Journal of Zhejiang University SCIENCE A
ISSN 1673-565X (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Feedback maximization of reliability of MDOF quasi integrable-Hamiltonian systems under combined harmonic and white noise excitations*

Lin-cong CHEN, Rong-hua HUAN, Wei-qiu ZHU†‡

(*Department of Mechanics, State Key Laboratory of Fluid Power Transmission and Control, Zhejiang University, Hangzhou 310027, China*)

†E-mail: wqzhu@yahoo.com

Received Apr. 26, 2008; Revision accepted Mar. 3, 2009; Crosschecked July 9, 2009

Abstract: We studied the feedback maximization of reliability of multi-degree-of-freedom (MDOF) quasi integrable-Hamiltonian systems under combined harmonic and white noise excitations. First, the partially averaged Itô equations are derived by using the stochastic averaging method for quasi integrable-Hamiltonian systems under combined harmonic and white noise excitations. Then, the dynamical programming equation and its boundary and final time conditions for the control problems of maximizing the reliability is established from the partially averaged equations by using the dynamical programming principle. The nonlinear stochastic optimal control for maximizing the reliability is determined from the dynamical programming equation and control constrains. The reliability function of optimally controlled systems is obtained by solving the final dynamical programming equation. Finally, the application of the proposed procedure and effectiveness of the control strategy are illustrated by using an example.

Key words: Stochastic optimal control, Dynamical programming, Quasi integrable-Hamiltonian system, Stochastic averaging, Combined harmonic and white noise excitation, Reliability

doi:10.1631/jzus.A0820316 **Document code:** A **CLC number:** O34

INTRODUCTION

The mathematical theory of stochastic optimal control has been quite well developed in the past decades. Bellman's dynamic programming and Pontryagin's maximum principle are the two principal approaches to stochastic optimal control problems. So far, the major application field of stochastic optimal control theory is economics and finance. In engineering field, however, only the linear quadratic Gaussian (LQG) control strategy has been widely applied so far. In the last few years, a set of nonlinear stochastic optimal control strategies for quasi-Hamiltonian systems have been proposed by Zhu and his coworkers (Zhu and Ying, 1999; Zhu *et al*., 2001), based on the stochastic averaging method for quasi-Hamiltonian systems and dynamical programming principle. At present, the strategies have been extended to the stochastic optimal control of partially observable systems (Zhu and Ying, 2004), the stochastic optimal semi-active control (Ying *et al*., 2007), feedback stabilization (Zhu, 2004), maximizing the reliability (e.g., Zhu and Deng, 2007) of multi-degree-of-freedom (MDOF) quasi-Hamiltonian systems. In addition, the proposed control strategy has been applied to civil engineering (e.g., Luo and Zhu, 2006) and proved to be more effective and efficient than LQG control strategy.

Although significant progress has been made in this area, to the present authors' knowledge, the work

‡ Corresponding author

* Project supported by the National Natural Science Foundation of China (No. 10772159), the Research Fund for the Doctoral Program of Higher Education of China (No. 20060335125), and the Zhejiang Provincial Nature Science Foundation of China (No. Y7080070)

on the control problem of MDOF systems subject to combined harmonic and white noise excitations is very limited. In the present paper, the feedback maximization of reliability of MDOF quasi integrable-Hamiltonian systems under combined harmonic and white noise excitations is studied. The stochastic averaging method for such systems is adopted to derive a set of partially averaged Itô equations, from which the dynamical programming equation and its associated boundary and final time conditions for maximizing the reliability is formulated via dynamical programming principle. The nonlinear stochastic control strategy is derived from the dynamical programming equation and control constrains. The reliability function of optimally controlled systems is obtained by solving the final dynamical programming equation. As an example, a controlled two coupled Duffing oscillators under combined harmonic and white noise excitations is worked out in detail.

STOCHASTIC AVERAGING

Consider a controlled quasi integrable-Hamiltonian system of n-degree-of-freedom governed by the following motion equations:

$$\begin{aligned}&\dot{Q}_i=\frac{\partial H'}{\partial P_i},\\&\dot{P}_i=-\frac{\partial H'}{\partial Q_i}-\varepsilon d'_{ij}(\boldsymbol{Q},\boldsymbol{P})\frac{\partial H'}{\partial P_j}+\varepsilon f_{ik}(\boldsymbol{Q},\boldsymbol{P})\cos\beta_k(t)\\&\qquad+\varepsilon u_i(\boldsymbol{Q},\boldsymbol{P})+\varepsilon^{1/2}h_{ie}(\boldsymbol{Q},\boldsymbol{P})\xi_e(t),\\&i,j=1,2,\dots,n;\ k=1,2,\dots,m;\ e=1,2,\dots,l,\end{aligned}\tag{1}$$

where $\boldsymbol{Q}=[Q_1, Q_2, \dots, Q_n]^{\mathrm{T}}$ and $\boldsymbol{P}=[P_1, P_2, \dots, P_n]^{\mathrm{T}}$; Q_i and P_i are generalized displacements and moments, respectively; $\beta_k=\Omega_k t+\varphi_k$; H' is a twice differentiable Hamiltonian; ε is a positive small parameter; $\varepsilon d'_{ij}(\boldsymbol{Q},\boldsymbol{P})$ denote the coefficients of quasi-linear damping; $\varepsilon f_{ik}(\boldsymbol{Q},\boldsymbol{P})$ denote the amplitudes of harmonic excitations; $\varepsilon u_i(\boldsymbol{Q},\boldsymbol{P})$ denotes the weak feedback control; $\varepsilon^{1/2}h_{ie}(\boldsymbol{Q},\boldsymbol{P})$ denote the amplitudes of random excitations; $\xi_e(t)$ are Gaussian white noises with correlation functions $E[\xi_e(t)\xi_f(t+\tau)]=2D_{ef}\delta(\tau)$ (e, f=1, 2, …, l).

