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Dynamics of a prestressed Timoshenko beam subject to arbitrary external load*

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Abstract: The free vibration and transient wave in a prestressed Rayleigh-Timoshenko beam subject to arbitrary transverse forces are analyzed by the newly developed method of reverberation-ray matrix (MRRM). The effects of shear deformation and rotational inertia are taken into consideration. With a Fourier transform technique, the general wave solutions with two sets of unknown amplitude coefficients are obtained in the transformed domain for an unbonded prestressed beam under the action of arbitrary external excitations. From the coupling at joints and the compatibility of displacements in each member, the free and forced vibration responses of a beam with various boundary conditions are finally evaluated through certain numerical algorithms. Results are presented for a simply-supported beam subject to either a point fixed load or moving load. Good agreement with the finite element method (FEM) is obtained. The present work is instructive for high-speed railway bridge design and structural health monitoring.

Key words: Reverberation-ray analysis, Prestressed Timoshenko beam, Free vibration, Dynamic responses
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1 Introduction

The prestressing technique, which is developed for the purpose of improving the overall performance of a structure to satisfy various service conditions, has been widely employed in most important civil engineering structures, especially in that of long-span bridges. Simply modeled as a single-span beam with an axial load, the dynamic characteristics of an unbonded bridge deck have been well investigated. Frýba (1972) studied the free and forced vibration of a simply-supported beam subject to an axial force and a moving force based on integral transformation methods. Bokaian (1988; 1990) presented the influence of a constant axial compressive and tensile force on natural frequencies and normal modes of a uniform single-span beam with different boundary conditions. Hamed and Frostig (2004) analyzed the effects of nonlinear material behavior of concrete, prestress force and cracks on the free vibration of a bonded prestressed beam. The work revealed that a great reduction in natural frequencies would be detected due to large cracking damage. Law and Lu (2005) investigated the dynamic responses of an unbonded simply-supported prestressed beam using the modal expansion method. The inverse problem of prestress identification was also studied and an effective identification approach was proposed. Kocatürk and Şimşek (2006a) employed the Lagrange equations to study the transverse vibration of a damped beam subjected to an eccentric compressive force and a moving harmonic force. Yang *et al.* (2008) discussed the free and forced vibration of cracked nonhomogeneous beams under the action of an axial constant force and a transverse moving load.

The researches aforementioned are all based on

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the Euler-Bernoulli beam theory, whilst studies employing the Rayleigh-Timoshenko beam theory by involving the effects of shear deformation and rotatory inertia were much fewer. Auciello and Ercolano (2004) proposed a general technique for determining the natural frequencies of non-uniform Timoshenko beams. Kocatürk and Şimşek (2006b) extended their previous work (Kocatürk and Şimşek, 2006a) to analyze dynamic responses of a damped Timoshenko beam under the combination of an eccentric compressive load and a moving harmonic force.

In a brief literature survey, the authors found that no work had been reported on the wave propagation of a prestressed beam subject to arbitrary transverse external loads. In this study, both the free vibration and transient wave in an unbonded prestressed Timoshenko beam are analyzed using the recently developed method of reverberation-ray matrix (MRRM) (Howard and Pao, 1998; Pao *et al*., 1999; Chen and Pao, 2003; Pao and Sun, 2003). In the MRRM, two sets of dual coordinates are introduced and solutions in the form of transient wave functions are obtained in the frequency domain for each member. The unknown amplitude coefficients in the solutions are determined by the continuity conditions at each joint and compatibility conditions of displacements in dual local coordinates for each member. The transient responses for arbitrary external forces are finally obtained by performing the inverse fast Fourier transform (IFFT) combined with the Neumann series expansion (Pao *et al*., 2007; Jiang and Chen, 2009). The natural frequencies and normal modes for free vibration can also be determined accurately with certain other numerical algorithms (Guo *et al*., 2008; Pao and Chen, 2009). Illustrative examples for an unbonded prestressed simply-supported beam are considered in this paper. The natural frequencies, the early time transient waves excited by an impact force and relatively long time responses under a moving load are calculated. It is concluded that the present method is effective and numerically accurate for evaluating dynamic responses of unbonded prestressed beam structures under arbitrary transverse forces. It is also an alternative to the widely used finite element method (FEM). Numerical results obtained in this work also show that the transverse displacement is the most sensitive variable to the prestress.

