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Perturbation spectrum method for seismic analysis of non-classically damped systems*

Wei GUO[†1], Hong-nan LI[1], Zhen GUO[2]

([1]State Key Laboratory of Coastal and Offshore Engineering, Faculty of Infrastructure Engineering, Dalian University of Technology, Dalian 116024, China)

([2]Department of Civil Engineering, Zhejiang University, Hangzhou 310027, China)

[†]E-mail: wei.guo.86@gmail.com

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Abstract: Fundamental principles from structural dynamics, random theory and perturbation methods are adopted to develop a new response spectrum combination rule for the seismic analysis of non-classically damped systems, such as structure-damper systems. The approach, which is named the perturbation spectrum method, can provide a more accurate evaluation of a non-classically damped system's mean peak response in terms of the ground response spectrum. To account for the effect of non-classical damping, all elements are included in the proposed method for seismic analysis of structure, which is usually approximated by ignoring the off-diagonal elements of the modal damping matrix. Moreover, as has been adopted in the traditional Complete Quadratic Combination (CQC) method, the white noise model is also used to simplify the expressions of perturbation correlation coefficients. Finally, numerical work is performed to examine the accuracy of the proposed method by comparing the approximate results with exact ones and to demonstrate the importance of the neglected off-diagonal elements of the modal damping matrix. In the examined cases, the proposed method shows good agreement with direct time-history integration. Also, the perturbation spectrum method leads to a more efficient and economical calculation by avoiding the integral and complex operation.

Key words: Seismic response, Non-classical damping, Perturbation technique, Response spectrum method, Forced decoupling method, Pseudo excitation

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1 Introduction

Non-classical damping usually occurs when the system consists of multiple subsystems with different damping characteristics and active or passive dampers are installed in the system. This characteristic gives rise to complex-valued mode shapes and makes the analysis more cumbersome than that of usual systems. For simplicity in engineering, the forced decoupling method which neglects the off-diagonal elements of the modal damping matrix, has been suggested (Elishakoff and Lyon, 1986). However, it may produce gross errors in some cases because of failure to properly account for the non-classical characteristic. The upper bound of the error introduced by the forced decoupling method has been also studied (Shahruz and Mahavamana, 1998). For this reason, the more accurate seismic analysis method of a non-classically damped system is of interest and arouses comprehensive concern especially in recent years following the increased adoption of supplemental damping elements in civil structures.

To account for the effect of non-classical damping, and simultaneously simplify dynamic analysis and reduce the computation times, several studies have been devoted to the evaluation of both deterministic and stochastic response of a non-classically

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damped system. For the time-history analysis of the structure with added dampers, the pseudo-force method is developed and then improved for rapid convergence (Ibrahtmbegovic and Wilson, 1989; Lin *et al*., 2003). Also, an approximate method, which accounts for the non-classical damping by constructing a corresponding classical damping, is presented for time-history analysis (Bilbao *et al*., 2006). Some researchers have focused on the frequency response analysis, and some fast analysis methods are presented (Heredia-Zavoni *et al*., 2006; Kim and Bennighof, 2006). As it is well known, the eigenvalue problem of non-classically damped systems is usually solved according to the state vector approach, but because of its complex operation several approximate methods have been presented in recent years (Lou *et al*., 2003; Khanlari and Ghafory-Ashtiany, 2005; Cortés and Elejabarrieta, 2006). However, since the time-history analysis is usually quite costly and time-consuming, as an alternative the response spectrum method is adopted for seismic design, which gives most response quantities of engineering interest. An improved response spectrum method available for the non-proportionally damped systems has been proposed (Zhou *et al*., 2004; Yu and Zhou, 2007), which needs the complex eigenvalue solution but has an expression in the real value form. A methodology based on the well-known state-space representation of dynamical systems has been adopted to analyze a number of design methods for supplemental damping systems (Antonio, 2009).

Based on the above research, this paper presents a new response spectrum combination rule for the seismic analysis of non-classically damped systems. This is referred to as the perturbation spectrum method. The proposed approach is based on the structural dynamics, random theory and perturbation methods, and the pseudo excitation method is also adopted for simplifying the derivation process (Lin, 1992). Simple formulas for perturbation coherent coefficients are also given using the same assumption as that of the traditional Complete Quadratic Combination (CQC) method. As shown in this study, the main advantage of the new method is its avoidance of the complex and integral operation, which means more computational efficiency. From a practical standpoint, the proposed method might be more easily accepted since it is an extension of the traditional CQC method for seismic analysis of non-classically damped systems. Finally, a numerical example is used to demonstrate the accuracy of the presented method.

2 Simple expression of perturbation method

The equations of motion for the non-classically damped system is written as

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{D}\dot{\boldsymbol{U}} + \boldsymbol{K}\boldsymbol{U} = -\boldsymbol{M}\boldsymbol{E}\ddot{u}_{\mathrm{g}}(t), \tag{1}$$

where $\boldsymbol{M}$, $\boldsymbol{D}$ and $\boldsymbol{K}$ are the mass, damping and stiffness matrices of the non-classically damped system with n degrees of freedom, respectively; $\boldsymbol{U}=[u_1(t), u_2(t)\cdots, u_n(t)]^{\mathrm{T}}$ is the displacement vector relative to the fixed base; $\boldsymbol{E}=[1,1,\cdots,1]^{\mathrm{T}}$ is the static displacement vector of the system when the base undergoes a unit displacement in the direction of the ground motion; $\ddot{u}_{\mathrm{g}}(t)$ is the seismic ground acceleration. The damping matrix $\boldsymbol{D}$ can be expressed in terms of two parts: classical damping matrix $\boldsymbol{C}$ and non-classical damping matrix $\varepsilon\Delta\boldsymbol{C}$, which is given as

$$\boldsymbol{D} = \boldsymbol{C} + \varepsilon\Delta\boldsymbol{C}, \tag{2}$$

where $\varepsilon\Delta\boldsymbol{C}$ represents a small perturbation of $\boldsymbol{C}$ and the system becomes classically damped when the perturbation quantity ε equals zero.