Using the stochastic averaging method (Huang and Zhu, 2004) in the external resonant case, one can derive the following partially averaged Itô stochastic differential equations (Chen and Zhu, 2009):

$$\begin{aligned}&\mathrm{d}A_i=\varepsilon\left\{\bar{m}_i^{(11)}(\boldsymbol{A},\boldsymbol{\Psi})+\left\langle F_i^{(12)}(\boldsymbol{A},\boldsymbol{\Theta}',\boldsymbol{\Psi},\boldsymbol{\beta},\boldsymbol{u})\right\rangle_t\right\}\mathrm{d}t\\&\qquad+\varepsilon^{1/2}\bar{\sigma}_{ie}^{(1)}(\boldsymbol{A},\boldsymbol{\Psi})\mathrm{d}B_e(t),\\&\mathrm{d}\Psi_s=\varepsilon\left\{\bar{m}_s^{(33)}(\boldsymbol{A},\boldsymbol{\Psi})+L_r^s\left\langle F_r^{(21)}(\boldsymbol{A},\boldsymbol{\Theta}',\boldsymbol{\Psi},\boldsymbol{\beta},\boldsymbol{u})\right\rangle_t\right\}\mathrm{d}t\\&\qquad+\varepsilon^{1/2}\bar{\sigma}_{se}^{(3)}(\boldsymbol{A},\boldsymbol{\Psi})\mathrm{d}B_e(t),\\&i,r=1,2,\dots,n;\ s=1,2,\dots,\eta;\ 1\le\eta\le m;\ e=1,2,\dots,l,\end{aligned}\tag{2}$$

where A_i is the amplitude of the ith degree-of-freedom of the system, Ψ_s is the combination of angle variables and phase angles of harmonic excitations, i.e.,

$$\begin{aligned}&\Psi_s=L_r^s\Theta_r+M_k^s\beta_k(t),\\&\quad s=1,2,\dots,\eta;\ r=1,2,\dots,n;\ k=1,2,\dots,m,\end{aligned}\tag{3}$$

$$\begin{aligned}&\boldsymbol{A}=[A_1,A_2,\dots,A_n]^{\mathrm{T}},\\&\boldsymbol{\Psi}=[\Psi_1,\Psi_2,\dots,\Psi_\eta]^{\mathrm{T}},\\&\bar{m}_i^{(11)}=\frac{1}{(2\pi)^{n-\eta+m}}\int_0^{2\pi}\int_0^{2\pi}F_i^{(11)}\mathrm{d}\boldsymbol{\Theta}'\mathrm{d}\boldsymbol{\beta},\\&\bar{m}_s^{(33)}=\chi_s+\frac{1}{(2\pi)^{n-\eta+m}}\int_0^{2\pi}\int_0^{2\pi}L_r^sF_r^{(22)}\mathrm{d}\boldsymbol{\Theta}'\mathrm{d}\boldsymbol{\beta},\\&F_i^{(12)}=\frac{A_i\nu_iu_i\sin\Theta_i}{g(A_i+B_i)(1+h_i)},F_i^{(21)}=\frac{\nu_i(\cos\Theta_i+h_i)u_i}{g(A_i+B_i)(1+h_i)},\\&\bar{b}_{ij}^{(1)}=\bar{\sigma}_{ie}^{(1)}\bar{\sigma}_{je}^{(1)}=\frac{1}{(2\pi)^{n-\eta+m}}\int_0^{2\pi}\int_0^{2\pi}\sigma_{ie}^{(1)}\sigma_{je}^{(1)}\mathrm{d}\boldsymbol{\Theta}'\mathrm{d}\boldsymbol{\beta},\\&\bar{b}_{s_1s_2}^{(3)}=\bar{b}_{s_2s_1}^{(3)}=\bar{\sigma}_{s_1e}^{(3)}\bar{\sigma}_{s_2e}^{(3)}\\&\quad=\frac{L_{r_1}^{s_1}L_{r_2}^{s_2}}{(2\pi)^{n-\eta+m}}\int_0^{2\pi}\int_0^{2\pi}\sigma_{r_1e}^{(2)}\sigma_{r_2e}^{(2)}\mathrm{d}\boldsymbol{\Theta}'\mathrm{d}\boldsymbol{\beta},\\&\bar{b}_{is}^{(0)}=\bar{b}_{si}^{(0)}=\bar{\sigma}_{ie}^{(1)}\bar{\sigma}_{se}^{(3)}\\&\quad=\frac{1}{(2\pi)^{n-\eta+m}}\int_0^{2\pi}\int_0^{2\pi}\sigma_{ie}^{(1)}L_r^s\sigma_{re}^{(2)}\mathrm{d}\boldsymbol{\Theta}'\mathrm{d}\boldsymbol{\beta},\\&\langle\cdot\rangle_t=\lim_{T\to\infty}\frac{1}{T}\int_0^T\langle\cdot\rangle\mathrm{d}t,\\&\quad i,j,r,r_1,r_2=1,2,\dots,n;\\&\quad s,s_1,s_2=1,2,\dots,\eta;\ e=1,2,\dots,l,\end{aligned}\tag{4}$$