2 Elastodynamic equations and formulations

2.1 Governing equations and solutions

As shown in Fig. 1a, the unbonded prestressed bridge deck is modeled as a simply-supported beam subject to a compressive axial force S_x and an arbitrary external load $q(x, t)$. The tendon eccentricity is not taken into account. In fact it can be replaced by an axial force and a couple, and would not affect the dynamic characteristics of the prestressed beam. According to the MRRM, two sets of local coordinate systems are introduced for each uniform beam element *JK* as shown in Fig. 1b. The origins of (x^{JK}, y^{JK}) and (x^{KJ}, y^{KJ}) are located at the two end joints *J* and *K*, respectively, with the directions of the dual coordinate systems opposite to each other. All physical variables pertaining to the local coordinates (x^{JK}, y^{JK}) will be denoted by superscripts *JK*. Based on the Rayleigh-Timoshenko beam theory, the governing equations of element *JK* can be written as (superscripts *JK* omitted)

$$\begin{cases} EI\dfrac{\partial^2\phi}{\partial x^2}+\kappa AG\left(\dfrac{\partial v}{\partial x}-\phi\right)=\rho I\dfrac{\partial^2\phi}{\partial t^2}, \\ \kappa AG\left(\dfrac{\partial^2 v}{\partial x^2}-\dfrac{\partial\phi}{\partial x}\right)-S_x\dfrac{\partial^2 v}{\partial x^2}=\rho A\dfrac{\partial^2 v}{\partial t^2}-q(x,t), \end{cases} \tag{1}$$

where $v(x, t)$ and $\phi(x, t)$ are the transverse displacement and rotatory angle, respectively. Young's modulus E, shear modulus G, mass density ρ, cross-sectional area A, and the moment of inertia of

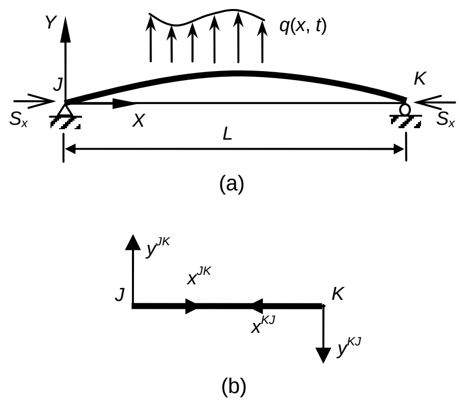

Fig. 1 Prestressed Timoshenko beam under an arbitrary external force (a) and dual local coordinates (b)

the cross-section I are constants for the uniform beam element. κ is the shear coefficient taken as $\pi^2/12$. If the prestressed beam is non-uniform, e.g., a tapered beam, a piece-wise uniform model can be adopted (Jiang and Chen, 2009). We divide equally or non-equally the non-uniform beam into M beam segments. Then, in each short segment, the properties can be regarded as approximately constant. The two ends of each uniform segment are considered as joints.

The pair of Fourier transforms which relate a variable in the time domain to that in the frequency domain are given by

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t,\quad \hat{f}(\omega)=\int_{-\infty}^{+\infty}f(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t. \tag{2}$$

After the Fourier transform of Eq. (1), we obtain

$$\begin{cases}EI\dfrac{\mathrm{d}^2\hat{\phi}}{\mathrm{d}x^2}+\kappa AG\left(\dfrac{\mathrm{d}\hat{v}}{\mathrm{d}x}-\hat{\phi}\right)=-\omega^2\rho I\hat{\phi},\\ \kappa AG\left(\dfrac{\mathrm{d}^2\hat{v}}{\mathrm{d}x^2}-\dfrac{\mathrm{d}\hat{\phi}}{\mathrm{d}x}\right)-S_x\dfrac{\mathrm{d}^2\hat{v}}{\mathrm{d}x^2}=-\omega^2\rho A\hat{v}-\hat{q}(x,\omega).\end{cases} \tag{3}$$

For free vibration (i.e., $\hat{q}=0$), the expressions for $\hat{v}$ and $\hat{\phi}$ can be easily obtained as

$$\hat{v}(x,\omega)=a_2\,\mathrm{e}^{\mathrm{i}k_2x}+d_2\,\mathrm{e}^{-\mathrm{i}k_2x}+a_3\,\mathrm{e}^{\mathrm{i}k_3x}+d_3\,\mathrm{e}^{-\mathrm{i}k_3x}, \tag{4}$$

$$\hat{\phi}(x,\omega)=g_2a_2\,\mathrm{e}^{\mathrm{i}k_2x}-g_2d_2\,\mathrm{e}^{-\mathrm{i}k_2x}+g_3a_3\,\mathrm{e}^{\mathrm{i}k_3x}-g_3d_3\,\mathrm{e}^{-\mathrm{i}k_3x}, \tag{5}$$

where a_j and d_j (j=2, 3) are unspecified amplitude coefficients of arriving and departing waves, respectively. The ratios of $\hat{v}$ and $\hat{\phi}$ are $g_j=\mathrm{i}\kappa c_2^2k_j/[\kappa c_2^2-R^2(\omega^2-c_1^2k_j^2)]$ ($j=2,\ 3$); $R=\sqrt{I/A}$ is the radius of gyration of the cross-section; $c_1=\sqrt{E/\rho}$ and $c_2=\sqrt{G/\rho}$ are the longitudinal and shear wave speeds, respectively; the wave number k_j (j=2, 3) is given by