The structural displacement response can be given by

$$\boldsymbol{U} = \boldsymbol{U}^{(0)} + \varepsilon\boldsymbol{U}^{(1)} + \varepsilon^2\boldsymbol{U}^{(2)} + \varepsilon^3\boldsymbol{U}^{(3)} + \cdots. \tag{3}$$

Substituting Eqs. (2) and (3) into Eq. (1), a sequence of equations is obtained in accordance with the order of magnitude as follows:

$$\begin{aligned}
&\varepsilon^0: \boldsymbol{M}\ddot{\boldsymbol{U}}^{(0)} + \boldsymbol{C}\dot{\boldsymbol{U}}^{(0)} + \boldsymbol{K}\boldsymbol{U}^{(0)} = -\boldsymbol{M}\boldsymbol{E}\ddot{u}_{\mathrm{g}}(t),\\
&\varepsilon^1: \boldsymbol{M}\ddot{\boldsymbol{U}}^{(1)} + \boldsymbol{C}\dot{\boldsymbol{U}}^{(1)} + \boldsymbol{K}\boldsymbol{U}^{(1)} = -\Delta\boldsymbol{C}\dot{\boldsymbol{U}}^{(0)},\\
&\varepsilon^2: \boldsymbol{M}\ddot{\boldsymbol{U}}^{(2)} + \boldsymbol{C}\dot{\boldsymbol{U}}^{(2)} + \boldsymbol{K}\boldsymbol{U}^{(2)} = -\Delta\boldsymbol{C}\dot{\boldsymbol{U}}^{(1)},\\
&\cdots\\
&\varepsilon^p: \boldsymbol{M}\ddot{\boldsymbol{U}}^{(p)} + \boldsymbol{C}\dot{\boldsymbol{U}}^{(p)} + \boldsymbol{K}\boldsymbol{U}^{(p)} = -\Delta\boldsymbol{C}\dot{\boldsymbol{U}}^{(p-1)}.
\end{aligned} \tag{4}$$

The above sequence of equations can be expressed in the general form:

$$\boldsymbol{M}\ddot{\boldsymbol{U}}^{(s)}+\boldsymbol{C}\dot{\boldsymbol{U}}^{(s)}+\boldsymbol{K}\boldsymbol{U}^{(s)}=\boldsymbol{p}^{(s)}, \tag{5a}$$

$$\boldsymbol{p}^{(0)}=-\boldsymbol{M}\boldsymbol{E}\ddot{u}_{\mathrm{g}}(t), \tag{5b}$$

$$\boldsymbol{p}^{(s+1)}=-\Delta\boldsymbol{C}\dot{\boldsymbol{U}}^{(s)},\quad s=0,1,2,\cdots, \tag{5c}$$

where $\boldsymbol{U}^{(s)}=[u_1^s(t),u_2^s(t),\cdots,u_n^s(t)]^{\mathrm{T}}$, and the initial solution $\boldsymbol{U}^{(0)}$ corresponds to the solution obtained by ignoring the off-diagonal elements of modal damping matrix. Hence, the result of the forced decoupling method actually corresponds to the 0th perturbation solution.

Let $\boldsymbol{\Phi}=[\boldsymbol{\varphi}_1\boldsymbol{\varphi}_2\cdots\boldsymbol{\varphi}_n]$ and ω_i $(i=1,2,\ldots,n)$ denote the modal matrix and natural frequencies of an undamped system, respectively. In terms of the undamped mode shapes, the *s*th perturbation solution of displacement in Eq. (5) can be written as

$$\boldsymbol{U}^{(s)}=\boldsymbol{\Phi}\boldsymbol{q}^{(s)}, \tag{6}$$

where $\boldsymbol{q}^{(s)}=[q_1^{(s)},q_2^{(s)},\cdots,q_n^{(s)}]^{\mathrm{T}}$ is the vector of normal coordinate response. Substituting Eq. (6) into Eq. (5) and pre-multiplying by $\boldsymbol{\varphi}_i^{\mathrm{T}}$, Eq. (5) can be simplified as

$$\ddot{q}_i^{(s)}+2\zeta_i\omega_i\dot{q}_i^{(s)}+\omega_i^2q_i^{(s)}=\sum_{b=1}^{n}\beta_{bi}^{(s)}\dot{u}_b^{(s-1)}(t), \tag{7}$$

where $i=1,2,\ldots,n$, $\beta_{bi}^{(0)}=\dfrac{\boldsymbol{\varphi}_i^{\mathrm{T}}(-\boldsymbol{M}\boldsymbol{I}_b)}{\boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{M}\boldsymbol{\varphi}_i}$ when *s*=0; $\beta_{bi}^{(s)}=\dfrac{\boldsymbol{\varphi}_i^{\mathrm{T}}(-\Delta\boldsymbol{C}\boldsymbol{I}_b)}{\boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{M}\boldsymbol{\varphi}_i}$ when $s\neq0$; $\boldsymbol{I}_b=[0,\cdots,\underset{b}{1},\cdots,0]^{\mathrm{T}}$; $\dot{u}_b^{(s-1)}(t)$ corresponds to the *b*th element of $\dot{\boldsymbol{U}}^{(s-1)}$. To unify the form of the right hand terms of Eq. (7), let $\dot{u}_b^{(-1)}(t)=\ddot{u}_{\mathrm{g}}(t)$. Also, an oscillator of frequency ω_i, damping ratio ζ_i, is defined as

$$\ddot{w}_{bi}^{(s)}+2\zeta_i\omega_i\dot{w}_{bi}^{(s)}+\omega_i^2w_{bi}^{(s)}=\dot{u}_b^{(s-1)}(t),\quad s=0,1,2,\cdots, \tag{8}$$

where $w_{bi}^{(s)}$, $\dot{w}_{bi}^{(s)}$, $\ddot{w}_{bi}^{(s)}$ correspond to the oscillator's displacement, velocity and acceleration, respectively. Obviously, $q_i^{(s)}(t)=\sum_{b=1}^{n}\beta_{bi}^{(s)}w_{bi}^{(s)}(t)$. Based on the modal superposition the displacement vector can be expressed as

$$\boldsymbol{U}^{(s)}=\sum_{i=1}^{n}\sum_{b=1}^{n}\boldsymbol{\varphi}_i\beta_{bi}^{(s)}w_{bi}^{(s)}(t),\quad s=0,1,2,\cdots. \tag{9}$$

