$\text{(i)}\ \boldsymbol{R}^{(i)}(t)=\boldsymbol{P}^{(i)}(\phi(t)),\quad t=0,\ i=0,\ 1,\ ...,\ \alpha, \tag{4}$$

where $\phi:\ t\in[0,1]\to u\in[0,1]$ is a strictly increasing function with $\phi(0)=0$, $\phi'(0)\neq0$.

$$\text{(ii)}\ \boldsymbol{R}^{(i)}(t)=\boldsymbol{Q}^{(i)}(\varphi(t)),\quad t=1,\ i=0,\ 1,\ ...,\ \alpha, \tag{5}$$

where $\varphi:\ t\in[0,1]\to v\in[0,1]$ is a strictly increasing function with $\varphi(1)=1$, $\varphi'(1)\neq0$.

Note that Hu *et al*.(2001) considered approximate merging of a Bézier curve by control points perturbation including C^α-continuity (α=0, 1), fixing the first α+1 control points of $\boldsymbol{P}(u)$ and the last α+1 control points of $\boldsymbol{Q}(v)$ as control points of the merged curve, while we take advantage of matrix form and additional parameters introduced by G^α-continuity to optimize the approximation error.

APPROXIMATE MERGING WITH G^2-CONTINUITY

G^2-continuity is a special case of G^1-continuity. Before G^2-continuity, G^1-continuity should be discussed.

G^1-constrained continuity

Clearly, for G^0-continuity, the left endpoint and right endpoint of $\boldsymbol{R}(t)$ should coincide with the left endpoint of $\boldsymbol{P}(u)$ and the right endpoint of $\boldsymbol{Q}(v)$, respectively. And for G^1-continuity, the coincidence of the oriented tangents is additionally needed. From Eqs.(4) and (5), we have: $\boldsymbol{R}(0)=\boldsymbol{P}(0)$, $\boldsymbol{R}^{(1)}(0)=\boldsymbol{P}^{(1)}(0)$, $\boldsymbol{R}(1)=\boldsymbol{Q}(1)$, $\boldsymbol{R}^{(1)}(1)=\boldsymbol{Q}^{(1)}(1)$. More precisely,

$$\begin{cases}\boldsymbol{r}_0=\boldsymbol{p}_0,\ \boldsymbol{r}_1=\boldsymbol{r}_0+n_1\delta_0^2(\boldsymbol{p}_1-\boldsymbol{p}_0)/n,\\ \boldsymbol{r}_n=\boldsymbol{q}_{n_2},\ \boldsymbol{r}_{n-1}=\boldsymbol{r}_n-n_2\delta_1^2(\boldsymbol{q}_{n_2}-\boldsymbol{q}_{n_2-1})/n.\end{cases} \tag{6}$$

Compared to C^1-continuity, Eq.(6) adds two parameters: squared term of δ_i (i=0, 1), where $\delta_0^2=\phi'(0)$, $\delta_1^2=\varphi'(1)$. When $\delta_0=\delta_1=1$, G^1-continuity degenerates to C^1-continuity. We can also imagine that G^1-continuity will lead to a smaller approximation error than C^1-continuity by parametric adjustment. In the following, we will propose our algorithm.

1. Regular case

For two Bézier curves $\boldsymbol{P}(u)$ and $\boldsymbol{Q}(v)$, the optimal merged curve $\boldsymbol{R}(t)$ can be determined through two stages. In the first stage, the n degree Bézier curve $\boldsymbol{R}(t)$ should satisfy the four equations in Eq.(6):

$$\boldsymbol{R}(t)=B_0^n(t)\boldsymbol{r}_0+B_1^n(t)\boldsymbol{r}_1+\sum_{i=2}^{n-2}B_i^n(t)\boldsymbol{r}_i+B_{n-1}^n(t)\boldsymbol{r}_{n-1}+B_n^n(t)\boldsymbol{r}_n,$$

where $\boldsymbol{r}_1$, $\boldsymbol{r}_{n-1}$ contain unknown terms δ_0, δ_1 respectively. Then we obtain the interior control points $\boldsymbol{r}_2$, $\boldsymbol{r}_3$, …, $\boldsymbol{r}_{n-2}$ by minimizing $d=\int_0^1\|\boldsymbol{R}(t)-\bar{\boldsymbol{R}}(t)\|^2\,\mathrm{d}t$.

Denote $\boldsymbol{A}_{11}=\boldsymbol{A}_1(\lambda)[0,1]$, $\boldsymbol{A}_{12}=\boldsymbol{A}_1(\lambda)[2,3,\ldots,n-2]$, $\boldsymbol{A}_{13}=\boldsymbol{A}_1(\lambda)[n-1,n]$, and $\boldsymbol{A}_{21}=\boldsymbol{A}_2(\lambda)[0,1]$, $\boldsymbol{A}_{22}=\boldsymbol{A}_2(\lambda)[2,3,\ldots,n-2]$, $\boldsymbol{A}_{23}=\boldsymbol{A}_2(\lambda)[n-1,n]$, where $\boldsymbol{A}_i(\lambda)[a,\ldots,b]$ (i=1, 2) is the submatrix of $\boldsymbol{A}_i(\lambda)$ obtained by extracting columns from a to b.