where η is the number of resonance relations,

$$\begin{aligned}&\boldsymbol{\Theta}'=[\Theta_{1+\eta},\Theta_{2+\eta},\dots,\Theta_n]^{\mathrm{T}},\\&\boldsymbol{\beta}=[\beta_1,\beta_2,\dots,\beta_m]^{\mathrm{T}},\\&F_i^{(11)}=\frac{A_i\nu_i\sin\Theta_i}{g(A_i+B_i)(1+h_i)}\\&\qquad\times\left[d_{ij}A_i\nu_i\sin\Theta_i+\sigma_{js}\frac{\partial\sigma_{is}}{\partial P_j}+f_{ik}\cos\beta_k(t)\right],\end{aligned}$$

$$F_i^{(22)} = \frac{\nu_i(\cos\Theta_i + h_i)}{g(A_i + B_i)(1+h_i)} \times \left[d_{ij} A_i \nu_i \sin\Theta_i + \sigma_{jc} \frac{\partial \sigma_{ic}}{\partial P_j} + f_{ik} \cos\beta_k(t) \right],$$
$$\sigma_{ie_1}^{(1)}\sigma_{je_1}^{(1)} = G_{ie_1}^{(1)}G_{je_1}^{(1)}, \quad \sigma_{ie_1}^{(2)}\sigma_{je_1}^{(2)} = G_{ie_1}^{(2)}G_{je_1}^{(2)},$$
$$\sigma_{ie_1}^{(1)}\sigma_{je_1}^{(2)} = G_{ie_1}^{(1)}G_{je_1}^{(2)},$$
$$G_{ie}^{(1)} = \frac{A_i \nu_i \sin\Theta_i}{g(A_i + B_i)(1+h_i)}\sigma_{ie},$$
$$G_{ie}^{(2)} = \frac{\nu_i(\cos\Theta_i + h_i)}{g(A_i + B_i)(1+h_i)}\sigma_{ie},$$
$$\sigma_{ie_1}\sigma_{je_2} = 2D_{e_1e_2}h_{ie_1}h_{je_2},$$
$$h_i = \frac{\mathrm{d}B_i}{\mathrm{d}A_i} = \frac{g(-A_i + B_i) + g(A_i + B_i)}{g(-A_i + B_i) - g(A_i + B_i)},$$
$$\nu_i(a_i, \theta_i) = \frac{\mathrm{d}\phi_i}{\mathrm{d}t} = \sqrt{\frac{2[U(a_i + b_i) - U(a_i\cos\theta_i + b_i)]}{a_i^2 \sin^2\theta_i}},$$
$$i, j = 1, 2, ..., n; \; k = 1, 2, ..., m; \; e, e_1, e_2 = 1, 2, ..., l. \tag{5}$$

Note that the terms containing control forces will be averaged later.

Since $H_i=U_i(A_i+B_i)$, the equations for A_i in Eq.(2) can be replaced by those for H_i by using Itô differential rule. The results are

$$\mathrm{d}H_i = \left[\bar{\bar{m}}_i^{(11)}(\boldsymbol{H},\boldsymbol{\Psi}) + \bar{\bar{m}}_i^{(12)}(\boldsymbol{H},\boldsymbol{\Psi},\boldsymbol{u})\right]\mathrm{d}t + \bar{\bar{\sigma}}_{ie}^{(1)}(\boldsymbol{H},\boldsymbol{\Psi})\mathrm{d}B_e(t),$$
$$\mathrm{d}\Psi_s = \left[\bar{\bar{m}}_s^{(33)}(\boldsymbol{H},\boldsymbol{\Psi}) + \bar{\bar{m}}_s^{(31)}(\boldsymbol{H},\boldsymbol{\Psi},\boldsymbol{u})\right]\mathrm{d}t + \bar{\bar{\sigma}}_{se}^{(3)}(\boldsymbol{H},\boldsymbol{\Psi})\mathrm{d}B_e(t),$$
$$s = 1, 2, ..., \eta; \; i, r = 1, 2, ..., n, 1 \le \eta \le m; \; e = 1, 2, ..., l, \tag{6}$$