3 Power spectral density function

The pseudo excitation method is adopted to obtain the power spectrum density function of displacement. The random ground motion is assumed to be a pseudo harmonic excitation: $\ddot{u}_{\mathrm{g}}(t)=\sqrt{G_{\ddot{u}_{\mathrm{g}}}(\omega)}\mathrm{e}^{j\omega t}$, $j=\sqrt{-1}$. Introducing the pseudo excitation in Eq. (9), the *s*th perturbation solution of displacement can be obtained:

$$\boldsymbol{U}^{(s)}(\omega)=\sum_{i^{(s)}=1}^{n}\sum_{b^{(s)}=1}^{n}\boldsymbol{\varphi}_i\beta_{b^{(s)}i^{(s)}}H_{i^{(s)}}(\omega)\dot{u}_b^{(s-1)}(\omega), \tag{10a}$$

$$H_{i^{(s)}}(\omega)=(\omega_{i^{(s)}}^2-\omega^2+2j\varsigma_{i^{(s)}}\omega\omega_{i^{(s)}})^{-1}, \tag{10b}$$

$$\dot{u}_b^{(-1)}(\omega)=\sqrt{G_{\ddot{u}_{\mathrm{g}}}(\omega)}, \tag{10c}$$

$$\dot{u}_b^{(s)}(\omega)=(j\omega)\times u_b^{(s)}(\omega),\quad s=0,1,2,\cdots, \tag{10d}$$

where $H_{i^{(s)}}(\omega)$ is the complex frequency response function corresponding to the mode $i^{(s)}$. Hence, the *k*th nodal displacement of the *s*th perturbation solution can be obtained as

$$\begin{aligned}u_k^{(s)}(\omega)&=\sum_{i^{(0)}=1}^{n}\sum_{b^{(0)}=1}^{n}\cdots\sum_{i^{(s)}=1}^{n}\sum_{b^{(s)}=1}^{n}\boldsymbol{\varphi}_{b^{(1)}i^{(0)}}\cdots\boldsymbol{\varphi}_{b^{(s)}i^{(s-1)}}\boldsymbol{\varphi}_{ki^{(s)}}\beta_{b^{(0)}i^{(0)}}\\&\quad\cdots\beta_{b^{(s)}i^{(s)}}H_{i^{(0)}}\cdots H_{i^{(s)}}\times(j\omega)^s\sqrt{G_{\ddot{u}_{\mathrm{g}}}(\omega)}\\&=\sum_{i^{(0)}=1}^{n}\sum_{b^{(0)}=1}^{n}\cdots\sum_{i^{(s)}=1}^{n}\sum_{b^{(s)}=1}^{n}A^{(s)}_{k;i^{(0)},\ldots,i^{(s)};b^{(0)},\ldots,b^{(s)}}H_{i^{(0)}}\cdots H_{i^{(s)}}\\&\quad\times(j\omega)^s\sqrt{G_{\ddot{u}_{\mathrm{g}}}(\omega)},\end{aligned} \tag{11}$$

where $\boldsymbol{\varphi}_{b^{(c-1)}i^{(c)}}$ $(c=1,2,\cdots,s)$ is the b^{c-1} element of mode shape $\boldsymbol{\varphi}_{i^{(c)}}$; $i^{(0)},\cdots,i^{(s)}$, $b^{(0)},\cdots,b^{(s)}$ denote different enumerative variables, whose values change in the range of 1–*n*. In Eq. (11), the coefficient $A^{(s)}_{k;i^{(0)},\dots,i^{(s)};b^{(0)},\dots,b^{(s)}}$ is defined as

$$A^{(s)}_{k;i^{(0)},\dots,i^{(s)};b^{(0)},\dots,b^{(s)}} = \boldsymbol{\varphi}_{b^{(1)}i^{(0)}} \cdots \boldsymbol{\varphi}_{b^{(s)}i^{(s-1)}} \boldsymbol{\varphi}_{ki^{(s)}} \beta_{b^{(0)}i^{(0)}} \cdots \beta_{b^{(s)}i^{(s)}}. \quad (12)$$

According to the pseudo excitation method (Lin, 1992), the power spectral density function matrix of the *s*th perturbation solution can be written as

$$\boldsymbol{G}_{s,r} = [G_{u_k^{(s)}u_l^{(r)}}(\omega)] = \boldsymbol{U}^{(s)}(\omega)^* \boldsymbol{U}^{(r)}(\omega)^{\mathrm{T}}, \quad (13)$$

where the superscript * denote the conjugate complex.

Introducing Eq. (11) into Eq. (13), the element of power spectral density function matrix $\boldsymbol{G}_{s,r}$ is given by

$$\begin{aligned} G_{u_k^{(s)}u_l^{(r)}}(\omega) = & \sum_{i_S^{(0)}=1}^{n}\sum_{b_S^{(0)}=1}^{n}\cdots\sum_{i_S^{(s)}=1}^{n}\sum_{b_S^{(s)}=1}^{n}\sum_{i_R^{(0)}=1}^{n}\sum_{b_R^{(0)}=1}^{n}\cdots\sum_{i_R^{(r)}=1}^{n}\sum_{b_R^{(r)}=1}^{n} \\ & A^{(s)}_{k;i_S^{(0)},\dots,i_S^{(s)};b_S^{(0)},\dots,b_S^{(s)}} A^{(r)}_{l;i_R^{(0)},\dots,i_R^{(r)};b_R^{(0)},\dots,b_R^{(r)}} \\ & (H_{i_S^{(0)}}\cdots H_{i_S^{(s)}})^* H_{i_R^{(0)}}\cdots H_{i_R^{(r)}} \\ & (-1)^s (j\omega)^{s+r} G_{\ddot{u}_g}(\omega). \end{aligned} \quad (14)$$

Define the coefficient,

$$\Lambda^{(s),(r)}_{k;l;i_S^{(0)},\dots,i_S^{(s)};i_R^{(0)},\dots,i_R^{(r)}} = \sum_{b_S^{(0)}=1}^{n}\cdots\sum_{b_S^{(s)}=1}^{n}\sum_{b_R^{(0)}=1}^{n}\cdots\sum_{b_R^{(r)}=1}^{n} A^{(s)}_{k;i_S^{(0)},\dots,i_S^{(s)};b_S^{(0)},\dots,b_S^{(s)}} A^{(r)}_{l;i_R^{(0)},\dots,i_R^{(r)};b_R^{(0)},\dots,b_R^{(r)}}. \quad (15)$$

Eq. (14) simplifies to

$$\begin{aligned} G_{u_k^{(s)}u_l^{(r)}}(\omega) = & \sum_{i_S^{(0)}=1}^{n}\cdots\sum_{i_S^{(s)}=1}^{n}\sum_{i_R^{(0)}=1}^{n}\cdots\sum_{i_R^{(r)}=1}^{n} \\ & \Lambda^{(s),(r)}_{k;l;i_S^{(0)},\dots,i_S^{(s)};i_R^{(0)},\dots,i_R^{(r)}} (H_{i_S^{(0)}}\cdots H_{i_S^{(s)}})^* \\ & H_{i_R^{(0)}}\cdots H_{i_R^{(r)}} (-1)^s (j\omega)^{s+r} G_{\ddot{u}_g}(\omega), \end{aligned} \quad (16)$$

where $i_S^{(0)},\cdots,i_S^{(s)}$, $b_S^{(0)},\cdots,b_S^{(s)}$, $i_R^{(0)},\cdots,i_R^{(r)}$, $b_R^{(0)},\cdots,b_R^{(r)}$ are different enumerative variables in the range of 1–*n*, which are distinguished by subscripts *S*, *R*, superscripts $(0),\cdots,(s)$ and $(0),\cdots,(r)$.