$$k_j=\frac{1}{c_1\sqrt{2(1-\hat{S}_x)}}\left((n+1)\omega^2-\hat{S}_x\left(\omega^2-\frac{c_1^2}{nR^2}\right)\pm\left(\left[(n+1)\omega^2-\hat{S}_x\left(\omega^2-\frac{c_1^2}{nR^2}\right)\right]^2-\frac{4c_1^2\omega^2}{R^2}(1-\hat{S}_x)\left(\frac{4R^2\omega^2}{c_1^2}-1\right)\right)^{1/2}\right)^{1/2}, \tag{6}$$

where $n=E/(\kappa G)$ is a dimensionless material parameter, and $\hat{S}_x=S_x/(\kappa AG)$ is the normalized axial force.

To obtain the transient wave solutions of a prestressed beam excited by arbitrary transverse forces, we perform the Fourier transform and inverse Fourier transform to Eq. (3) in spatial variable x. Combined with the convolution theorem and residue theorem, the general solutions in the frequency domain can be finally derived after an arduous manipulation,

$$\hat{v}(x,\omega)=a_2\,\mathrm{e}^{\mathrm{i}k_2x}+d_2\,\mathrm{e}^{-\mathrm{i}k_2x}+a_3\,\mathrm{e}^{\mathrm{i}k_3x}+d_3\,\mathrm{e}^{-\mathrm{i}k_3x}-\frac{A_2}{k_2}\int_0^x\hat{q}(s,\omega)\sin k_2(x-s)\mathrm{d}s+\frac{A_3}{k_3}\int_0^x\hat{q}(s,\omega)\sin k_3(x-s)\mathrm{d}s, \tag{7}$$

$$\hat{\phi}(x,\omega)=g_2a_2\,\mathrm{e}^{\mathrm{i}k_2x}-g_2d_2\,\mathrm{e}^{-\mathrm{i}k_2x}+g_3a_3\,\mathrm{e}^{\mathrm{i}k_3x}-g_3d_3\,\mathrm{e}^{-\mathrm{i}k_3x}-A_4\int_0^x\hat{q}(s,\omega)\cos k_2(x-s)\mathrm{d}s+A_4\int_0^x\hat{q}(s,\omega)\cos k_3(x-s)\mathrm{d}s, \tag{8}$$

where $A_j=\dfrac{k_j^2+1/(nR^2)-\omega^2/c_1^2}{\kappa AG(1-\hat{S}_x)(k_2^2-k_3^2)}$ $(j=2,\ 3)$, and $A_4=1/[EI(1-\hat{S}_x)(k_2^2-k_3^2)]$.

2.2 Formulations of the reverberation-ray matrix method

Consider the boundary conditions at joints J and K. For a simply-supported beam, the following equations should be satisfied:

$$\hat{v}^{JK}=\hat{M}^{JK}=0,\ \text{at}\ x^{JK}=0,$$

$$\text{and } \hat{v}^{KJ} = \hat{M}^{KJ} = 0, \text{ at } x^{KJ} = 0, \tag{9}$$

where $\hat{M}(x,\omega) = EI\dfrac{\mathrm{d}\hat{\phi}(x,\omega)}{\mathrm{d}x}$ is the Fourier transform of the bending moment. Substituting Eqs. (7) and (8) into Eq. (9), we obtain two sets of linear equations which can be summarized in a matrix form as

$$\boldsymbol{d}^J = \boldsymbol{S}^J\boldsymbol{a}^J, \text{ and } \boldsymbol{d}^K = \boldsymbol{S}^K\boldsymbol{a}^K, \tag{10}$$

where $\boldsymbol{a}^J = [a_2^{JK}, a_3^{JK}]^\mathrm{T}$ and $\boldsymbol{a}^K = [a_2^{KJ}, a_3^{KJ}]^\mathrm{T}$ are unspecified arriving-wave vectors at joint J and joint K, respectively; $\boldsymbol{d}^J = [d_2^{JK}, d_3^{JK}]^\mathrm{T}$ and $\boldsymbol{d}^K = [d_2^{KJ}, d_3^{KJ}]^\mathrm{T}$ are unspecified departing-wave vectors at joint J and joint K, respectively; $\boldsymbol{S}^J$ and $\boldsymbol{S}^K$ are the joint scattering matrices which relate the incident waves to the transmitted and reflected ones.

Note that the present method can be applied in general to various boundary conditions, e.g., for a fixed-free beam, Eq. (9) is rewritten as

$$\hat{v}^{JK} = \hat{\phi}^{JK} = 0, \text{ at } x^{JK} = 0,$$
$$\text{and } \hat{Q}^{KJ} = \hat{M}^{KJ} = 0 \text{ at } x^{KJ} = 0, \tag{11}$$

where $\hat{Q}(x,\omega) = \kappa AG\left[\dfrac{\mathrm{d}\hat{v}(x,\omega)}{\mathrm{d}x} - \hat{\phi}(x,\omega)\right]$ is the transformed shear force. The scattering relation in Eq. (10) can then be derived similarly. If the prestressed beam is non-uniform, the continuity conditions at each middle joint where the neighboring piece-wise uniform elements meet together should also be considered.

