Journal of Zhejiang University SCIENCE A
ISSN 1673-565X (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Solution of nonlinear cubic-quintic Duffing oscillators using He's Energy Balance Method

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Received Sept. 8, 2008; Revision accepted Oct. 27, 2008; Crosschecked July 21, 2009

Abstract: In this study, He's Energy Balance Method (EBM) was applied to solve strong nonlinear Duffing oscillators with cubic-quintic nonlinear restoring force. The complete EBM solution procedure of the cubic-quintic Duffing oscillator equation is presented. For illustration of effectiveness and convenience of the EBM, different cases of cubic-quintic Duffing oscillator with different parameters of α, β and γ were compared with the exact solution. We found that the solutions were valid for small as well as large amplitudes of oscillation. The results show that the EBM is very convenient and precise, so it can be widely applicable in engineering and other sciences.

Key words: Energy Balance Method (EBM), Cubic-quintic Duffing equation, Oscillator
doi:10.1631/jzus.A0820651 **Document code:** A **CLC number:** O34

INTRODUCTION

Basically in different fields of science, there are few issues occurring linearly, whereas a great number of problems result in the nonlinear systems. One of the nonlinear cases may be found in the oscillatory problems because of some nonlinear assumptions for the devices used to construct the corresponding system. The first flash that passes through one's mind is to solve them using traditional analytical methods; however, if someone has got previously involved in using those methods, this person will probably find that those methods are quite useless because of the existence of some nonlinear terms. In this view, some other approximate methods have been proposed by different authors to overcome the corresponding casualties encountered in the traditional analytical methods. These methods include the perturbation technique (Nayfeh and Mook, 1995; Ganji and Rafei, 2006; Ganji and Rajabi, 2006; Ganji and Sadighi, 2006; Gorji *et al*., 2007), the harmonic balance method (Itovich and Moiola, 2006; Gottlieb, 2006; Hu, 2006; Hu and Tang, 2006; Chen and Lui, 2007), the Lindstedt-Poincaré method (He, 2001; 2002a), non-perturbative methods (He, 2006b), the parameter-expansion method (He, 2006c; Wang and He, 2008) and the parameterized perturbation method (He, 1999a; 2000). Recently, some other approximately variational methods, including the approximate energy method (He, 2002b; 2006a; D′Acunto, 2006), the variational iteration method (He, 1999b; 2006b; Ganji *et al*., 2007; Rafei *et al*., 2007; Varedi *et al*., 2007) and variational approach (He, 2007; Wu, 2007; Bormashenko and Whyman, 2008; Xu, 2008), have been used to solve nonlinear governing equations. These methods give successive approximations of high accuracy of the solution. The Energy Balance Method (EBM) is one of the well-known methods to solve the nonlinear equations. This method was established by He (2002b; 2003; 2006a) and has been used in many researches (Stoker, 1992; Mickens, 1996; Jordan and Smith, 1999; Lim and Lai, 2006; Ozis and Yıldırım, 2007; Ganji *et al*., 2008; Lai *et al*., 2008; Pashaei *et al*., 2008).

The aim of this paper was to determine the periodic solutions to the nonlinear oscillators by applying the EBM. By comparing the results with its exact solution, we illustrated the high accuracy of this method.

HE'S ENERGY BALANCE METHOD

In He's EBM, a variational principle for the nonlinear oscillation is established and then a Hamiltonian is constructed, from which the angular frequency can be readily obtained by the collocation method. This method will be useful for differential equations with strong nonlinearity. To show this case, we consider a general nonlinear oscillator as follows:

$$X'' + f(X(t)) = 0, \tag{1}$$

where X and t are generalized dimensionless displacement and time variables, respectively. The initial conditions are:

$$X(0) = X_0,\ X'(0) = 0, \tag{2}$$

where the derivative operator denotes the derivation with respect to time, t. Its variational principle can be easily obtained:

$$J(X) = \int_0^t \left(-X'^2/2 + F(X)\right)\mathrm{d}t, \tag{3}$$

where $T=2\pi/\omega$ is the period of the nonlinear oscillator, $F(X) = \int f(X)\mathrm{d}X$.