where

$$\boldsymbol{H} = [H_1, H_2, ..., H_n]^{\mathrm{T}},$$
$$\bar{\bar{m}}_i^{(11)} = \varepsilon\left\{\bar{m}_i^{(1)} g_i(A_i)(1+h_i) + \frac{1}{2}\frac{\mathrm{d}[g_i(A_i)(1+h_i)]}{\mathrm{d}A_i}\bar{\sigma}_{ie}^{(1)}\bar{\sigma}_{ie}^{(1)}\right\}_{A_i=U_i^{-1}(H_i)-B_i},$$
$$\bar{\bar{m}}_i^{(12)} = \varepsilon\left\{\left\langle F_i^{(12)}\right\rangle_t g_i(A_i)(1+h_i)\right\}_{A_i=U_i^{-1}(H_i)-B_i} = \varepsilon\left\langle u_i P_i\right\rangle_t,$$
$$\bar{\bar{m}}_s^{(33)} = \varepsilon \bar{m}_s^{(3)}\Big|_{A_i=U_i^{-1}(H_i)-B_i},$$
$$\bar{\bar{m}}_s^{(31)} = \varepsilon\left\{\left\langle L_r^s F_r^{(21)}\right\rangle_t\right\}_{A_i=U_i^{-1}(H_i)-B_i},$$
$$\bar{\bar{b}}_{ij}^{(1)} = \varepsilon\bar{\bar{\sigma}}_{ie}^{(1)}\bar{\bar{\sigma}}_{je}^{(1)} = \varepsilon\left[g_i(A_i)g_j(A_i)(1+h_i)(1+h_j)\bar{\sigma}_{ie}^{(1)}\bar{\sigma}_{je}^{(1)}\right]_{A_i=U_i^{-1}(H_i)-B_i},$$
$$\bar{\bar{b}}_{s_1s_2}^{(3)} = \bar{\bar{b}}_{s_2s_1}^{(3)} = \varepsilon\bar{\bar{\sigma}}_{s_1e}^{(3)}\bar{\bar{\sigma}}_{s_2e}^{(3)} = \varepsilon\left[\bar{\sigma}_{s_1e}^{(3)}\bar{\sigma}_{s_2e}^{(3)}\right]_{A_i=U_i^{-1}(H_i)-B_i},$$
$$\bar{\bar{b}}_{is}^{(0)} = \bar{\bar{b}}_{si}^{(0)} = \varepsilon\bar{\bar{\sigma}}_{ie}^{(1)}\bar{\bar{\sigma}}_{se}^{(3)} = \varepsilon\left[\bar{\sigma}_{ie}^{(1)}\bar{\sigma}_{se}^{(3)} g_i(A_i)(1+h_i)\right]_{A_i=U_i^{-1}(H_i)-B_i}, \tag{7}$$
$$i, j = 1, 2, ..., n; \; s, s_1, s_2 = 1, 2, ..., \eta; \; e = 1, 2, ..., l.$$

NONLINEAR STOCHASTIC OPTIMAL CONTROL

For most mechanical and structural dynamical systems, Hamiltonian H represents the total energy of the system while H_i represents the energy of the ith degree-of-freedom of the system. Each H_i may vary in some sub-interval of $[0,\infty)$. For the partially averaged System Eq.(5), the safety domain Ω_s is bounded with boundaries Γ_0 (at least one of H_i vanishes) and critical boundary Γ_c. It is reasonable to assume that the first passage failure occurs when $\boldsymbol{H}(t)$ exceeds certain critical boundary Γ_c for the first time. Let $J(\boldsymbol{u})$ denotes the reliability function of System Eq.(5), which is defined as the probability of process $[\boldsymbol{H}^{\mathrm{T}}(t), \boldsymbol{\Psi}^{\mathrm{T}}(t)]^{\mathrm{T}}$ being in the safety domain Ω_s within the time interval $[0,t_f)$, i.e.,

$$J(\boldsymbol{u}) = \mathrm{Prob}\left\{\left(\boldsymbol{H}(\tau,\boldsymbol{u}), \boldsymbol{\Psi}(\tau,\boldsymbol{u})\right) \in \Omega_s, \; \tau \in [t, t_f]\right\}. \tag{8}$$

For the control problem of reliability maximization of System Eq.(5), introduce the value function

$$V(t, \boldsymbol{H}, \boldsymbol{\Psi}) = \sup_{\boldsymbol{u}\in U} J(\boldsymbol{u}). \tag{9}$$

Based on the stochastic dynamical programming principle, one can derive the following dynamical programming equation:

$$\frac{\partial V}{\partial t} = \sum_{i=1}^{n}\left[\bar{\bar{m}}_i^{(11)} + \bar{\bar{m}}_i^{(12)}\right]\frac{\partial V}{\partial H_i} + \sum_{s=1}^{\eta}\left[\bar{\bar{m}}_s^{(33)} + \bar{\bar{m}}_s^{(31)}\right]\frac{\partial V}{\partial \Psi_s} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\bar{\bar{b}}_{ij}^{(1)}\frac{\partial^2 V}{\partial H_i \partial H_j} + \sum_{i=1}^{n}\sum_{s=1}^{\eta}\bar{\bar{b}}_{is}^{(0)}\frac{\partial^2 V}{\partial H_i \partial \Psi_s} + \frac{1}{2}\sum_{s_1=1}^{\eta}\sum_{s_2=1}^{\eta}\bar{\bar{b}}_{s_1s_2}^{(3)}\frac{\partial^2 V}{\partial \Psi_{s_1}\partial \Psi_{s_2}}. \tag{10}$$

The boundary conditions associated with Eq.(10) are

$$V(t, \boldsymbol{H}, \boldsymbol{\Psi}) = 0 \text{ at } \Gamma_c, \tag{11}$$

$$V(t,\boldsymbol{H},\boldsymbol{\Psi}) = \text{finite at } \Gamma_0. \quad (12)$$

Since $V(t,\boldsymbol{H},\boldsymbol{\Psi})$ is a periodic function with respect to $\boldsymbol{\Psi}$, the boundary condition of Eq.(9) with respect to $\boldsymbol{\Psi}$ is

$$V(t,\boldsymbol{H},2\pi+\Psi_1,2\pi+\Psi_2,...,2\pi+\Psi_\eta)=V(t,\boldsymbol{H},\boldsymbol{\Psi}),\quad \boldsymbol{H}<H_c. \quad (13)$$

The final time condition is

$$V(t_f,\boldsymbol{H},\boldsymbol{\Psi})=1,\ \boldsymbol{H}<H_c. \quad (14)$$

Eqs.(10)~(14) are the mathematical formulations for the problem of feedback maximization of reliability of System Eq.(5).