Assume that

$$\boldsymbol{C}=(c_{ij})\ (0\le i,j\le n),\quad c_{ij}=\int_0^1B_i^n(t)B_j^n(t)\mathrm{d}t;$$

$$\boldsymbol{D}=(d_{ij})\ (0\le i\le n_1,\ 0\le j\le n_1),\quad d_{ij}=\int_0^1B_i^n(t)B_j^{n_1}(t)\mathrm{d}t;$$

$$\boldsymbol{E}=(e_{ij})\ (0\le i\le n_2,\ 0\le j\le n_2),\quad e_{ij}=\int_0^1B_i^n(t)B_j^{n_2}(t)\mathrm{d}t.$$

Control points sequence $\boldsymbol{R}_n$ of $\boldsymbol{R}(t)$ is divided into three parts: $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, $\boldsymbol{R}_3$, where $\boldsymbol{R}_1=[\boldsymbol{r}_0, \boldsymbol{r}_1]^{\mathrm{T}}$, $\boldsymbol{R}_2=[\boldsymbol{r}_2, \boldsymbol{r}_3, \ldots, \boldsymbol{r}_{n-2}]^{\mathrm{T}}$, $\boldsymbol{R}_3=[\boldsymbol{r}_{n-1}, \boldsymbol{r}_n]^{\mathrm{T}}$.

According to Theorem 1, the curve $\boldsymbol{R}(t)$ is scattered into two sub-Bézier curves, where $t=\lambda\in[0, 1]$.

$$\begin{aligned} d &= \int_0^1 \| \boldsymbol{R}(t)-\bar{\boldsymbol{R}}(t)\|^2\,\mathrm{d}t \\ &= \int_0^1 \| \boldsymbol{R}_{\mathrm{l}}(t)-\boldsymbol{P}(t)\|^2\,\mathrm{d}t + \int_0^1 \| \boldsymbol{R}_{\mathrm{r}}(t)-\boldsymbol{Q}(t)\|^2\,\mathrm{d}t \\ &= \int_0^1 \| \boldsymbol{B}_n\boldsymbol{A}_1\boldsymbol{R}_n-\boldsymbol{B}_{n_1}\boldsymbol{P}_{n_1}\|^2\,\mathrm{d}t + \int_0^1 \| \boldsymbol{B}_n\boldsymbol{A}_2\boldsymbol{R}_n-\boldsymbol{B}_{n_2}\boldsymbol{Q}_{n_2}\|^2\,\mathrm{d}t. \end{aligned}$$

In order to obtain a minimum of d, it is necessary that the derivatives of d with respect to each element of $\boldsymbol{R}_2$ be zero. We write these equations in the matrix form:

$$\begin{aligned} &(\boldsymbol{A}_{12}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{A}_{12}+\boldsymbol{A}_{22}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{A}_{22})\boldsymbol{R}_2 \\ &\quad=\boldsymbol{A}_{12}^{\mathrm{T}}(\boldsymbol{D}\boldsymbol{P}_{n_1}-\boldsymbol{C}\boldsymbol{A}_{11}\boldsymbol{R}_1-\boldsymbol{C}\boldsymbol{A}_{13}\boldsymbol{R}_3) \\ &\qquad+\boldsymbol{A}_{22}^{\mathrm{T}}(\boldsymbol{E}\boldsymbol{Q}_{n_2}-\boldsymbol{C}\boldsymbol{A}_{21}\boldsymbol{R}_1-\boldsymbol{C}\boldsymbol{A}_{23}\boldsymbol{R}_3). \end{aligned}$$

Denote $\boldsymbol{G}=\boldsymbol{A}_{12}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{A}_{12}+\boldsymbol{A}_{22}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{A}_{22}$. Since $\boldsymbol{C}$ is invertible (Theorem 2), $\boldsymbol{G}$ is invertible.

$$\begin{aligned} \boldsymbol{R}_2=\boldsymbol{G}^{-1}[&\boldsymbol{A}_{12}^{\mathrm{T}}(\boldsymbol{D}\boldsymbol{P}_{n_1}-\boldsymbol{C}\boldsymbol{A}_{11}\boldsymbol{R}_1-\boldsymbol{C}\boldsymbol{A}_{13}\boldsymbol{R}_3) \\ &+\boldsymbol{A}_{22}^{\mathrm{T}}(\boldsymbol{E}\boldsymbol{Q}_{n_2}-\boldsymbol{C}\boldsymbol{A}_{21}\boldsymbol{R}_1-\boldsymbol{C}\boldsymbol{A}_{23}\boldsymbol{R}_3)]. \end{aligned} \tag{7}$$

Note that $\boldsymbol{r}_1$ and $\boldsymbol{r}_{n-1}$ are quadratic functions of the parameters δ_0 and δ_1, so are the interior control points $\boldsymbol{r}_i$ (i=2, 3, …, n−2). That is to say, different δ_0 and δ_1 will lead to different $\boldsymbol{R}_n$ and different errors d (Fig.1).