Its Hamiltonian, therefore, can be written as

$$H = X'^2/2 + F(X) - F(X_0), \tag{4}$$

or
$$R(t) = X'^2/2 + F(X) - F(X_0) = 0. \tag{5}$$

Oscillatory systems contain two important physical parameters, i.e., the frequency ω and the amplitude of oscillation X_0. Assume that its initial approximate guess can be expressed as

$$X(t) = X_0\cos(\omega t). \tag{6}$$

Substituting Eq.(6) into Eq.(5), yields

$$\begin{aligned} R(t) = {} & \omega^2 X_0^2 \sin^2(\omega t)/2 \\ & + F(X_0\cos(\omega t)) - F(X_0) = 0. \end{aligned} \tag{7}$$

If, by chance, the exact solution had been chosen as the trial function, then it would be possible to make $H=0$ for all values of t by an appropriate choice of ω. Collocation at $\omega t=\pi/4$ gives:

$$\omega = \sqrt{\frac{2(F(X_0) - F(X_0\cos(\omega t)))}{X_0^2\sin^2(\omega t)}}. \tag{8}$$

Its period can be written as

$$T = 2\pi\sqrt{\frac{X_0^2\sin^2(\omega t)}{2(F(X_0) - F(X_0\cos(\omega t)))}}. \tag{9}$$

GENERAL DEFINITION OF CUBIC-QUINTIC DUFFING OSCILLATORS

A cubic-quintic Duffing oscillator has the general form of

$$x'' + f(x) = 0, \tag{10}$$

with initial conditions of

$$x(0) = X_0,\ x'(0) = 0, \tag{11}$$

where $f(x)=\alpha x+\beta x^3+\gamma x^5$. x and t are generalized dimensionless displacement and time variable, respectively, and x is the function of t.

The exact frequency ω_e by imposing the initial conditions is (Lai *et al.*, 2008)

$$\omega_e(X_0) = \frac{\pi k_1}{2\int_0^{\pi/2}(1+k_2\sin^2 t + k_3\sin^4 t)^{-1/2}\mathrm{d}t}, \tag{12a}$$

$$k_1 = \sqrt{\alpha + \frac{\beta X_0^2}{2} + \frac{\gamma X_0^4}{4}}, \tag{12b}$$

$$k_2 = \frac{3\beta X_0^2 + 2\gamma X_0^4}{6\alpha + 3\beta X_0^2 + 2\gamma X_0^4}, \tag{12c}$$

$$k_3 = \frac{2\gamma X_0^4}{6\alpha + 3\beta X_0^2 + 2\gamma X_0^4}. \tag{12d}$$

APPLYING EBM ON CUBIC-QUINTIC DUFFING OSCILLATORS

The complete solution procedure of cubic-quintic equation (Eq.(10)) with $\alpha=\beta=\gamma=1$ is presented via the Energy Balance Method. Furthermore, the comparison between the Energy Balance solution and exact solution together with the corresponding results of angular frequencies are shown in Figs.1a~1d and Tables 1~4. These figures and tables correspond to small as well as large amplitudes of oscillation for different parameters of α, β and γ, which were obtained via the same procedure as described in the following example.

Example Assuming $\alpha=\beta=\gamma=1$, one obtains (Eq.(10)):

$$x'' + x + x^3 + x^5 = 0. \tag{13}$$

If $X_0=3$, the initial conditions for this equation are (Eq.(11)):

$$x(0) = 3, x'(0) = 0. \tag{14}$$

The first step to solve Eq.(13) is to obtain the Hamiltonian. So, we multiply the value of x' in Eq.(13):

$$x'x'' + x'x + x'x^3 + x'x^5 = 0. \tag{15}$$

By integrating Eq.(15), we can readily obtain Hamiltonian formulation as follows:

$$H = x^6/6 + x^4/4 + x^2/2 + x'^2/2 - 146.25, \tag{16}$$

where $F(X_0)=146.25$. We use the following trial function to determine the angular frequency:

$$x = 3\cos(\omega t). \tag{17}$$

Substituting Eq.(17) into Eq.(16) and make a simplification, we obtain the following residual equation:

$$R(t) = 243\cos^6(\omega t)/2 + 81\cos^4(\omega t)/4 + 9\cos^2(\omega t)/2 + 9\sin^2(\omega t)\omega^2/2 - 146.25. \tag{18}$$

In this method according to basic idea of the EBM, if $X=0$, it shows that the whole energy is in form of kinetic energy; if $X=\pi/2$, it shows that the whole energy is in form of potential energy; if $X=\pi/4$, there is a balance between the potential energy and kinetic energy. So we can benefit from this point.