It should be mentioned that the objective of adopting the pseudo excitation method in this section is to make the derivation more simple and compact. In fact, the computational efficiency of Eq. (16) is the same as the traditional random vibration method due to multiple pre-positive summation symbols.

It is helpful in engineering applications to generalize Eq. (16) for response quantities other than nodal displacement. The displacement-related response quantity *z*(*t*) such as an internal force or stress, can be expressed in terms of the vector of nodal relative displacement $\boldsymbol{U}$

$$z(t) = \boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{U}, \quad (17)$$

where $\boldsymbol{\Gamma}$ is a vector of constants. For the internal force in a member, for example, $\boldsymbol{\Gamma}$ is given in terms of the elements of the stiffness matrix of the member. The response quantity $z(t)$ of interest can be written in the perturbation form

$$z(t) = z^{(0)}(t) + \varepsilon z^{(1)}(t) + \varepsilon^2 z^{(2)}(t) + \varepsilon^3 z^{(3)}(t) + \cdots, \quad (18)$$

where $z^{(s)}(t) = \boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{U}^{(s)}$. Thus, the power spectral density function of $z(t)$ can be obtained:

$$\begin{aligned} G_{zz} = & G_{z^{(0)}z^{(0)}} + \varepsilon[G_{z^{(0)}z^{(1)}} + G_{z^{(1)}z^{(0)}}] \\ & + \varepsilon^2[G_{z^{(0)}z^{(2)}} + G_{z^{(1)}z^{(1)}} + G_{z^{(2)}z^{(0)}}] \\ & + \varepsilon^3[G_{z^{(0)}z^{(3)}} + G_{z^{(1)}z^{(2)}} + G_{z^{(2)}z^{(1)}} + G_{z^{(3)}z^{(0)}}] \\ & + \varepsilon^4[\cdots] + \cdots, \end{aligned} \quad (19)$$

where not only auto-power spectral density function $G_{z^{(s)}z^{(s)}}$ of the same perturbation order but also the cross-power spectrum density function $G_{z^{(s)}z^{(r)}}$ ($s \neq r$) of different perturbation order are involved.

In general, the power spectrum density function $G_{z^{(s)}z^{(r)}}$ of $z^{(s)}(t)$ and $z^{(r)}(t)$ can be calculated as

$$G_{z^{(s)}z^{(r)}} = \boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{U}^{(s)}(\omega)^*\boldsymbol{U}^{(r)}(\omega)^{\mathrm{T}}\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{G}_{s,r}\boldsymbol{\Gamma}, \quad (20)$$

where $G_{u_k^{(s)}u_l^{(r)}}(\omega)$ can be obtained from Eq. (16).

4 Perturbation spectrum method

4.1 Equations

The mean square of the response quantity $z(t)$ is computed by integrating the power spectral density function Eq. (19) over the frequency domain

$$I_{zz}=\sigma_z^2=\sigma_{z^{(0)}z^{(0)}}+\varepsilon[\sigma_{z^{(0)}z^{(1)}}+\sigma_{z^{(1)}z^{(0)}}]+\varepsilon^2[\sigma_{z^{(0)}z^{(2)}}+\sigma_{z^{(1)}z^{(1)}}+\sigma_{z^{(2)}z^{(0)}}]+\varepsilon^3[\sigma_{z^{(0)}z^{(3)}}+\sigma_{z^{(1)}z^{(2)}}+\sigma_{z^{(2)}z^{(1)}}+\sigma_{z^{(3)}z^{(0)}}]+\varepsilon^4[\cdots]+\cdots, \tag{21}$$

where

$$\sigma_{z^{(s)}z^{(r)}}=\int_{-\infty}^{\infty}G_{z^{(s)}z^{(r)}}(\omega)\mathrm{d}\omega=\boldsymbol{\Gamma}^{\mathrm{T}}\left[\int_{-\infty}^{\infty}G_{u_k^{(s)}u_l^{(r)}}(\omega)\mathrm{d}\omega\right]\boldsymbol{\Gamma}=\boldsymbol{\Gamma}^{\mathrm{T}}[\sigma_{u_k^{(s)}u_l^{(r)}}]\boldsymbol{\Gamma}. \tag{22}$$

Now, considering the general term $\sigma_{u_k^{(s)}u_l^{(r)}}$, the perturbation correlation coefficient can be defined as

$$\lambda_{i_S^{(0)},\ldots,i_S^{(s)};i_R^{(0)},\ldots,i_R^{(r)}}=\frac{1}{(\sigma_{i_S^{(0)}}\cdots\sigma_{i_S^{(s)}}\sigma_{i_R^{(0)}}\cdots\sigma_{i_R^{(r)}})^{2/(s+r+2)}}\int_{-\infty}^{\infty}\mathrm{Re}[(H_{i_S^{(0)}}\cdots H_{i_S^{(s)}})^*H_{i_R^{(0)}}\cdots H_{i_R^{(r)}}(-1)^s(j\omega)^{s+r}]G_{\ddot{u}_g}(\omega)\mathrm{d}\omega, \tag{23}$$

where

$$\sigma_{i_S^{(a)}}^2=\int_{-\infty}^{\infty}|H_{i_S^{(a)}}(\omega)|^2\,G_{\ddot{u}_g}(\omega)\mathrm{d}\omega,\ \text{for}\ a=1,2,\ldots,s, \tag{24a}$$

$$\sigma_{i_R^{(b)}}=\int_{-\infty}^{\infty}|H_{i_R^{(b)}}(\omega)|^2\,G_{\ddot{u}_g}(\omega)\mathrm{d}\omega,\ \text{for}\ b=1,2,\ldots,r. \tag{24b}$$