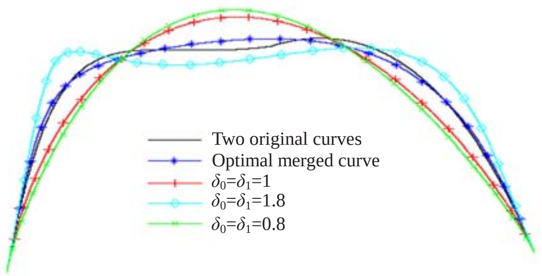

Fig.1 Different δ_0, δ_1 and different merged curves

The second stage is to determine two unknown terms δ_0, δ_1. After $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, $\boldsymbol{R}_3$ containing δ_0, δ_1 being replaced into Eq.(3), d will be a quartic polynomial with two parameters, that is, $d=d(\delta_0, \delta_1)$. In this way, the problem will be transformed into solving the minimum of a multiple non-linear function. That is,

$$\frac{\partial d(\delta_0,\delta_1)}{\partial\delta_0}=0, \quad \frac{\partial d(\delta_0,\delta_1)}{\partial\delta_1}=0. \tag{8}$$

Though these equations are very complicated, we can resort to the quasi-Newton method by using Matlab. After replacing all control points in $\boldsymbol{R}_n$ with the solved values δ_0, δ_1, we obtain the optimal merged Bézier curve $\boldsymbol{R}(t)$ with G^1-continuity at left and right endpoints.

2. Improvement

We investigate the result in Eq.(8). Since $\delta_i^2\geq 0$ (i=0, 1), Eq.(8) holds. But if the solved value δ_i (i=0, 1) equals or nearly equals 0, $\boldsymbol{r}_0$ and $\boldsymbol{r}_n$ nearly equal $\boldsymbol{r}_1$ and $\boldsymbol{r}_{n-1}$, respectively. In such a case, the merged curve is singular at the endpoints. This is not always a desirable merged Bézier curve. So we adjust Eq.(3) by adding one regularization term:

$$\begin{aligned} d' = \int_0^1 \| \boldsymbol{R}(t)-\bar{\boldsymbol{R}}(t)\|^2\,\mathrm{d}t + \mu[&\operatorname{avg}(\boldsymbol{P})\cdot(1-\delta_0^2)^2 \\ &+\operatorname{avg}(\boldsymbol{Q})\cdot(1-\delta_1^2)^2], \end{aligned} \tag{9}$$

where μ is a small positive number, and avg($\boldsymbol{P}$) and avg($\boldsymbol{Q}$) are the average edge length of control polygons $\boldsymbol{P}(u)$ and $\boldsymbol{Q}(v)$, respectively.

In Eq.(9), μ is uncertain. Different μ will lead to different δ_0, δ_1. μ should be a small value, because if μ is large enough, δ_i (i=0, 1) will tend to 1 (that is C^1-approximation), while μ extends the regularization term's influence to avoid the singularities. So we should set a fair value for μ. Due to numerous experiments, a frequently used value for μ is 10^{-4}. Then we minimize Eq.(9) to obtain the parametric values δ_0 and δ_1.

Now we can summarize the G^1-continuous merging algorithm as follows:

Algorithm 1

Input: two control points sequences $\{\boldsymbol{p}_i\}$ (i=0, 1, …, n_1) and $\{\boldsymbol{q}_i\}$ (i=0, 1, …, n_2), the merged degree n.

Output: control points sequence of the merged curve $\{\boldsymbol{r}_i\}$ (i=0, 1, …, n).

Step 1: Set δ_0 and δ_1 with the initial value 1.

Step 2: Express $\boldsymbol{r}_0$, $\boldsymbol{r}_1$, $\boldsymbol{r}_{n-1}$, $\boldsymbol{r}_n$ by Eq.(6) and $\boldsymbol{r}_i$ (i=2, 3, …, n−2) by Eq.(7).

Step 3: Use the quasi-Newton method to obtain δ_0, δ_1 by minimizing the function Eq.(3) or Eq.(9).

Step 4: Compute $\boldsymbol{r}_i$ (i=0, 1, …, n) by Eqs.(6) and (7) and the approximation error d by Eq.(3).

G^2-constrained approximate merging

It is obvious that G^2-continuity implies G^1-continuity. In addition to the condition in Eq.(6), the coincidence of the oriented curvatures is additionally satisfied:

$$\boldsymbol{R}''(0) = \boldsymbol{P}''(\phi(0)),\ \boldsymbol{R}''(1) = \boldsymbol{Q}''(\varphi(1)). \tag{10}$$

From Eq.(10), it is deduced that

$$\begin{cases} \boldsymbol{r}_2 = 2\boldsymbol{r}_1 - \boldsymbol{r}_0 + \dfrac{n_1(n_1-1)}{n(n-1)}\delta_0^{\ 4}(\Delta\boldsymbol{p}_0)^2 + \dfrac{n_1}{n(n-1)}\eta_0\Delta\boldsymbol{p}_0, \\ \boldsymbol{r}_{n-2} = 2\boldsymbol{r}_{n-1} - \boldsymbol{r}_n + \dfrac{n_2(n_2-1)}{n(n-1)}\delta_1^{\ 4}(\Delta\boldsymbol{q}_{n_2-2})^2 + \dfrac{n_2}{n(n-1)}\eta_1\Delta\boldsymbol{q}_{n_2-1}, \end{cases} \tag{11}$$

where $\delta_0^2 = \phi'(0), \eta_0 = \phi''(0), \delta_1^2 = \varphi'(1), \eta_1 = \varphi''(1)$.

Now we know that G^2-continuity is equivalent to Eqs.(6) and (11). In the following, the method is the same as in the previous subsection. We only outline the main steps.