The optimal control law can be determined from maximizing the right-hand of Eq.(10) with respect to $\boldsymbol{u}\in U$. Since the $V(\boldsymbol{H},\boldsymbol{\Psi})$ is the periodic functions of $\boldsymbol{\Psi}$, the right-hand of Eq.(10) will be maximized when the term $\bar{\bar{m}}_i^{(12)}\frac{\partial V}{\partial H_i}$ (no summation with respect to i) reaches maximum. Suppose that control constraints are of the form

$$-u_{i0}\le u_i\le u_{i0},\quad i=1,2,...,n, \quad (15)$$

where u_{i0} are position constants. The optimal control forces can be determined when $|u_i|=u_{i0}$ and $\left[u_iP_i\frac{\partial V}{\partial H_i}\right]$ is positive, i.e.,

$$u_i^{\text{opt}}=u_{i0}\,\text{sgn}\left[P_i\frac{\partial V}{\partial H_i}\right]. \quad (16)$$

Since the reliability function is the monotonously decreasing functions of H_i, i.e., $\frac{\partial V}{\partial H_i}<0$, Eq.(16) is reduced as

$$u_i^{\text{opt}}=-u_{i0}\,\text{sgn}(P_i). \quad (17)$$

Inserting u_i^{opt} into Eq.(10) to replace u_i and complete averaging, one can obtain the final dynamical programming equation for the control problem of reliability maximization as follows:

$$\begin{aligned}\frac{\partial V}{\partial t}=&\sum_{i=1}^{n}\bar{\bar{m}}_i^{(1)}\frac{\partial V}{\partial H_i}+\sum_{s=1}^{\eta}\bar{\bar{m}}_s^{(3)}\frac{\partial V}{\partial \Psi_s}\\&+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\bar{\bar{b}}_{ij}^{(1)}\frac{\partial^2 V}{\partial H_i\partial H_j}+\sum_{i=1}^{n}\sum_{s=1}^{\eta}\bar{\bar{b}}_{is}^{(0)}\frac{\partial^2 V}{\partial H_i\partial \Psi_s}\\&+\frac{1}{2}\sum_{s_1=1}^{\eta}\sum_{s_2=1}^{\eta}\bar{\bar{b}}_{s_1s_2}^{(3)}\frac{\partial^2 V}{\partial \Psi_{s_1}\partial \Psi_{s_2}},\end{aligned} \quad (18)$$

where

$$\begin{aligned}\bar{\bar{m}}_i^{(1)}&=\bar{\bar{m}}_i^{(11)}(\boldsymbol{H},\boldsymbol{\Psi})+\bar{\bar{m}}_i^{(12)}(\boldsymbol{H},\boldsymbol{\Psi},u_i^{\text{opt}}),\\ \bar{\bar{m}}_i^{(3)}&=\bar{\bar{m}}_i^{(33)}(\boldsymbol{H},\boldsymbol{\Psi})+\bar{\bar{m}}_i^{(31)}(\boldsymbol{H},\boldsymbol{\Psi},u_i^{\text{opt}}).\end{aligned} \quad (19)$$

The boundary and final time conditions are still those in Eqs.(11)~(14). The reliability function of optimally controlled system can be obtained by solving the final dynamical programming Eq.(18) with boundary conditions Eqs.(11)~(13) and final time condition Eq.(14).

EXAMPLE

Consider the feedback maximization of reliability of two nonlinearly coupled Duffing oscillators subjected to external harmonic excitations and both external and parametric excitations of Gaussian white noises. The equations of motion are of the form

$$\begin{aligned}&\ddot{X}_1+(d_{11}+d_{12}X_1^2+d_{13}X_2^2)\dot{X}_1+\omega_{10}^2X_1+\alpha_1X_1^3\\&\quad=E_1\cos(\Omega t)+u_1+\xi_1(t)+X_1\xi_2(t),\\&\ddot{X}_2+(d_{21}+d_{22}X_1^2+d_{23}X_2^2)\dot{X}_2+\omega_{20}^2X_2+\alpha_2X_2^3\\&\quad=E_2\cos(\Omega t)+u_2+\xi_3(t)+X_2\xi_4(t),\end{aligned} \quad (20)$$

where d_{ij} (i=1, 2; j=1, 2, 3) are coefficients of linear and nonlinear dampings; ω_{i0}, α_i, E_i, Ω and u_i (i=1, 2) are positive constants denoting the natural frequency of associated degenerate linear oscillator, intensity of nonlinearity, small damping coefficient, amplitude, frequency of harmonic excitation of oscillators and bounded control, respectively; $\xi_k(t)$ (k=1, 2, 3, 4) are independent Gaussian white noises with intensities $2D_k$; ω_{i0}, α_i, E_i, D_k and u_i are assumed to be of the same order of ε.