Grouping all joint scattering relations, we can obtain

$$\boldsymbol{d} = \boldsymbol{S}\boldsymbol{a}, \tag{12}$$

where $\boldsymbol{a}$ and $\boldsymbol{d}$ are the global arriving- and departing-wave vectors, respectively, and $\boldsymbol{S}$ is the global scattering matrix.

It is shown that for a uniform prestressed beam, there are only 4 algebraic equations contained in Eq. (12) for unknown amplitude vectors $\boldsymbol{a}$ and $\boldsymbol{d}$, a column matrix of 4 elements each. Therefore, additional equations must be supplemented. Note that in the MRRM two dual local coordinates are adopted for each structural member, and two displacements or forces in the dual coordinates at a common cross-section should be the same. For the transverse displacement $\hat{v}$, this compatibility condition is expressed as

$$\hat{v}^{JK}(x^{JK},\omega) = -\hat{v}^{KJ}(L - x^{JK},\omega), \tag{13}$$

where L is the length of beam. Substituting the expression of Eq. (7) into the previous equation, we find the following local phase relations:

$$\begin{aligned} a_2^{JK} &= -d_2^{KJ}\,\mathrm{e}^{-\mathrm{i}k_2^{JK}L} + q_2^{JK}, \\ a_3^{JK} &= -d_3^{KJ}\,\mathrm{e}^{-\mathrm{i}k_3^{JK}L} + q_3^{JK}, \\ a_2^{KJ} &= -d_2^{JK}\,\mathrm{e}^{-\mathrm{i}k_2^{JK}L} + q_2^{KJ}, \\ a_3^{KJ} &= -d_3^{JK}\,\mathrm{e}^{-\mathrm{i}k_3^{JK}L} + q_3^{KJ}, \end{aligned} \tag{14}$$

where

$$\begin{aligned} q_2^{JK} &= \frac{A_2^{JK}}{2\mathrm{i}k_2^{JK}}\int_0^L \hat{q}^{JK}(s,\omega)\,\mathrm{e}^{-\mathrm{i}k_2^{JK}s}\,\mathrm{d}s, \\ q_3^{JK} &= -\frac{A_3^{JK}}{2\mathrm{i}k_3^{JK}}\int_0^L \hat{q}^{JK}(s,\omega)\,\mathrm{e}^{-\mathrm{i}k_3^{JK}s}\,\mathrm{d}s, \\ q_2^{KJ} &= -\frac{A_2^{JK}\,\mathrm{e}^{-\mathrm{i}k_2^{JK}L}}{2\mathrm{i}k_2^{JK}}\int_0^L \hat{q}^{JK}(s,\omega)\,\mathrm{e}^{\mathrm{i}k_2^{JK}s}\,\mathrm{d}s, \\ q_3^{KJ} &= \frac{A_3^{JK}\,\mathrm{e}^{-\mathrm{i}k_3^{JK}L}}{2\mathrm{i}k_3^{JK}}\int_0^L \hat{q}^{JK}(s,\omega)\,\mathrm{e}^{\mathrm{i}k_3^{JK}s}\,\mathrm{d}s. \end{aligned} \tag{15}$$

From Eq. (14), we can obtain the global phase relation in a matrix form as

$$\boldsymbol{a} = \boldsymbol{P}\bar{\boldsymbol{d}} + \boldsymbol{q} = \boldsymbol{P}\boldsymbol{U}\boldsymbol{d} + \boldsymbol{q}, \tag{16}$$

where $\boldsymbol{P} = \mathrm{diag}\{-\mathrm{e}^{-\mathrm{i}k_2^{JK}L}, -\mathrm{e}^{-\mathrm{i}k_3^{JK}L}, -\mathrm{e}^{-\mathrm{i}k_2^{JK}L}, -\mathrm{e}^{-\mathrm{i}k_3^{JK}L}\}$ is the global phase matrix; $\boldsymbol{q} = [q_2^{JK}, q_3^{JK}, q_2^{KJ}, q_3^{KJ}]^\mathrm{T}$ is the source vector; $\bar{\boldsymbol{d}}$ is a new global departing-wave vector which contains the same elements of $\boldsymbol{d}$ but in different sequences. The permutation matrix $\boldsymbol{U}$ is introduced to relate the two vectors, $\bar{\boldsymbol{d}} = \boldsymbol{U}\boldsymbol{d}$, where

$$\boldsymbol{U} = \{\{0,0,1,0\},\{0,0,0,1\},\{1,0,0,0\},\{0,1,0,0\}\}. \tag{17}$$

Using a combination of Eqs. (12) and (16), the unknown amplitude vector $\boldsymbol{d}$ can be finally determined from

$$\boldsymbol{d} = (\boldsymbol{I} - \boldsymbol{R})^{-1}\boldsymbol{s}, \tag{18}$$

where $\boldsymbol{R}=\boldsymbol{SPU}$ is named the reverberation-ray matrix, and $\boldsymbol{s}=\boldsymbol{Sq}$ is the global source vector. Once the vector $\boldsymbol{d}$ is obtained, the other unknown vector $\boldsymbol{a}$ can also be determined from Eq. (12) or Eq. (16).