Utilizing the coefficient defined in Eq. (23), Eq. (21) is simplified as

$$\sigma_z=[\sigma_{z(0,0)}^2+\varepsilon\sigma_{z(0,1)}^2+\varepsilon^2\sigma_{z(0,2)}^2+\varepsilon^3\sigma_{z(0,3)}^2+\varepsilon^4\sigma_{z(0,4)}^2+\cdots]^{1/2}, \tag{25}$$

where

$$\sigma_{z(0,m)}^2=\sum_{s=0}^{m}\sigma_{z^{(s)}z^{(r)}}^2\ ,\ \text{for}\ s+r=m, \tag{26a}$$

$$\sigma_{z^{(s)}z^{(r)}}^2=\boldsymbol{\Gamma}^{\mathrm{T}}[\sigma_{u_k^{(s)}z_l^{(r)}}^2]\boldsymbol{\Gamma}, \tag{26b}$$

$$\sigma_{u_k^{(s)}z_l^{(r)}}^2=\sum_{i_S^{(0)}=1}^{n}\cdots\sum_{i_S^{(s)}=1}^{n}\sum_{i_R^{(0)}=1}^{n}\cdots\sum_{i_R^{(r)}=1}^{n}\Lambda_{k;l;i_S^{(0)},\ldots,i_S^{(s)};i_R^{(0)},\ldots,i_R^{(r)}}^{(s),(r)}\lambda_{i_S^{(0)},\ldots,i_S^{(s)};i_R^{(0)},\ldots,i_R^{(r)}}\left(\sigma_{i_S^{(0)}}\cdots\sigma_{i_S^{(s)}}\sigma_{i_R^{(0)}}\cdots\sigma_{i_R^{(r)}}\right)^{2/(s+r+2)}. \tag{26c}$$

The parameters in Eq. (26) have been defined above.

Let $S_{i_R^{(a)}}$ and $S_{i_R^{(b)}}$ denote the response spectrum values representing the mean peak displacement responses of two oscillators of frequency $\omega_{i_S^{(a)}}$, damping ratio $\zeta_{i_S^{(a)}}$, and frequency $\omega_{i_R^{(b)}}$, damping ratio $\zeta_{i_S^{(b)}}$, respectively. The relationship between the mean square and the mean of the peak of a stationary process can be expressed as follows:

$$S_{i_R^{(a)}}=p_{i_R^{(a)}}\sigma_{i_S^{(s)}}^a,\ \ S_{i_R^{(b)}}=p_{i_R^{(b)}}\sigma_{i_S^{(s)}}^b. \tag{27}$$

Assuming S_z denotes the mean peak value of the response quantity $z(t)$, and it can be written as

$$S_z=p_z\sigma_z. \tag{28}$$

It has been shown that peak factors are relatively insensitive to characteristics of response processes and the ratios $p_{i_R^{(a)}}/p_z$ and $p_{i_R^{(b)}}/p_z$ are near unity. Thus, the above expression in Eq. (25) can be written in the form of response spectrum

$$S_z=[S_{z(0,0)}^2+\varepsilon S_{z(0,1)}^2+\varepsilon^2S_{z(0,2)}^2+\varepsilon^3S_{z(0,3)}^2+\varepsilon^4(\cdots)+\cdots]^{1/2}, \tag{29}$$

where

$$S_{z(0,m)}^2=\sum_{s=0}^{m}S_{z^{(s)}z^{(r)}}^2,\ \ \text{for}\ s+r=m, \tag{30a}$$

$$S_{z^{(s)}z^{(r)}}^2=\boldsymbol{\Gamma}^{\mathrm{T}}[S_{u_k^{(s)}z_l^{(r)}}^2]\boldsymbol{\Gamma}, \tag{30b}$$

$$S_{u_k^{(s)}z_l^{(r)}}^2=\sum_{i_S^{(0)}=1}^{n}\cdots\sum_{i_S^{(s)}=1}^{n}\sum_{i_R^{(0)}=1}^{n}\cdots\sum_{i_R^{(r)}=1}^{n}\Lambda_{k;l;i_S^{(0)},\ldots,i_S^{(s)};i_R^{(0)},\ldots,i_R^{(r)}}^{(s),(r)}\lambda_{i_S^{(0)},\ldots,i_S^{(s)};i_R^{(0)},\ldots,i_R^{(r)}}\left(S_{i_S^{(0)}}\cdots S_{i_S^{(s)}}S_{i_R^{(0)}}\cdots S_{i_R^{(r)}}\right)^{2/(s+r+2)}. \tag{30c}$$

The above equation represents a modal combination rule for the response of non-classical systems in terms of the ground response spectrum. From Eq. (29), it can be seen that the perturbation response spectrum formulation reduces to the traditional CQC formulation while the perturbation orders *s* and *r* are equal to zero. Thus, the perturbation spectrum method is an extension of the traditional CQC method based on perturbation techniques. Meanwhile, the response spectrum in seismic design code is based on the quasi acceleration response, thus the displacement spectrum can be obtained easily:

$$S_{i_S^{(a)}}=\alpha_{i_S^{(b)}}g\Big/\omega^2_{i_S^{(a)}},\quad S_{i_R^{(b)}}=\alpha_{i_R^{(b)}}g\Big/\omega^2_{i_R^{(b)}},\tag{31}$$

where the earthquake affecting coefficients $\alpha_{i_S^{(a)}}$ and $\alpha_{i_R^{(b)}}$ have been given in the seismic design code.

4.2 Perturbation correlation coefficient

Since the perturbation correlation coefficient is given by Eq. (23) in the integral form, the calculation may be too time-consuming and encounter difficulties. To reduce the computation time, the ground motion can be assumed to be a Gauss white noise process because earthquakes are generally wide-band, which can be defined as

$$E[\ddot{u}_g(t)\ddot{u}_g(t+\Delta t)]=2\pi S_0\delta(\Delta t),\tag{32}$$

where $E[]$ is the mean value operator, $\delta()$ is the dirac delta function, and S_0 is the intensity of the Gauss white noise. Based on this assumption, a simplified integral expression is give by

$$\lambda_{i_S^{(0)},\ldots,i_S^{(s)};i_R^{(0)},\ldots,i_R^{(r)}}=\frac{1}{\left(\Omega_{i_S^{(0)}}\cdots\Omega_{i_S^{(s)}}\Omega_{i_R^{(0)}}\cdots\Omega_{i_R^{(r)}}\right)^{2/(s+r+2)}}\int_{-\infty}^{\infty}\mathrm{Re}\left[\left(H_{i_S^{(0)}}\cdots H_{i_S^{(s)}}\right)^*H_{i_R^{(0)}}\cdots H_{i_R^{(r)}}(-1)^s(j\omega)^{s+r}\right]\mathrm{d}\omega,\tag{33}$$