Let $X_1=Q_1$, $\dot{X}_1=P_1$, $X_2=Q_2$, $\dot{X}_2=P_2$, Eq.(20) can be rewritten as equations in the form of Eq.(1). The

Hamiltonian associated with Eq.(20) is

$$\begin{aligned} &H = H_1 + H_2, \\ &H_i = \frac{1}{2}p_i^2 + \frac{1}{2}\omega_{i0}^2 q_i^2 + \frac{1}{4}\alpha_i q_i^4 = \frac{1}{2}p_i^2 + U_i(q_i). \end{aligned} \tag{21}$$

In this example, the case of primary external resonance between the harmonic excitation and the first oscillator of System Eq.(20) and no internal resonance between the two oscillators is considered. The partial averaged Itô equations for H_1, H_2 and Ψ_1 are of the following forms:

$$\begin{aligned} &\mathrm{d}H_1 = \left[\bar{\bar{m}}_1^{(11)} + \bar{\bar{m}}_1^{(12)}\right]\mathrm{d}t + \bar{\bar{\sigma}}_{1i}^{(1)}\mathrm{d}B_i(t), \\ &\mathrm{d}H_2 = \left[\bar{\bar{m}}_2^{(11)} + \bar{\bar{m}}_2^{(12)}\right]\mathrm{d}t + \bar{\bar{\sigma}}_{2i+2}^{(1)}\mathrm{d}B_{i+2}(t), \\ &\mathrm{d}\Psi_1 = \left[\bar{\bar{m}}_1^{(33)} + \bar{\bar{m}}_1^{(31)}\right]\mathrm{d}t + \bar{\bar{\sigma}}_{1i}^{(3)}\mathrm{d}B_i(t),\ i = 1, 2, \end{aligned} \tag{22}$$

where

$$\begin{aligned} &\bar{\bar{m}}_i^{(11)} = \left[g(A_i)\bar{m}_i^{(1)} + \frac{1}{2}\frac{\mathrm{d}g(A_i)}{\mathrm{d}A_i}\bar{b}_{ii}^{(1)}\right]_{A_i=\sqrt{(\sqrt{4\alpha_i H_i+\omega_{i0}^4}-\omega_{i0}^2)/\alpha_i}}, \\ &\bar{\bar{m}}_i^{(12)} = \left[g(A_i)\left\langle F_i^{(12)}\right\rangle_t\right]_{A_i=\sqrt{(\sqrt{4\alpha_i H_i+\omega_{i0}^4}-\omega_{i0}^2)/\alpha_i}} = \left\langle u_i P_i\right\rangle_t, \\ &\bar{\bar{m}}_1^{(33)} = \left[\bar{m}_1^{(3)}\right]_{A_i=\sqrt{(\sqrt{4\alpha_i H_i+\omega_{i0}^4}-\omega_{i0}^2)/\alpha_i}}, \\ &\bar{\bar{m}}_1^{(31)} = -\left[\left\langle F_1^{(21)}\right\rangle_t\right]_{A_i=\sqrt{(\sqrt{4\alpha_i H_i+\omega_{i0}^4}-\omega_{i0}^2)/\alpha_i}}, \\ &\bar{\bar{b}}_{ii}^{(1)} = \left[g^2(A_i)\bar{b}_{ii}^{(1)}\right]_{A_i=\sqrt{(\sqrt{4\alpha_i H_i+\omega_{i0}^4}-\omega_{i0}^2)/\alpha_i}}, \\ &\bar{\bar{b}}_{11}^{(3)} = \left[\bar{b}_{11}^{(3)}\right]_{A_i=\sqrt{(\sqrt{4\alpha_i H_i+\omega_{i0}^4}-\omega_{i0}^2)/\alpha_i}}, \\ &\bar{\bar{b}}_{1i}^{(0)} = \bar{\bar{b}}_{i1}^{(0)} = \bar{\bar{b}}_{12}^{(1)} = \bar{\bar{b}}_{21}^{(1)} = 0, \end{aligned} \tag{23}$$

where $\bar{m}_i^{(1)}$, $\bar{m}_1^{(3)}$, $\bar{b}_{ii}^{(1)}$ are given in Eq.(A1) of Appendix.

As shown in Eq.(21), the energies of two sub-systems vary from 0 to ∞. Suppose that the limit state of the system is $H=H_1+H_2=H_c$, i.e.,

$$\Gamma_c: H_1 + H_2 = H_c,\ \ H_1, H_2 \geq 0. \tag{24}$$

In the space of (H_1, H_2, Ψ_1), the safety domain of System Eq.(22) is inside of a region with boundaries consisting of absorbing boundary Γ_c and reflecting boundary Γ_0, where Γ_0 consists of Γ_{10} and Γ_{20} (Fig.1), i.e.,

$$\begin{aligned} &\Gamma_{10}: H_1 = 0,\ \ 0 \leq H_2 < H_c, \\ &\Gamma_{20}: H_2 = 0,\ \ 0 < H_1 < H_c. \end{aligned} \tag{25}$$

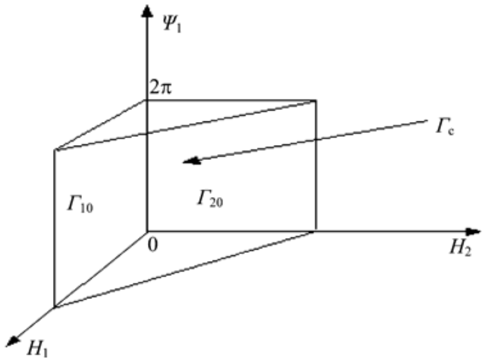

Fig.1 Safety domain of System Eq.(22). Γ_{10} and Γ_{20} are the reflecting boundaries while Γ_c is the absorbing boundary