Let $\boldsymbol{R}_n$ be composed of three parts: $\boldsymbol{R}_1=[\boldsymbol{r}_0, \boldsymbol{r}_1, \boldsymbol{r}_2]^{\mathrm{T}}$, $\boldsymbol{R}_2=[\boldsymbol{r}_3, \boldsymbol{r}_4, \ldots, \boldsymbol{r}_{n-3}]^{\mathrm{T}}$, $\boldsymbol{R}_3=[\boldsymbol{r}_{n-2}, \boldsymbol{r}_{n-1}, \boldsymbol{r}_n]^{\mathrm{T}}$.

Denote $\boldsymbol{A}_{11}=\boldsymbol{A}_1(\lambda)[0, 1, 2]$, $\boldsymbol{A}_{12}=\boldsymbol{A}_1(\lambda)[3, 4, \ldots, n-3]$, $\boldsymbol{A}_{13}=\boldsymbol{A}_1(\lambda)[n-2, n-1, n]$, and $\boldsymbol{A}_{21}=\boldsymbol{A}_2(\lambda)[0, 1, 2]$, $\boldsymbol{A}_{22}=\boldsymbol{A}_2(\lambda)[3, 4, \ldots, n-3]$, $\boldsymbol{A}_{23}=\boldsymbol{A}_2(\lambda)[n-2, n-1, n]$, where $\boldsymbol{A}_i(\lambda)[a, \ldots, b]$ (i=1, 2) is the submatrix of $\boldsymbol{A}_i(\lambda)$ obtained by extracting columns from a to b; $\boldsymbol{G} = \boldsymbol{A}_{12}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{A}_{12} + \boldsymbol{A}_{22}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{A}_{22}$.

$$\begin{aligned} \boldsymbol{R}_2 = \boldsymbol{G}^{-1}[&\boldsymbol{A}_{12}^{\mathrm{T}}(\boldsymbol{D}\boldsymbol{P}_{n_1} - \boldsymbol{C}\boldsymbol{A}_{11}\boldsymbol{R}_1 - \boldsymbol{C}\boldsymbol{A}_{13}\boldsymbol{R}_3) \\ &+ \boldsymbol{A}_{22}^{\mathrm{T}}(\boldsymbol{E}\boldsymbol{Q}_{n_2} - \boldsymbol{C}\boldsymbol{A}_{21}\boldsymbol{R}_1 - \boldsymbol{C}\boldsymbol{A}_{23}\boldsymbol{R}_3)]. \end{aligned} \tag{12}$$

It is obvious that $\boldsymbol{r}_i$ (i=0, 1, …, n) can be regarded as quartic functions containing four parameters δ_0, δ_1, η_0, η_1. Then the four additional parameters are provided to optimize the approximation error. Considering the singular point, we may add the regularization term to Eq.(3).

Again, we use the quasi-Newton method to solve these four parameters and obtain the optimal approximate merging Bézier curve with G^2-continuity. The initial values of δ_0, δ_1, η_0, η_1 can be chosen as 1, 1, 0, 0 respectively because now G^2-continuity degenerates to C^2-continuity.

Since the algorithm for G^2-continuity is similar to that for G^1-continuity, we omit it.

Examples

We show several examples for the above algorithms.

Example 1 Control points of $\boldsymbol{P}(u)$: (−10, −10), (−8, 2), (−6, 1), (−1, 0); control points of $\boldsymbol{Q}(v)$: (−1, 0), (4, 1), (6, 2), (8, −10). We find one merged Bézier curve. The results are given in Figs.2~5.

Remark 1 Figs.2 and 3 show that G^1 and G^2 approximation errors are much smaller than C^1 and C^2 approximation errors with the same degree, since geometric continuity is less dependent on the parameters. In Fig.2, the error of C^1 is d=11.571, Hoschek's G^1 error is 3.345, while our G^1 error is d=2.776. Our result is much better than theirs, and the running time is less than Hoschek's. When we set θ=89.999° as recurrence's termination (Hoschek, 1987), Hoschek's running time is 14 s while ours is 7 s. In Fig.3, the error of C^2 is d=12.803 while the error of G^2 is d=0.220.

Remark 2 In Fig.4, the G^2 approximation error is still smaller than the C^2-continuity error when the degree of G^2 approximation is lower. Here the error is 1.208 for C^2 and 0.169 for G^2. This means that we can obtain the merged curve with less data communication by geometric continuity for a given error. Fig.5 shows that raising the merged curve's degree can produce a higher degree merged curve instead of degree elevation of the original Bézier curves (Hu *et al.*, 2001).

Example 2 Control points of $\boldsymbol{P}(u)$: (1, 1), (2, −2), (2.5, −1), (3.5, 0), (4.5, 1.5), (5, 3.5), (5.7, 4), (6, 4); control points of $\boldsymbol{Q}(v)$: (6, 4), (7, 3), (7.5, 3), (8.5, 4.5), (9, 3), (9.5, 4), (10, 6), (11, −3), (12, −1), (13, 2). We find one merged Bézier curve. The results are given in Figs.6 and 7.

Remark 3 Figs.6 and 7 show that even if the degree of $\boldsymbol{P}(u)$ is different from the degree of $\boldsymbol{Q}(v)$, we can still obtain the merged Bézier curve directly, and that G^α approximation error is still smaller than C^α-error (α=1, 2).