where

$$\Omega^2_{i_S^{(a)}}=\int_{-\infty}^{\infty}|H_{i_S^{(a)}}(\omega)|^2\,\mathrm{d}\omega,$$

$$\Omega_{i_R^{(b)}}=\int_{-\infty}^{\infty}|H_{i_R^{(b)}}(\omega)|^2\,\mathrm{d}\omega,$$

$$\text{for } a=1,2,\ldots,s,\ \ b=1,2,\ldots,r.\tag{34}$$

In the above equation, the numerator and denominator are all the integration form of product of the multiple frequency response function. According to the integral method mentioned in (Elishakaff and Lyon, 1986) numerator and denominator in Eq. (33) can be transformed to

$$I=\int_0^{\infty}\frac{\sum_{k=0}^{2n-1}a_k\omega^{4n-2-2k}}{\prod_{j=1}^{n}\{(\omega_j^2-\omega^2)^2+4\zeta_j^2\omega_j^2\omega^2\}}\mathrm{d}\omega.\tag{35}$$

The above integral has the closed form solution, which is

$$I=\sum_{j=1}^{n}\left\{\frac{\pi}{4\zeta_j\omega_j^3}(A_j\omega_j^2+B_j)\right\}.\tag{36}$$

Based on Eq. (35) and Eq. (36), the integration can be avoided in the calculation of the perturbation correlation coefficient, and the flow chart is shown in Fig. 1. Since it is a closed form solution without integration, the calculation is simple and convenient in application. Utilizing Fig. 1, the numerator and denominator of correlation coefficient for 0th perturbation are given by

$$\int_0^{\infty}real\{H_{i_S^{(0)}}(\omega)\}^*H_{i_S^{(0)}}(\omega)\mathrm{d}\omega=\frac{\pi}{4\zeta_{i_S^{(0)}}\omega^3_{i_S^{(0)}}},\tag{37a}$$

$$\int_0^{\infty}real\{H_{i_R^{(0)}}(\omega)\}^*H_{i_R^{(0)}}(\omega)\mathrm{d}\omega=\frac{\pi}{4\zeta_{i_R^{(0)}}\omega^3_{i_R^{(0)}}},\tag{37b}$$

$$\begin{aligned}&\int_0^{\infty}real\{H_{i_S^{(0)}}(\omega)\}^*H_{i_R^{(0)}}(\omega)\mathrm{d}\omega=\\&\frac{\pi}{2}\times\left[\zeta_{i_S^{(0)}}\omega_{i_S^{(0)}}+\zeta_{i_R^{(0)}}\omega_{i_R^{(0)}}\right]\Big/\\&\left[\omega^2_{i_S^{(0)}}\omega^2_{i_R^{(0)}}\zeta^2_{i_S^{(0)}}+\zeta_{i_S^{(0)}}\zeta_{i_R^{(0)}}\omega^3_{i_S^{(0)}}\omega_{i_R^{(0)}}\right.\\&+\zeta_{i_S^{(0)}}\zeta_{i_R^{(0)}}\omega_{i_S^{(0)}}\omega^3_{i_R^{(0)}}+\omega^4_{i_R^{(0)}}\\&\left.-2\omega^2_{i_S^{(0)}}\omega^2_{i_R^{(0)}}+\omega^4_{i_S^{(0)}}+4\zeta^2_{i_R^{(0)}}\omega^2_{i_S^{(0)}}\omega^2_{i_R^{(0)}}\right].\end{aligned}\tag{37c}$$

Introducing these above equations into Eq. (33), the correlation coefficient is rewritten as

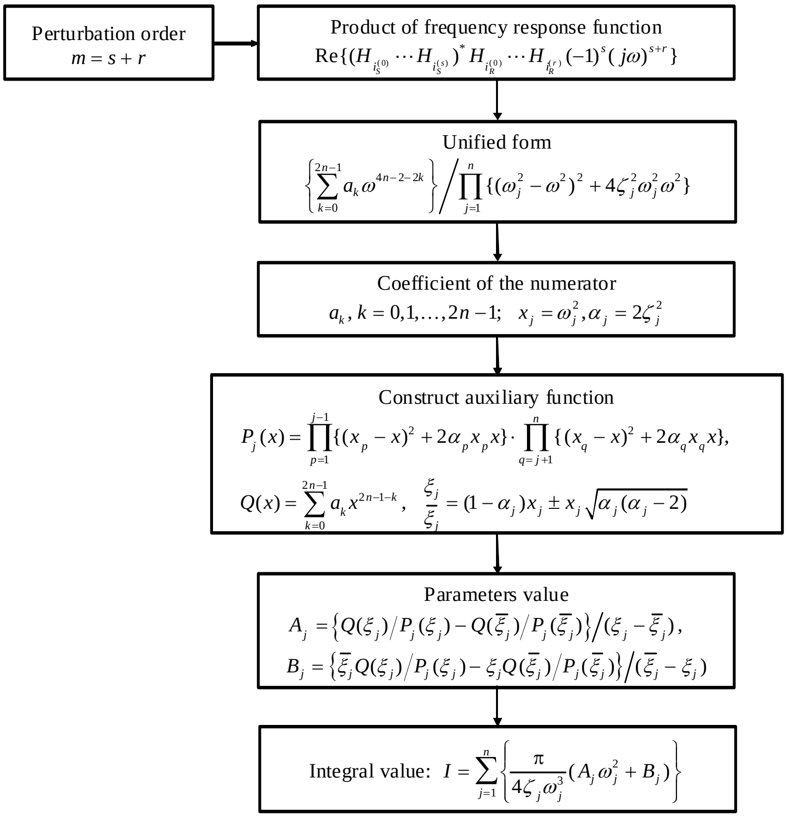

Fig. 1 Calculation flow chart of numerator and denominator of perturbation correlation

$$\lambda_{i_S^{(0)};i_R^{(0)}} = \left[8\sqrt{\zeta_{i_S^{(0)}}\zeta_{i_R^{(0)}}}(\zeta_{i_S^{(0)}}\beta + \zeta_{i_R^{(0)}})\beta^{3/2}\right] \Big/ \left[(1-\beta^2)^2 + 4\zeta_{i_S^{(0)}}\zeta_{i_R^{(0)}}\beta(1-\beta^2) + 4(\zeta_{i_S^{(0)}}^2 + \zeta_{i_R^{(0)}}^2)\beta^2\right], \quad \beta = \frac{\omega_{i_s^{(0)}}}{\omega_{i_r^{(0)}}}. \tag{38}$$

This is identical to the correlation coefficient formulation of the traditional CQC method (Kiureghian and Nakamura, 1993). Therefore, the perturbation spectrum method can be considered to be an extension of the traditional CQC method.