The dynamical programming equation for control problem of reliability maximization is of the same as Eq.(10) with drift and diffusion coefficients defined by Eq.(23). Suppose that control constrains are of the same form as those in Eq.(15). The optimal control strategy is of the same form as Eq.(17) and the final dynamical programming equation is of the form of Eq.(18) with $\bar{\bar{m}}_i^{(1)} = \bar{\bar{m}}_i^{(11)} + \bar{\bar{m}}_i^{(12)}$ and $\bar{\bar{m}}_1^{(3)} = \bar{\bar{m}}_1^{(33)} + \bar{\bar{m}}_1^{(31)}$, where

$$\begin{aligned} &\bar{\bar{m}}_i^{(12)} = \left[2u_{i0}\left(-C_0^i + \frac{1}{3}C_2^i + \frac{1}{15}C_4^i + \frac{1}{35}C_6^i\right)A_i\right]_{A_i=\sqrt{(\sqrt{4\alpha_i H_i+\omega_{i0}^4}-\omega_{i0}^2)/\alpha_i}}, \\ &\bar{\bar{m}}_1^{(31)} = 0,\quad i = 1, 2, \end{aligned} \tag{26}$$

where C_0^i, C_2^i, C_4^i and C_6^i are given in Eq.(A2) of Appendix. The associated boundary condition on Γ_c, periodic boundary condition with respect to Ψ_1 and final condition are defined by Eqs.(11), (13) and (14), respectively. Note that the boundary conditions on Γ_0 defined by Eq.(12) are qualitative and should be replaced by quantitative ones for solving Eq.(18) numerically. Since

at Γ_{10},

$$\begin{aligned} &\bar{\bar{m}}_1^{(1)} \to D_1,\ \ \bar{\bar{m}}_2^{(1)} \to \text{finite},\ \ \bar{\bar{m}}_1^{(3)} \to \infty, \\ &\bar{\bar{b}}_{11}^{(1)} \to 0,\ \bar{\bar{b}}_{22}^{(1)} \to \text{finite and } \bar{\bar{b}}_{11}^{(3)} \to \infty; \end{aligned} \tag{27}$$

at Γ_{20},

$$\bar{\bar{m}}_1^{(1)} \to \text{finite},\ \bar{\bar{m}}_2^{(1)} \to D_3,\ \bar{\bar{m}}_1^{(3)} \to \text{finite},$$
$$\bar{\bar{b}}_{11}^{(1)} \to \text{finite},\ \bar{\bar{b}}_{22}^{(1)} \to 0 \text{ and } \bar{\bar{b}}_{11}^{(3)} \to \text{finite}, \quad (28)$$

the quantitative boundary conditions at Γ_0 can be derived from Eq.(18) as follows:

at Γ_{10},

$$\frac{\partial V}{\partial \Psi_1} = \frac{\partial^2 V}{\partial \Psi_1^2} = 0,$$
$$\frac{\partial V}{\partial t} = D_1 \frac{\partial V}{\partial H_1} + \bar{\bar{m}}_2^{(1)} \frac{\partial V}{\partial H_2} + \frac{1}{2}\bar{\bar{b}}_{22}^{(1)} \frac{\partial^2 V}{\partial H_2^2}; \quad (29)$$

at Γ_{20},

$$\frac{\partial V}{\partial t} = \bar{\bar{m}}_1^{(1)} \frac{\partial V}{\partial H_1} + D_3 \frac{\partial V}{\partial H_2} + \bar{\bar{m}}_1^{(3)} \frac{\partial V}{\partial \Psi_1} + \frac{1}{2}\bar{\bar{b}}_{11}^{(1)} \frac{\partial^2 V}{\partial H_1^2} + \frac{1}{2}\bar{\bar{b}}_{11}^{(3)} \frac{\partial^2 V}{\partial \Psi_1^2}. \quad (30)$$

The numerical result for the reliability function of uncontrolled and optimally controlled system is shown in Fig.2. It denotes that the theoretical results and those from simulation are in good agreement, the optimal control indeed improves the reliability of system and larger control forces lead to more reliable system.

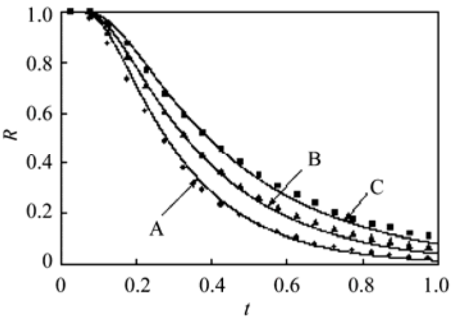

Fig.2 Reliability function of uncontrolled and optimally controlled system Eq.(20). $u_{10}=u_{20}=0$ **for A;** $u_{10}=0.15$, $u_{20}=0$ **for B;** $u_{10}=0.3$, $u_{20}=0$ **for C.** $d_{11}=d_{21}=-0.01$, $d_{12}=d_{13}=d_{22}=d_{23}=0.01$, $\alpha_{11}=\alpha_{21}=-0.01$, $\omega_{10}=1.0$, $\omega_{21}=(2.0)^{1/2}$, $\Omega=1.2$, $E_1=E_2=0.2$, $H_c=0.01$, $D_1=D_2=0.01$. **Solid lines denote the analytical results; ●, ▲, ■ denote the results from Monte Carlo simulation of original system**

CONCLUSION

In this paper a procedure for designing the optimally bounded control of quasi integrable-Hamiltonian systems under combined harmonic and white noise excitations for maximizing the reliability has been proposed. The procedure consists of applying the stochastic averaging method for such systems, establishing the dynamical programming equation from the partially averaged equations using dynamical programming principle, determining the optimal control from the dynamical programming equation and control constraints, and obtaining the reliability function of the optimally controlled system by solving the final dynamical programming equation. The numerical result obtained for the controlled two coupled Duffing oscillators has shown that the optimal control really improves the reliability of original system.