3 Free and forced vibration responses

3.1 Free vibration analysis

If the prestressed beam is free of any external forces, the source vector $\boldsymbol{s}$ in Eq. (18) vanishes. The solutions so determined describe the free vibration of the beam. To obtain non-trivial solutions for amplitude vectors $\boldsymbol{a}$ and $\boldsymbol{d}$, the determinant of matrix $(\boldsymbol{I}-\boldsymbol{R})$ must equal zero. Thus, we can obtain the following characteristic equation:

$$\det[\boldsymbol{I} - \boldsymbol{R}(\omega)] = 0, \tag{19}$$

the roots of which are the natural frequencies of the beam, and can be obtained through a numerical searching algorithm. The corresponding values for $\boldsymbol{d}$ are then determined up to one arbitrary factor as the characteristic vector of $(\boldsymbol{I}-\boldsymbol{R})$, and the global arriving-wave amplitudes $\boldsymbol{a}$ is determined by Eq. (12) or (16) with $\boldsymbol{q}=0$. Thus, the normal modes of the prestressed beam can be obtained from Eqs. (7) and (8).

3.2 Transient responses

For a prestressed beam under arbitrary external forces, the transient response is calculated by performing the inverse Fourier transform as

$$v(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{v}(x,\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\,\omega, \tag{20}$$

where $v(x, t)$ is the transverse displacement in the time domain. Other variables, for example, the rotatory angle ϕ and internal forces and moments, can be determined in the same way.

Note that the integration in Eq. (20) cannot be performed directly, as it contains an infinite number of singularities corresponding to the natural frequencies by the direct inverse of $(\boldsymbol{I}-\boldsymbol{R})$. To overcome this difficulty, the following Neumann series expansion is used (Howard and Pao, 1998; Pao *et al.*, 1999; Chen and Pao, 2003)

$$(\boldsymbol{I} - \boldsymbol{R})^{-1} = \boldsymbol{I} + \boldsymbol{R} + \boldsymbol{R}^2 + \ldots + \boldsymbol{R}^N + \ldots \tag{21}$$

In the traditional MRRM, only the case of joint concentrated forces is considered, and the Neumann series is usually truncated up to the term $\boldsymbol{R}^{N_{\max}}$ in numerical calculation. The power index $N_{\max}$ represents the scattering times of the fastest transient wave propagating along the shortest path from the loading point to the calculated position. Using a proper selection of $N_{\max}$ and the time window in the Fast Fourier Transform (FFT) which is used to numerically perform the integration in Eq. (20), the exact early time transient responses of a structure under joint forces can be determined. Recent research (Jiang and Chen, 2009) shows that the Neumann series technique and the FFT algorithm can also be employed for calculating the responses of structures subjected to arbitrary distributive or moving loads.

4 Numerical examples

First, a short investigation of free vibration of a simply-supported Timoshenko beam is made for validating the present method. The Poisson's ratio and the shear coefficient of the beam are taken as $\upsilon=0.3$ and $\kappa=5/6$, respectively. The dimensionless frequencies of the Timoshenko beam $\lambda_i^4=\rho AL^4\omega^2/(EI)$ are calculated by the MRRM for different thickness-to-length ratios (h/L). As shown in Table 1, the results agree extremely well with the solutions of Lee and Schultz (2004) and Kocatürk and Şimşek (2006b).

Next, we consider a simply-supported prestressed beam with an axial force S_x. The length of the beam is taken as $L=15$ m. The other material and geometrical parameters of the beam are: Young's modulus $E=3\times10^{10}$ N/m^2, shear modulus $G=1.1\times10^{10}$ N/m^2, mass density $\rho=2400$ kg/m^3, cross-sectional area $A=0.3$ m^2, and the second moment of inertia $I=0.025$ m^4.