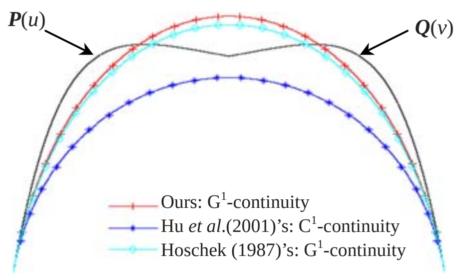

Fig.2 Degree 3 with G^1- and C^1-continuity

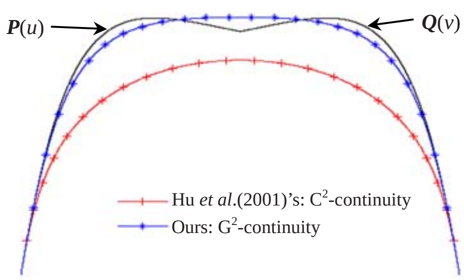

Fig.3 Degree 5 with G^2- and C^2-continuity

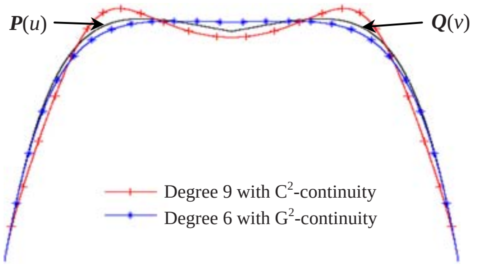

Fig.4 Degree 9 with C^2-continuity and degree 6 with G^2-continuity

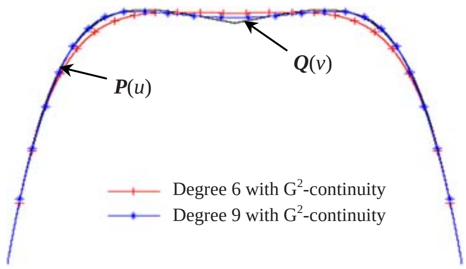

Fig.5 Degrees 6 and 9 with G^2-continuity

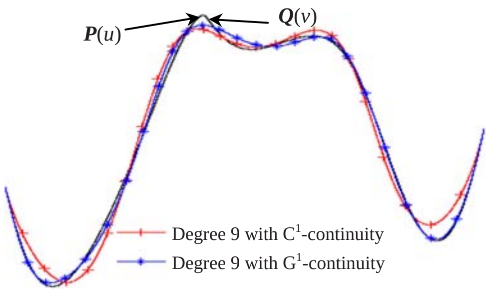

Fig.6 Degree 9 with C^1- and G^1-continuity

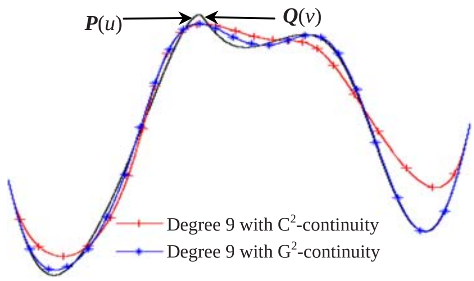

Fig.7 Degree 9 with C^2- and G^2-continuity

APPROXIMATE MERGING WITH POINT CONSTRAINTS AND G^2-CONTINUITY

Problem 2 For two adjacent Bézier curves: an n_1 degree Bézier curve $\boldsymbol{P}(u)$ and an n_2 degree Bézier curve $\boldsymbol{Q}(v)$, merging of $\boldsymbol{P}(u)$ and $\boldsymbol{Q}(v)$ is a process that amounts to finding an n ($\geq\max(n_1, n_2)$) degree Bézier curve $\boldsymbol{R}(t)$. Besides G^{α}-continuity, $\boldsymbol{R}(t)$ should pass through l+1 points on the curve $\boldsymbol{P}(u)$, where corresponding parametric values are u_j (j=0, 1, …, l), and through m+1 points on the curve $\boldsymbol{Q}(v)$, where corresponding parametric values are v_k (k=0, 1, …, m). It is certain that $0\leq l\leq n_1$, $0\leq m\leq n_2$ and $l+m+1\leq n$ (Hu *et al.*, 2001).

Approximate merging with point constraints and G^1-continuity

Using parametric transformation, the values $x_j=\lambda u_j$ (j=0, 1, …, l) are parametric values of u_j (j=0, 1, …, l) corresponding to $\boldsymbol{R}(t)$ and $y_k=\lambda+(1-\lambda)v_k$ (k=0, 1, …, m) are parametric values of v_k (k=0, 1, …, m) corresponding to $\boldsymbol{R}(t)$. The method is the same as in the subsection "G^1-constrained continuity". First, scatter the merged curve $\boldsymbol{R}(t)$ into two sub-curves. Then use two sub-curves to approximate the two original curves.