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APPENDIX

$$\begin{aligned}\bar{m}_1^{(1)} =& -\frac{\alpha_1 A_1^3(10d_{11}+3d_{12}A_1^2+5d_{13}A_2^2)}{32(\alpha_1 A_1^2+\omega_{10}^2)} -\frac{\omega_{01}^2 A_1(4d_{11}+d_{12}A_1^2+2d_{13}A_2^2)}{8(\alpha_1 A_1^2+\omega_{10}^2)} \\ &+\frac{E_1(2C_0^1-C_2^1)\times\sin\Psi_1}{4(\alpha_1 A_1^2+\omega_{10}^2)} -\frac{\alpha_1 D_1 A_1(3\omega_{10}^2+1.5\alpha_1 A_1^2)}{4(\omega_{10}^2+\alpha_1 A_1^2)^3} +\frac{D_1(\omega_{10}^2+7\alpha_1 A_1^2/8)}{2(\omega_{10}^2+\alpha_1 A_1^2)^2 A_1} \\ &+\frac{D_2\omega_{10}^2 A_1(\omega_{10}^2+0.5\alpha_1 A_1^2)}{8(\omega_{10}^2+\alpha_1 A_1^2)^3} +\frac{D_2 A_1(\omega_{10}^2+7\alpha_1 A_1^2/8)}{4(\omega_{10}^2+\alpha_1 A_1^2)^2},\\ \bar{m}_2^{(1)} =& -\frac{\alpha_2 A_2^3(10d_{21}+3d_{22}A_1^2+5d_{23}A_2^2)}{32(\alpha_2 A_2^2+\omega_{20}^2)} -\frac{\omega_{20}^2 A_2(4d_{21}+d_{22}A_1^2+2d_{23}A_2^2)}{8(\alpha_2 A_2^2+\omega_{20}^2)} -\frac{\alpha_2 D_3 A_2(3\omega_{20}^2+1.5\alpha_2 A_2^2)}{4(\omega_{20}^2+\alpha_2 A_2^2)^3} \\ &+\frac{D_3(\omega_{20}^2+7\alpha_2 A_2^2/8)}{2(\omega_{20}^2+\alpha_2 A_2^2)^2 A_2} +\frac{D_4\omega_{20}^2 A_2(\omega_{20}^2+\alpha_2 A_2^2/2)}{8(\omega_{20}^2+\alpha_2 A_2^2)^3} +\frac{D_4 A_2(\omega_{20}^2+7\alpha_2 A_2^2/8)}{4(\omega_{20}^2+\alpha_2 A_2^2)^2},\\ \bar{m}_1^{(3)} =& \Omega - C_0^1 + E_1(2C_0^1+C_2^1)\times\cos\Psi\left[4A_1(\omega_{10}^2+\alpha_1 A_1^2)\right],\\ \bar{b}_{11}^{(1)} =& \frac{D_1(\omega_{10}^2+5\alpha_1 A_1^2/8)}{(\omega_{10}^2+\alpha_1 A_1^2)^2}+\frac{D_2 A_1^2(\omega_{10}^2+0.75\alpha_1 A_1^2)}{4(\omega_{10}^2+\alpha_1 A_1^2)^2},\\ \bar{b}_{22}^{(1)} =& \frac{D_3(\omega_{20}^2+5\alpha_2 A_2^2/8)}{(\omega_{20}^2+\alpha_2 A_2^2)^2}+\frac{D_4 A_2^2(\omega_{20}^2+3\alpha_2 A_2^2/4)}{4(\omega_{20}^2+\alpha_2 A_2^2)^2},\\ \bar{b}_{11}^{(3)} =& \frac{4D_1(\omega_{10}^2+7\alpha_1 A_1^2/8)}{A_1^2(\omega_{10}^2+\alpha_1 A_1^2)^2}+\frac{D_2(3\omega_{10}^2+11\alpha_1 A_1^2/4)}{(\omega_{10}^2+\alpha_1 A_1^2)^2}, \end{aligned} \tag{A1}$$

where

$$\begin{aligned} C_0^i &= (\omega_{i0}^2+3\alpha_i A_i^2/4)^{1/2}(1-\varpi_i^2/16),\\ C_2^i &= (\omega_{i0}^2+3\alpha_i A_i^2/4)^{1/2}(\varpi_i/2+3\varpi_i^3/64),\\ C_4^i &= (\omega_{i0}^2+3\alpha_i A_i^2/4)^{1/2}(-\varpi_i^2/16),\\ C_6^i &= (\omega_{i0}^2+3\alpha_i A_i^2/4)^{1/2}(3\varpi_i^3/64),\\ \varpi_i &= \alpha_i A_i^2/(4\omega_{i0}^2+3\alpha_i A_i^2),\ i=1,2. \end{aligned} \tag{A2}$$