4.1 Influence of prestress on natural frequencies

The natural frequencies of the beam are calculated by the present method and FEM for S_x=0, 2000 and 6000 kN, respectively. The commercial software ANSYS is used to perform the FEM with the beam discretized equally into 600 or 4000 elements (the corresponding results are denoted by FEM-600 or FEM-4000). Note that the MRRM adopts the continuum model and hence has no discretization error. As shown in Table 2, the first six frequencies of the beam obtained by the FEM-600 and MRRM coincide with each other perfectly when there is no axial force acted on the supports. However, a much finer mesh is required for the FEM to obtain a more exact solution for higher modes. Also, as shown in Table 2, the frequency of the beam decreases with the increasing prestress, which is commonly known as the "compression softening" effect (Tse *et al.*, 1978). The effect of the prestress seems to be more obvious on the lower frequencies than on the higher order ones.

4.2 Transient response due to impulsive force

Assume that a time-stepped concentrated force with magnitude 1500 kN is applied at $x_p=L/3$. The

Table 1 Comparison of the dimensionless frequency of the simply-supported Timoshenko beam for different thickness-to-length ratios

h/L	Method	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
0.05	Present	3.13498	6.23136	9.25537	12.1813	14.9926	17.6810	20.2447	22.6862
	Lee and Schultz (2004)	3.13498	6.23136	9.25537	12.1813	14.9926	17.6810	20.2447	22.6862
	Kocatürk and Şimşek (2006b)	3.13498	6.23136	9.25536	12.1812	14.9926	17.6802	20.2445	22.6809
0.1	Present	3.11568	6.09066	8.84052	11.3431	13.6132	15.6790	17.5705	19.3142
	Lee and Schultz (2004)	3.11568	6.09066	8.84052	11.3431	13.6132	15.6790	17.5705	19.3142
	Kocatürk and Şimşek (2006b)	3.11567	6.09066	8.84048	11.3430	13.6131	15.6769	17.5700	19.1928
0.2	Present	3.04533	5.67155	7.83952	9.65709	11.2220	12.6022	13.0323	13.4443
	Lee and Schultz (2004)	3.04533	5.67155	7.83952	9.65709	11.2220	12.6022	13.0323	13.4443
	Kocatürk and Şimşek (2006b)	3.04533	5.67155	7.83949	9.65693	11.2219	12.5971	13.0323	13.4442

Table 2 Natural frequencies corresponding to different prestressing forces (rad/s)

Mode No.	S_x=0			S_x=2000 kN		S_x=6000 kN	
	FEM-600	FEM-4000	MRRM	FEM-4000	MRRM	FEM-4000	MRRM
1	44.42	44.42	44.42	43.03	43.03	40.11	40.11
2	173.74	173.74	173.74	172.35	172.35	169.53	169.53
3	377.57	377.57	377.57	376.15	376.15	373.30	373.30
4	642.39	642.39	642.41	640.95	640.95	638.00	638.03
5	954.73	954.73	954.74	953.22	953.23	950.14	950.20
6	1302.82	1302.82	1302.80	1301.19	1301.22	1297.98	1298.06
7	1677.17	1677.12	1677.14	1675.47	1675.48	1672.02	1672.17
8	2070.62	2070.56	2070.55	2068.74	2068.81	2065.16	2065.32
9	2477.71	2477.65	2477.63	2475.70	2475.80	2471.87	2472.13
10	2894.54	2894.41	2894.40	2892.40	2892.47	2888.32	2888.60
11	3318.15	3317.96	3317.94	3315.76	3315.90	3311.43	3311.82
12	3746.48	3746.10	3746.11	3743.84	3743.96	3739.19	3739.65
13	4177.82	4177.31	4177.33	4174.93	4175.06	4170.02	4170.53
14	4611.10	4610.48	4610.46	4607.90	4608.08	4602.69	4603.30
15	5045.52	5044.64	5044.66	5041.94	5042.15	5036.41	5037.13

Note: FEM-600 and FEM-4000 mean the beam is discretized equally into 600 and 4000 elements in the FEM, respectively duration of the force is taken to be $4t_0$, where $t_0=1/\sqrt{E/\rho}$ is the time interval for the flexural wave propagates one meter at the faster group speed. The early time transient waves of displacement, acceleration, shear force, and flexural strain at $x=x_p$ are calculated respectively for illustration. As shown in Fig. 2, the displacement wave is the most sensitive response to the effect of prestress, while the waves of acceleration and shear force, as well as the very early time flexural wave, are hardly affected. It is also seen from Fig. 2c that, for $0\leq t\leq 4t_0$ the response of shear force is exactly a stepped function with the amplitude being one half of the external force. After $t>4t_0$, the excitation vanishes and the shear wave also becomes zero accordingly. Later, at the instant of $t=10t_0$, the shear wave first reflected from the left support of the beam arrives at the observed point, resulting in an obvious change in the waveform. It can be concluded that the present method can clearly predict the propagating of waves along the beam and the scattering at supports, i.e., the transient responses of structures especially at the early stages can be determined accurately by the MRRM. Here, the transient waves at the mid-span of the beam are also calculated and similar conclusions can be obtained (Fig. 3).