5 Numerical examples

To examine the accuracy of the perturbation spectrum method, an example of the non-classically damped system shown in Fig. 2 is analyzed. The non-classically damped system is highly idealized and may not resemble typical non-classically damped systems encountered in practice. However, it still possesses the essential mechanical features of non-classically damping systems and is simple enough to provide a clear demonstration of the proposed new method. The system in Fig. 2 consists of a structure and a damper connecting the first floor and the ground. The structure has properties with floor masses m_1=1500 kg, m_2=m_3=1000 kg, and interstorey stiffnesses k_1=200 kN/m, k_2=k_3=200 kN/m. Adopting the Rayleigh damping supposition, the first two modal damping ratios are assumed to be 0.05. As shown in Fig. 2, the damping is divided into two parts: mass classical damping and stiffness classical damping, which are distinguished respectively by superscripts $^{(m)}$ and $^{(k)}$. Based on the above description, the modal properties of the structure are shown in Table 1. Furthermore, it is assumed that the stiffness of the damper is negligible and the damping coefficient is denoted by c_{damper}. Assuming the modal matrix of the structure is $\boldsymbol{\Phi}$, the modal damping matrix of the non-classically damped system, which is usually a full rank matrix, can be written as

$$\boldsymbol{D}^m = \boldsymbol{\Phi}^{\mathrm{T}} \boldsymbol{D} \boldsymbol{\Phi} = \begin{bmatrix} d_{11}^m & d_{12}^m & d_{13}^m \\ d_{21}^m & d_{22}^m & d_{23}^m \\ d_{31}^m & d_{32}^m & d_{33}^m \end{bmatrix}. \tag{39}$$

The perturbation quantity ε is defined as

$$\varepsilon = \max_i \left\{ \sum_{j \neq i} \frac{|d_{ij}^m|}{d_{ii}^m} \right\}. \tag{40}$$

In some sense, it can represent the non-classical degree of the system.

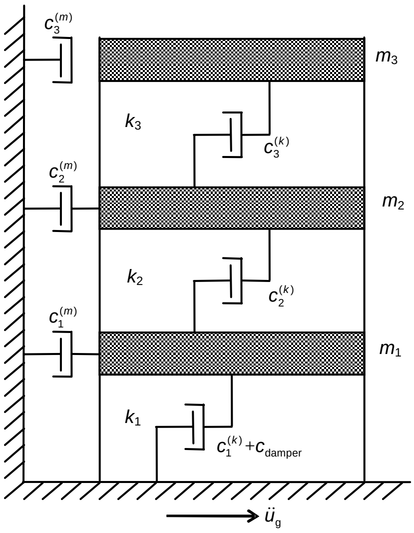

Fig. 2 Model of structure-damper system

Table 1 Modal properties of the structure

Mode	Period (s)	Modal shape	Damping ratio
1st	1.05	[0.44; 0.80; 1.00]	0.050
2nd	0.42	[1.00; 0.22; −0.84]	0.050
3rd	0.27	[−0.43; 1.00; −0.51]	0.064

In the numerical example, the El Centro earthquake record is taken as the excitation, whose time-history and corresponding response spectrum are shown in Fig. 3. To examine the effect of non-classical damping, four values of damping coefficients of the damper, $c_{\mathrm{damper}}=C(1, 1)$, $c_{\mathrm{damper}}=5C(1, 1)$, $c_{\mathrm{damper}}=10C(1, 1)$, $c_{\mathrm{damper}}=15C(1, 1)$, are considered, where $C(1, 1)$ represents the element located on column 1 and row 1 of the damping matrix of the structure. Table 2 summarizes the structure-damper system's results, which are obtained respectively based on the direct time-history integration, the forced decoupling method, and the perturbation spectrum method. The direct time-history integral results are taken as the relatively exact values, which are listed in the 3rd column. The data in the 4th and 5th columns correspond to the results of the time-history analysis and the traditional CQC method based on the forced decoupling method, respectively. It also shows that the error of the traditional CQC method is small for the light non-classical damping, and large for the strong non-classical damping. The results based on the perturbation spectrum method are listed in the 7th–15th columns, where the 0th perturbation results equal to the results of the traditional CQC method. Moreover, from the column 16 it can be noticed that, compared with the CQC method the results based on the perturbation spectrum method show a better agreement with the exact integration. Therefore, the perturbation spectrum method is more accurate for the dynamic analysis of non-classically damped system.

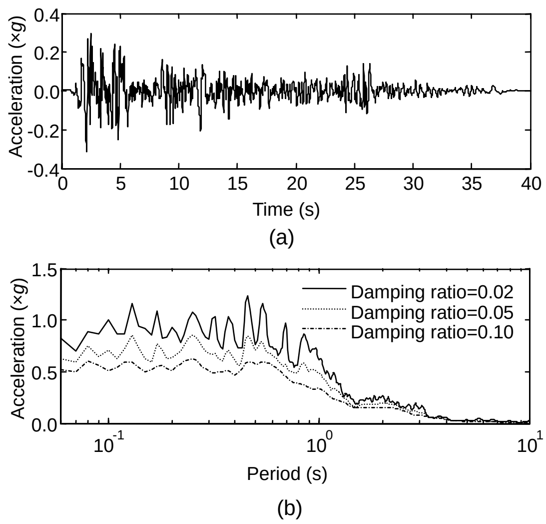

Fig. 3 Time-history (a) and response spectrum (b) of El Centro earthquake record

Also, the errors corresponding to different perturbation orders are listed in Fig. 4 for a detailed study on the accuracy of the perturbation spectrum method. Fig. 4 shows that the result based on the perturbation spectrum method will converge to a relatively accurate one with the perturbation order increasing. Finally, it should be mentioned that the error of