We write the Lagrange function as

$$\mathcal{L}=\int_0^1(\|\boldsymbol{R}_{\mathrm{l}}(t)-\boldsymbol{P}(t)\|^2+\|\boldsymbol{R}_{\mathrm{r}}(t)-\boldsymbol{Q}(t)\|^2)\mathrm{d}t$$

$$+\sum_{i=0}^{l}\eta_i(\boldsymbol{R}(x_i)-\boldsymbol{P}(u_i))+\sum_{i=0}^{m}\eta_{l+1+i}(\boldsymbol{R}(y_i)-\boldsymbol{Q}(v_i))$$

$$=\int_0^1\|\boldsymbol{B}_n\boldsymbol{A}_1\boldsymbol{R}_n-\boldsymbol{B}_{n_1}\boldsymbol{P}_{n_1}\|^2\,\mathrm{d}t+\int_0^1\|\boldsymbol{B}_n\boldsymbol{A}_2\boldsymbol{R}_n-\boldsymbol{B}_{n_2}\boldsymbol{Q}_{n_2}\|^2\,\mathrm{d}t$$

$$+\sum_{i=0}^{l}\eta_i(\boldsymbol{R}(x_i)-\boldsymbol{P}(u_i))+\sum_{i=0}^{m}\eta_{l+1+i}(\boldsymbol{R}(y_i)-\boldsymbol{Q}(v_i)).$$

Three parts of $\boldsymbol{R}_n$ are: $\boldsymbol{R}_1=[\boldsymbol{r}_0, \boldsymbol{r}_1]^{\mathrm{T}}$, $\boldsymbol{R}_2=[\boldsymbol{r}_2, \boldsymbol{r}_3, \ldots, \boldsymbol{r}_{n-2}]^{\mathrm{T}}$, $\boldsymbol{R}_3=[\boldsymbol{r}_{n-1}, \boldsymbol{r}_n]^{\mathrm{T}}$.

Denote $\boldsymbol{A}_{11}=\boldsymbol{A}_1(\lambda)[0, 1]$, $\boldsymbol{A}_{12}=\boldsymbol{A}_1(\lambda)[2, 3, \ldots, n-2]$, $\boldsymbol{A}_{13}=\boldsymbol{A}_1(\lambda)[n-1, n]$, and $\boldsymbol{A}_{21}=\boldsymbol{A}_2(\lambda)[0, 1]$, $\boldsymbol{A}_{22}=\boldsymbol{A}_2(\lambda)[2, 3, \ldots, n-2]$, $\boldsymbol{A}_{23}=\boldsymbol{A}_2(\lambda)[n-1, n]$, where $\boldsymbol{A}_i(\lambda)[a, \ldots, b]$ (i=1, 2) is the submatrix of $\boldsymbol{A}_i(\lambda)$ obtained by extracting columns from a to b.

It is necessary that the derivatives of L with respect to each element of $\boldsymbol{R}_2$ and each η_i (i=0, 1, …, l+1+m) be zero. We write the l+m+n−1 equations in the matrix form:

$$2(\boldsymbol{A}_{12}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{A}_{11}\boldsymbol{R}_1+\boldsymbol{A}_{22}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{A}_{21}\boldsymbol{R}_1+\boldsymbol{A}_{12}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{A}_{12}\boldsymbol{R}_2 +\boldsymbol{A}_{22}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{A}_{22}\boldsymbol{R}_2+\boldsymbol{A}_{12}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{A}_{13}\boldsymbol{R}_3+\boldsymbol{A}_{22}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{A}_{23}\boldsymbol{R}_3 -\boldsymbol{A}_{12}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{P}_{n_1}-\boldsymbol{A}_{22}^{\mathrm{T}}\boldsymbol{E}\boldsymbol{Q}_{n_2})+\boldsymbol{H}_{12}^{\mathrm{T}}\boldsymbol{F}=\boldsymbol{0}, \quad (13)$$

$$\boldsymbol{H}_{11}\boldsymbol{R}_1+\boldsymbol{H}_{12}\boldsymbol{R}_2+\boldsymbol{H}_{13}\boldsymbol{R}_3=\boldsymbol{L}, \quad (14)$$

where

$$\boldsymbol{F}=[\eta_0,\cdots,\eta_l,\eta_{l+1},\cdots,\eta_{l+m+1}]^{\mathrm{T}},$$

$$\boldsymbol{L}=\left[\sum_{i=0}^{n_1}B_i^{n_1}(u_0)\boldsymbol{p}_i,\cdots,\sum_{i=0}^{n_1}B_i^{n_1}(u_l)\boldsymbol{p}_i,\ \sum_{i=0}^{n_2}B_i^{n_2}(v_0)\boldsymbol{q}_i,\cdots,\sum_{i=0}^{n_2}B_i^{n_2}(v_m)\boldsymbol{q}_i\right]^{\mathrm{T}},$$

$$\boldsymbol{H}=\begin{bmatrix} B_0^n(x_0) & B_1^n(x_0) & \cdots & B_n^n(x_0)\\ \vdots & \vdots & & \vdots\\ B_0^n(x_l) & B_1^n(x_l) & \cdots & B_n^n(x_l)\\ B_0^n(y_0) & B_1^n(y_0) & \cdots & B_n^n(y_0)\\ \vdots & \vdots & & \vdots\\ B_0^n(y_m) & B_1^n(y_m) & \cdots & B_n^n(y_m)\end{bmatrix},$$

$\boldsymbol{H}_{11}=[\boldsymbol{H}_0, \boldsymbol{H}_1]$, $\boldsymbol{H}_{12}=[\boldsymbol{H}_2, \boldsymbol{H}_3, \ldots, \boldsymbol{H}_{n-2}]$, $\boldsymbol{H}_{13}=[\boldsymbol{H}_{n-1}, \boldsymbol{H}_n]$, and $\boldsymbol{H}_i$ is the (i+1)th column vector of $\boldsymbol{H}$.