4.3 Transient response due to point moving force

Moving load is one of the most important loads for bridges. In this numerical example, the effect of prestress to the vibration of a simply-supported beam traversed by a single force moving at a constant speed is also investigated. The moving force is taken as q_δ=200 kN with the speed v_0=30 m/s. All formulations derived in Sections 2 and 3 stay unaltered except that the external force $q(x, t)$ is substituted by $q_\delta\delta(x-v_0t)$. The results in Fig. 4 show that the sensitivity of various responses to the prestress is almost the same as that under a point fixed load. That means, the load type considered here does not have significant influence on the analysis of prestressing effect. The displacement response according to different crossing speeds of the moving force at the mid-span of the beam is also calculated. As shown in Fig. 5, the divergence between the responses of unprestressed and prestressed beam becomes less obvious with the increasing moving speed v_0.

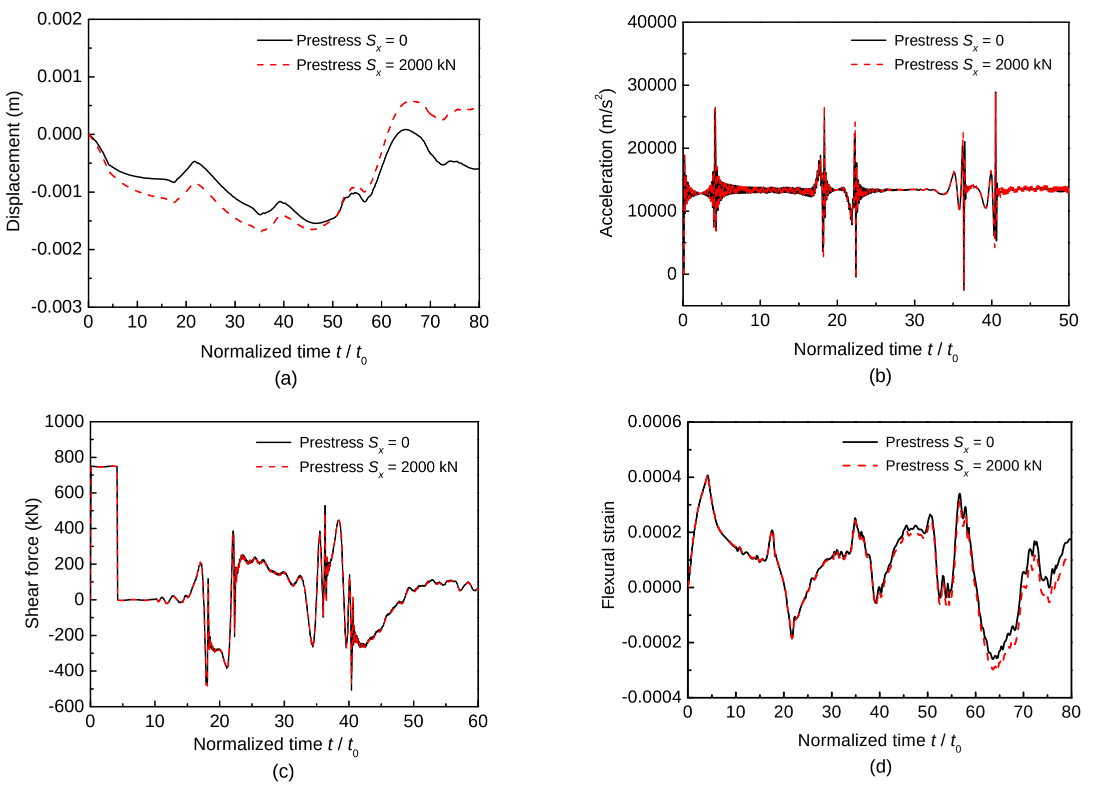

Fig. 2 Transient waves of displacement (a), acceleration (b), shear force (c), and flexural strain (d) at $x=L/3$