Table 2 Error analysis of several methods for evaluation of maximum structural displacements relative to the ground in different damper situations

c_{damper}	Floor	Direct time-history integration (cm)	Forced decoupling method			Perturbation spectrum method									
			Time-history (cm)	CQC (cm)	Error (%)	0th (cm)	1st (cm)	2nd (cm)	3rd (cm)	4th (cm)	5th (cm)	6th (cm)	7th (cm)	8th (cm)	Error (%)
$C(1,1)$	1	5.97	6.05	6.00	0.5	6.00	5.97	5.98	5.98	5.98	5.98	5.98	5.98	5.98	0.2
	2	10.79	10.80	10.72	−0.7	10.72	10.71	10.72	10.72	10.72	10.72	10.72	10.72	10.72	−0.7
	3	13.32	13.27	13.38	0.5	13.38	13.39	13.40	13.40	13.40	13.40	13.40	13.40	13.40	0.6
$5C(1,1)$	1	3.82	3.95	3.80	−0.5	3.80	3.57	3.69	3.67	3.68	3.69	3.70	3.72	3.72	−2.6
	2	6.88	6.72	6.64	−3.5	6.64	6.55	6.67	6.68	6.69	6.70	6.72	6.73	6.73	−2.2
	3	8.40	8.07	8.22	−2.1	8.22	8.27	8.40	8.41	8.42	8.42	8.42	8.42	8.42	0.2
$10C(1,1)$	1	2.79	2.77	2.84	1.8	2.84	2.25	2.67	2.54	2.64	2.63	2.67	2.70	2.71	−2.9
	2	5.31	4.71	4.76	−10.4	4.76	4.56	4.90	4.87	4.96	4.95	4.96	5.04	5.05	−4.9
	3	7.03	5.90	5.77	−17.9	5.77	5.95	6.28	6.36	6.44	6.46	6.48	6.51	6.52	−7.3
$15C(1,1)$	1	2.32	2.23	2.30	−0.9	2.30	1.66	2.15	1.90	2.09	1.96	2.11	2.13	2.15	−7.3
	2	4.71	3.69	3.73	−20.8	3.73	3.46	3.85	3.76	4.92	4.82	4.83	4.81	4.80	1.9
	3	6.59	4.57	4.43	−32.8	4.43	4.66	5.10	5.24	5.40	5.56	5.72	5.86	5.90	−10.5

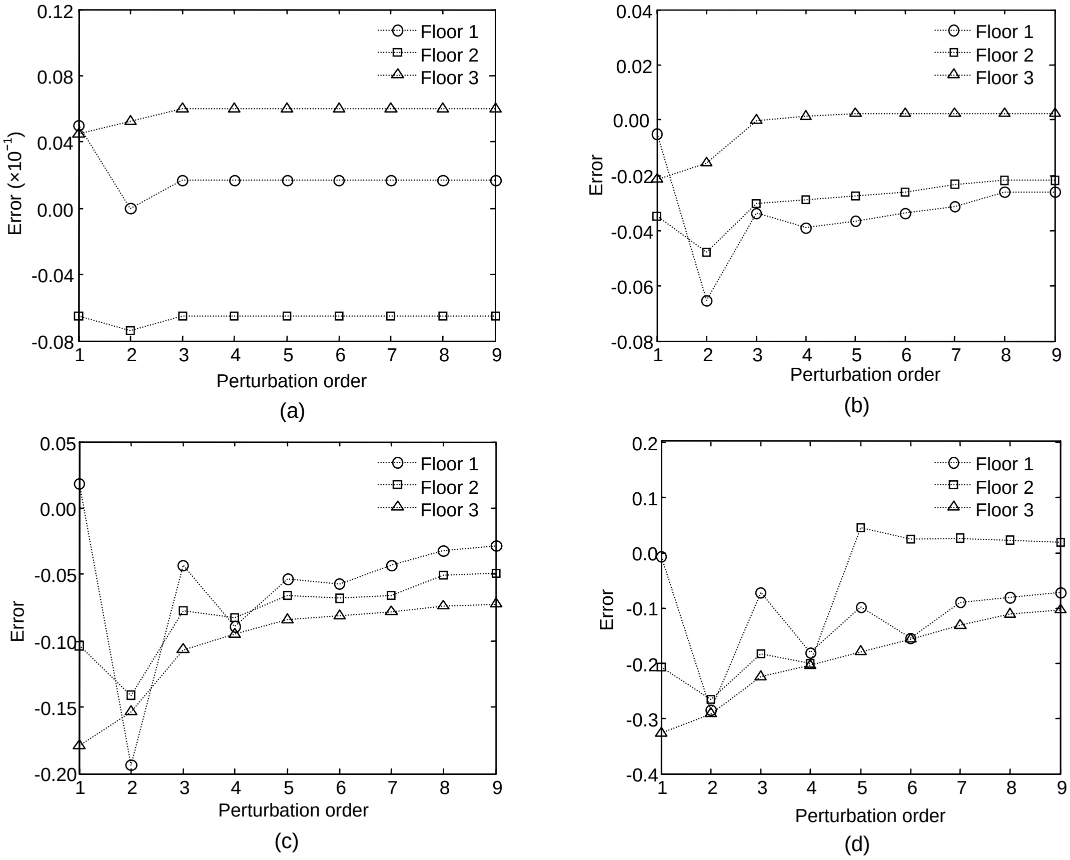

Fig. 4 Error of the perturbation spectrum method for different perturbation orders. (a) c_{damper}=C(1,1); (b) c_{damper}=5C(1,1); (c) c_{damper}=10C(1,1); (d) c_{damper}=20C(1,1)

perturbation spectrum method increases slightly while the degree of non-classical damping increases. According to the derivation of the new method, this is due to the assumption that the excitation is considered as a white noise for the evaluation of the correlation coefficient. However, it is possible to remove this assumption, obtaining more accurate results (but probably more complicated, too). These considerations will certainly be the object of future work.

6 Conclusion

The perturbation spectrum method developed in this study is based on fundamental principles of structural dynamics, random processes and perturbation theory. The method presents an obvious improvement and advantages over the forced coupling method. The major contributions of this study are in introducing the novel idea of combining perturbation method and response spectrum, and in developing the response spectrum combination rule for non-classically damped systems in Eq. (30). Furthermore, a simple method is provided for computing the perturbation correlation coefficients by which the integral and complex calculation can be avoided. The accuracy of the proposed method has been verified by comparisons with the exact results for an example of a non-classically damped system. In all cases the proposed approach agrees well with the exact results, which are obtained through time-history integration of the non-classically damped system. Also, the results show that the forced decoupling method, which improperly handled the damping property, is likely to cause gross error for the strong non-classical damping, and provides a result accurate enough for the light non-classical damping.

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