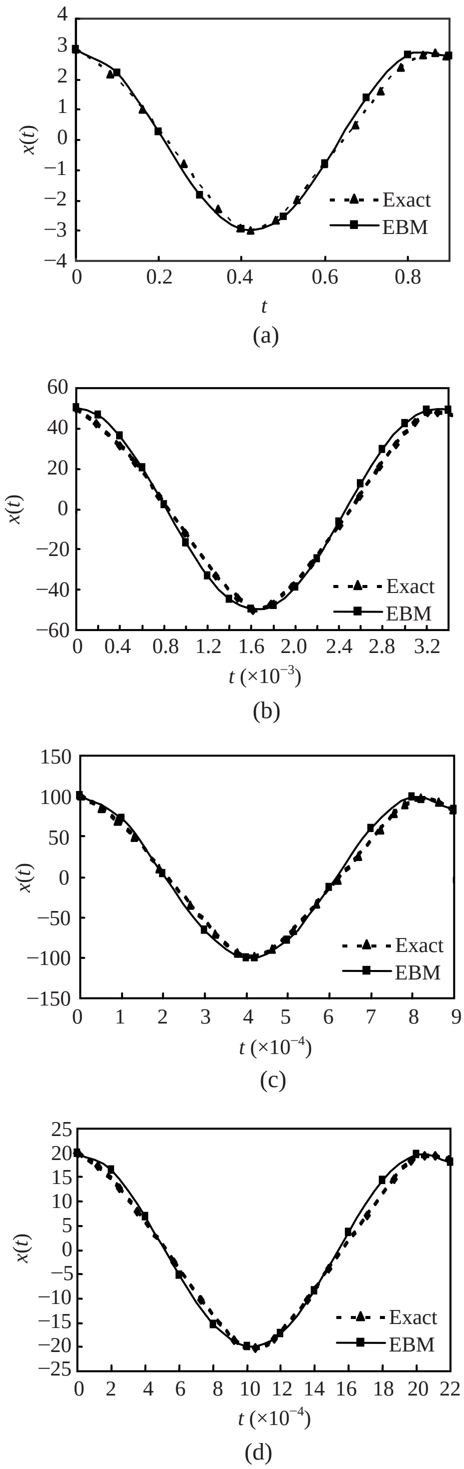

Fig.1 Comparison of the EBM solution with the exact solution for the cubic-quintic Duffing oscillator. (a) $\alpha=\beta=\gamma=1$ and $X_0=3$; (b) $\alpha=2$, $\beta=\gamma=1$ and $X_0=50$; (c) $\alpha=5$, $\beta=3$, $\gamma=1$ and $X_0=100$; (d) $\alpha=1$, $\beta=10$, $\gamma=100$ and $X_0=20$

Table 1 Percentage errors for comparison of EBM frequencies ω_{EBM} with exact frequency ω_e when $\alpha=\beta=\gamma=1$

X_0	ω_e	ω_{EBM}	Error (%)
0.1	1.003 770 000	1.003 773 057	0.03×10^{-4}
0.3	1.035 540 000	1.035 492 669	0.04×10^{-1}
0.5	1.106 540 000	1.106 356 497	0.10
1	1.523 590 000	1.527 720 966	0.27
3	7.268 630 000	7.417 768 993	2.05
5	19.181 500 0	19.608 793 4	2.23
8	48.294 600 00	49.390 624 12	2.26
10	75.177 400 00	76.889 585 29	2.27
20	299.223 00	306.066 81	2.28
50	1867.570 000	1 910.332 216	2.29
70	3659.980 000	3743.779 487	2.29
100	7468.830 000	7639.855 092	2.29
300	67 215.570 00	68 754.767 31	2.29
500	186 709.0400	190 984.5919	2.29
700	365 949.2500	374 329.3286	2.29
1000	746 834.6900	763 936.8945	2.29

Table 2 Percentage errors for comparison of EBM frequencies ω_{EBM} with exact frequency ω_e when $\alpha=2$, $\beta=\gamma=1$