Denote $\boldsymbol{X}=\boldsymbol{A}_{12}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{A}_{11}+\boldsymbol{A}_{22}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{A}_{21}$, $\boldsymbol{Y}=\boldsymbol{A}_{12}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{A}_{12}+\boldsymbol{A}_{22}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{A}_{22}$, $\boldsymbol{Z}=\boldsymbol{A}_{12}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{A}_{13}+\boldsymbol{A}_{22}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{A}_{23}$, $\boldsymbol{S}=\boldsymbol{H}_{12}\boldsymbol{Y}^{-1}\boldsymbol{H}_{12}^{\mathrm{T}}$. $\boldsymbol{W}=\boldsymbol{A}_{12}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{P}_{n_1}+\boldsymbol{A}_{22}^{\mathrm{T}}\boldsymbol{E}\boldsymbol{Q}_{n_2}$. Based on Eq.(13),

$$\boldsymbol{R}_2=\boldsymbol{Y}^{-1}(\boldsymbol{W}-\boldsymbol{H}_{12}^{\mathrm{T}}\boldsymbol{F}/2-\boldsymbol{X}\boldsymbol{R}_1-\boldsymbol{Z}\boldsymbol{R}_3).$$

Since $\boldsymbol{H}$ is a full-rank matrix, $\boldsymbol{H}_{12}\boldsymbol{Y}^{-1}\boldsymbol{H}_{12}^{\mathrm{T}}$ is invertible. Thus we can deduce

$$\boldsymbol{F}=2\boldsymbol{S}^{-1}[\boldsymbol{H}_{12}\boldsymbol{Y}^{-1}\boldsymbol{W}-\boldsymbol{L}+(\boldsymbol{H}_{11}-\boldsymbol{H}_{12}\boldsymbol{Y}^{-1}\boldsymbol{X})\boldsymbol{R}_1 +(\boldsymbol{H}_{13}-\boldsymbol{H}_{12}\boldsymbol{Y}^{-1}\boldsymbol{Z})\boldsymbol{R}_3].$$

Then through Eq.(13),

$$\boldsymbol{R}_2=\boldsymbol{Y}^{-1}[\boldsymbol{W}-\boldsymbol{H}_{12}^{\mathrm{T}}\boldsymbol{S}^{-1}\boldsymbol{H}_{12}\boldsymbol{Y}^{-1}\boldsymbol{W}+\boldsymbol{H}_{12}^{\mathrm{T}}\boldsymbol{S}^{-1}\boldsymbol{L} -(\boldsymbol{H}_{12}^{\mathrm{T}}\boldsymbol{S}^{-1}\boldsymbol{H}_{11}-\boldsymbol{H}_{12}^{\mathrm{T}}\boldsymbol{S}^{-1}\boldsymbol{H}_{12}\boldsymbol{Y}^{-1}\boldsymbol{X}+\boldsymbol{X})\boldsymbol{R}_1 -(\boldsymbol{H}_{12}^{\mathrm{T}}\boldsymbol{S}^{-1}\boldsymbol{H}_{13}-\boldsymbol{H}_{12}^{\mathrm{T}}\boldsymbol{S}^{-1}\boldsymbol{H}_{12}\boldsymbol{Y}^{-1}\boldsymbol{Z}+\boldsymbol{Z})\boldsymbol{R}_3]. \quad (15)$$

In light of Eqs.(6) and (15), $d=d(\delta_0, \delta_1)$ will be a quartic function of δ_0 and δ_1. Considering the singular point, we could replace the d in Eq.(3) with d' in Eq.(9). Next solve the minimum of d' by the quasi-Newton method.

The algorithm is summarized as follows:

Algorithm 2

Input: two control points sequences $\{\boldsymbol{p}_i\}$ (i=0, 1, …, n_1) and $\{\boldsymbol{q}_i\}$ (i=0, 1, …, n_2), the merged degree n and the interpolated points' parametric values $\{u_i\}_{i=0}^{l}$, $\{v_i\}_{i=0}^{m}$.

Output: the merged curve's control points $\{\boldsymbol{r}_i\}$ (i=0, 1, …, n).

Step 1: Set δ_0 and δ_1 with the initial value 1.

Step 2: Express $\boldsymbol{r}_0$, $\boldsymbol{r}_1$, $\boldsymbol{r}_{n-1}$, $\boldsymbol{r}_n$ by Eq.(6) and $\boldsymbol{r}_i$ (i=2, 3, …, n−2) by Eq.(15).

Step 3: Use the quasi-Newton method to obtain δ_0 and δ_1 by minimizing the function Eq.(3) or Eq.(9).

Step 4: Compute $\boldsymbol{r}_i$ (i=0, 1, …, n) by Eqs.(6) and (15) and the approximation error d by Eq.(3).

Point constraints and G^2-continuity

G^2-continuity is equivalent to Eqs.(6) and (11). The main steps are easy to deduce. We only give the result.

Three parts of $\boldsymbol{R}_n$ are: $\boldsymbol{R}_1=[\boldsymbol{r}_0, \boldsymbol{r}_1, \boldsymbol{r}_2]^{\mathrm{T}}$, $\boldsymbol{R}_2=[\boldsymbol{r}_3, \boldsymbol{r}_4, \ldots, \boldsymbol{r}_{n-3}]^{\mathrm{T}}$, $\boldsymbol{R}_3=[\boldsymbol{r}_{n-2}, \boldsymbol{r}_{n-1}, \boldsymbol{r}_n]^{\mathrm{T}}$.

Denote $\boldsymbol{A}_{11}=\boldsymbol{A}_1(\lambda)[0, 1, 2]$, $\boldsymbol{A}_{12}=\boldsymbol{A}_1(\lambda)[3, 4, \ldots, n-3]$, $\boldsymbol{A}_{13}=\boldsymbol{A}_1(\lambda)[n-2, n-1, n]$, and $\boldsymbol{A}_{21}=\boldsymbol{A}_2(\lambda)[0, 1, 2]$, $\boldsymbol{A}_{22}=\boldsymbol{A}_2(\lambda)[3, 4, \ldots, n-3]$, $\boldsymbol{A}_{23}=\boldsymbol{A}_2(\lambda)[n-2, n-1, n]$, where $\boldsymbol{A}_i(\lambda)[a, \ldots, b]$ (i=1, 2) is the submatrix of $\boldsymbol{A}_i(\lambda)$ obtained by extracting columns from a to b.

$$\boldsymbol{R}_2=\boldsymbol{Y}^{-1}[\boldsymbol{W}-\boldsymbol{H}_{12}^{\mathrm{T}}\boldsymbol{S}^{-1}\boldsymbol{H}_{12}\boldsymbol{Y}^{-1}\boldsymbol{W}+\boldsymbol{H}_{12}^{\mathrm{T}}\boldsymbol{S}^{-1}\boldsymbol{L} -(\boldsymbol{H}_{12}^{\mathrm{T}}\boldsymbol{S}^{-1}\boldsymbol{H}_{11}-\boldsymbol{H}_{12}^{\mathrm{T}}\boldsymbol{S}^{-1}\boldsymbol{H}_{12}\boldsymbol{Y}^{-1}\boldsymbol{X}+\boldsymbol{X})\boldsymbol{R}_1 -(\boldsymbol{H}_{12}^{\mathrm{T}}\boldsymbol{S}^{-1}\boldsymbol{H}_{13}-\boldsymbol{H}_{12}^{\mathrm{T}}\boldsymbol{S}^{-1}\boldsymbol{H}_{12}\boldsymbol{Y}^{-1}\boldsymbol{Z}+\boldsymbol{Z})\boldsymbol{R}_3]. \quad (16)$$

In the following, it is the problem of the minimum of a non-linear function with four parameters. Since the algorithm for G^2-continuity and point constraints is simple, we do not describe it any more.

Example 3 Control points of $\boldsymbol{P}(u)$: (1, 1), (2, −2), (2.5, −1), (3.5, 0), (4.5, 1.5), (5, 3.5), (5.7, 4), (6, 4);

control points of $\boldsymbol{Q}(v)$: (6, 4), (7, 3), (7.5, 3), (8.5, 4.5), (9, 3), (9.5, 4), (10, 6), (11, −3), (12, −1), (13, 2). Passed points' parametric values are: (u_0, u_1, u_2)=(0.2, 0.5, 0.8), (v_0, v_1, v_2)=(0.1, 0.5, 0.9). We find one merged Bézier curve. The results are given in Figs.8 and 9.

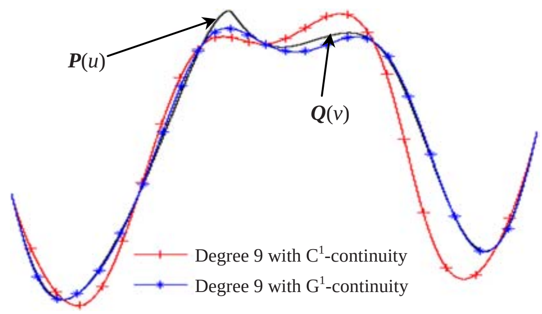

Fig.8 C¹- and G¹-continuity and point constraints

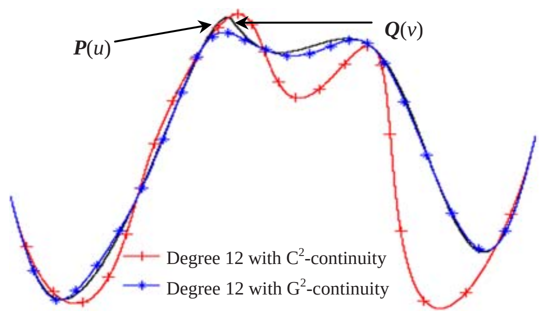

Fig.9 C²- and G²-continuity and point constraints

CONCLUSION AND FUTURE WORK

This paper presents approximate merging of two adjacent Bézier curves with the same or different degrees under G^2-continuity. Our method uses matrix representation of a discrete Bézier curve and can directly obtain the merged curve, regardless of degrees of the two original curves. Compared with traditional methods, our method achieves better results. However, geometric continuity is inevitably influenced by parameters. To overcome this influence is our future work. Pottmann *et al*.(2002) proposed the concept of 'active B-spline' and successfully applied it to degree reduction of a Bézier curve. Maybe we can have a try.

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