X_0	ω_e	ω_{EBM}	Error (%)
0.1	1.416 880 000	1.416 884 029	0.0028×10^{-3}
0.3	1.439 600 00	1.439 529 46	0.005
0.5	1.491 770 000	1.491 316 431	0.03
1	1.826 820 000	1.825 869 084	0.05
3	7.343 860 000	7.484 871 198	1.92
5	19.210 400 00	19.634 262 02	2.2
8	48.306 100 00	49.400 744 35	2.26
10	75.184 800 00	76.896 086 99	2.27
20	299.225 000 0	306.068 443 7	2.29
50	1867.570 000	1910.332 478	2.29
70	3659.980 00	3743.779 62	2.29
100	7468.830 000	7639.855 158	2.29
300	67 215.570 00	68754.767 31	2.29
500	186 709.0400	190 984.5919	2.29
700	365 949.2500	374 329.3286	2.29
1000	746 834.6900	763 936.8945	2.29

Table 3 Percentage errors for comparison of EBM frequencies ω_{EBM} with exact frequency ω_e when $\alpha=5$, $\beta=3$, $\gamma=1$

X_0	ω_e	ω_{EBM}	Error (%)
0.1	2.241 110 000	2.241 102 478	0.0003
0.3	2.281 930 000	2.281 946 735	0.0007
0.5	2.366 150 000	2.366 246 867	0.004
1	2.796 270 000	2.798 963 393	0.1
3	8.378 770 000	8.516 271 502	1.64
5	20.216 400 00	20.640 111 42	2.10
8	49.295 500 00	50.393 045 41	2.23
10	76.169 800 00	77.884 838 19	2.25
20	300.204 000 0	307.052 196 7	2.28
50	1868.550 000	1911.314 776	2.29
70	3 660.950 000	3744.761 781	2.29
100	7469.810 000	7640.837 246	2.29
300	67 216.540 00	68 755.749 35	2.29
500	186 710.0100	190 985.5701	2.29
700	365 950.2200	374 330.2997	2.29
1000	746 835.6600	763 937.8765	2.29

Table 4 Percentage errors for comparison of EBM frequencies ω_{EBM} with exact frequency ω_e when $\alpha=1$, $\beta=10$, $\gamma=100$

X_0	ω_e	ω_{EBM}	Error (%)
0.1	1.039 700 000	1.039 642 196	0.006
0.3	1.462 590 00	1.465 569 59	0.2
0.5	2.524 690 000	2.554 014 562	1.16
1	8.010 050 000	8.176 911 017	2.04
3	67.709 700 0	69.250 764 8	2.28
5	187.199 000 0	191.477 091 5	2.28
8	478.463 000 0	489.411 082 4	2.29
10	747.323 000 0	764.427 908 7	2.29
20	2987.830 000	3056.236 747	2.29
50	18 671.340 00	19 098.901 11	2.29
70	36 595.360 00	37 433.374 78	2.29
100	74 683.910 00	76 394.131 36	2.29
300	672 151.2700	687 543.2539	2.29
500	1 867 085.990	1 909 841.499	2.29
700	3 659 488.070	3 743 288.867	2.29
1000	7 468 342.490	7 639 364.525	2.29

If we collocate at $\omega t=\pi/4$, we obtain:

$$R(t) = -123.703\,830\,7 + 2.248\,208\,265\omega^2. \quad (19)$$

By using the Maple package, we solve Eq.(19) and obtain the frequencies of the oscillator:

$$\omega = 7.417\,768\,993. \quad (20)$$

By considering Eq.(17), we obtain:

$$x = 3\cos(7.417\,768\,993t). \quad (21)$$

We compared the results that are obtained by the EBM and exact solution in Figs.1a~1d and found that, in spite of inconvenience of this method, it is very powerful to solve nonlinear equations.

The Duffing equation is a nonlinear differential equation with many variations including the cubic-quintic Duffing equation which relates to high order of nonlinearity, and thus the method of solving the cubic-quintic Duffing equation should be very accurate. Here the EBM approach has been applied for predicting the solution leading to very accurate solution together with convenience and high relief. Different cubic-quintic oscillators with different coefficients were investigated. Furthermore, different initial conditions from very small to very high were applied.

The results were in an excellent accordance with the exact solution where the maximum error was 2.29% for the very high initial condition.

CONCLUSION

In this study, the EBM was employed to solve the nonlinear differential equations governing on Duffing oscillators. Some examples of Duffing oscillators with different parameters of α, β and γ were presented and the solution procedure is completely explained through an example. EBM is also applied for approaching frequency of the system. Comparisons with the results of exact solution were done via figures and tables. These examples showed that EBM is in excellent agreement with the corresponding exact solutions. Especially the tables show that the maximum error was 2.29% in very high amplitude. This method can be easily extended to many nonlinear oscillators, and it is accurate, fast and reliable for such problems.

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