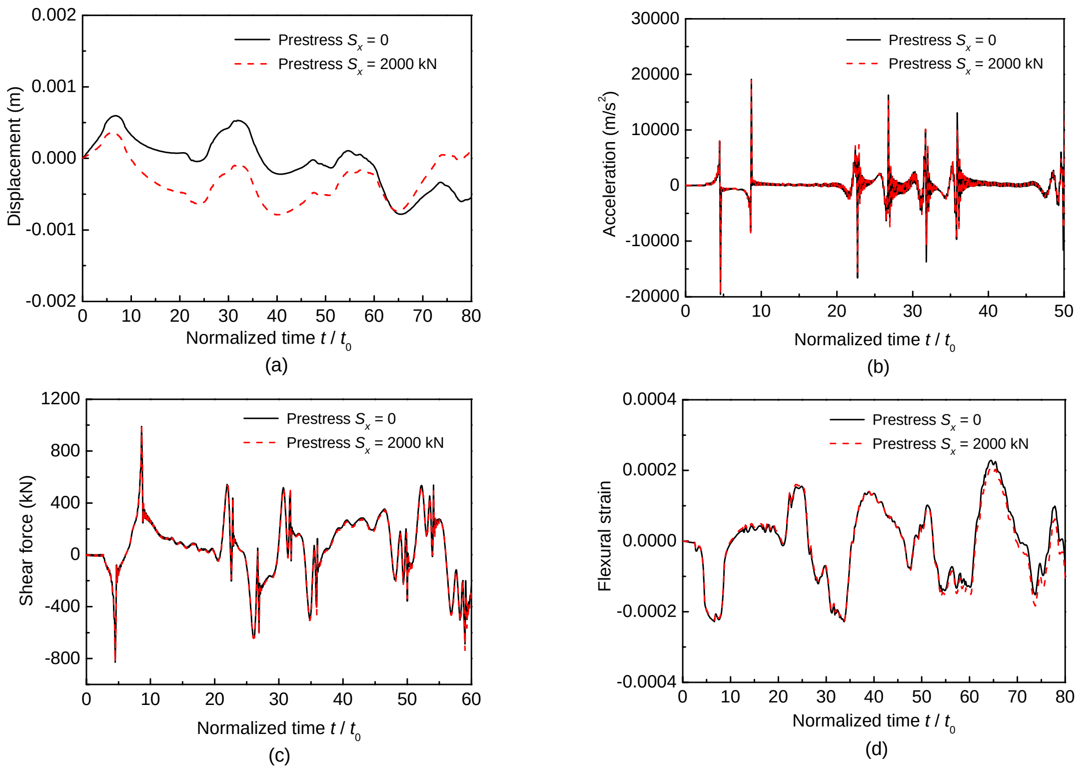

Fig. 3 Transient waves of displacement (a), acceleration (b), shear force (c), and flexural strain (d) at $x=L/2$

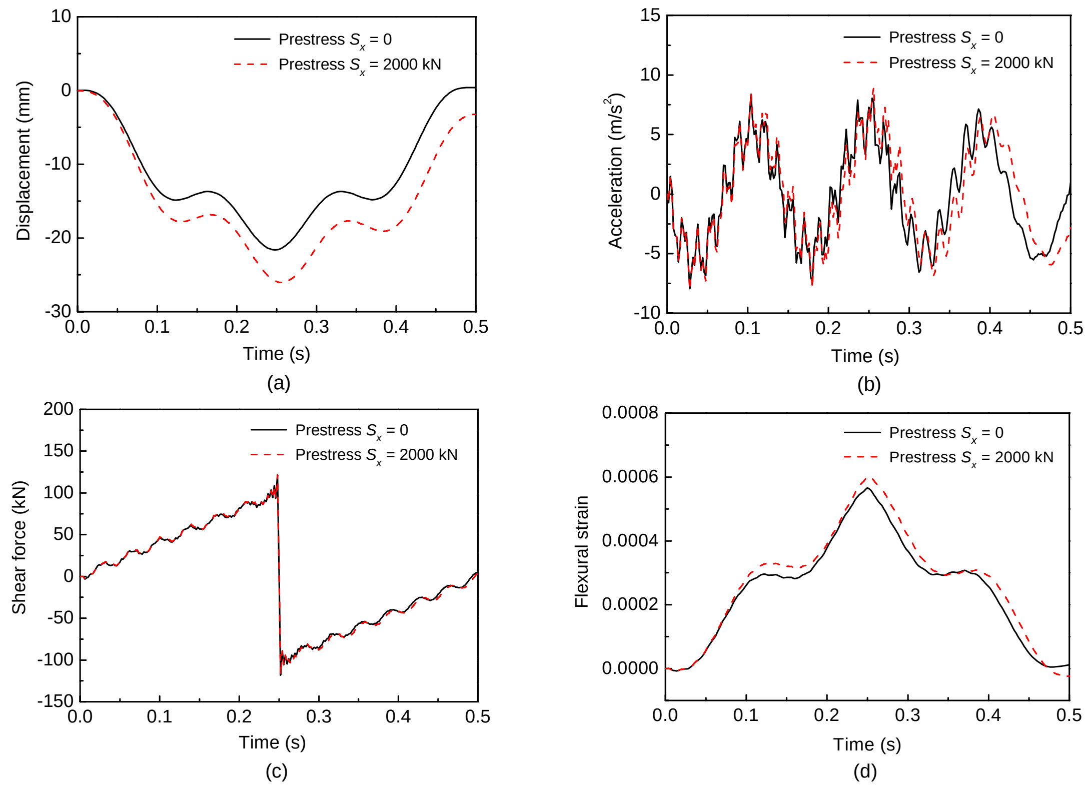

Fig. 4 Responses of displacement (a), acceleration (b), shear force (c), and flexural strain (d) at $x=L/2$ for the beam subjected to a single moving load at v_0=30 m/s

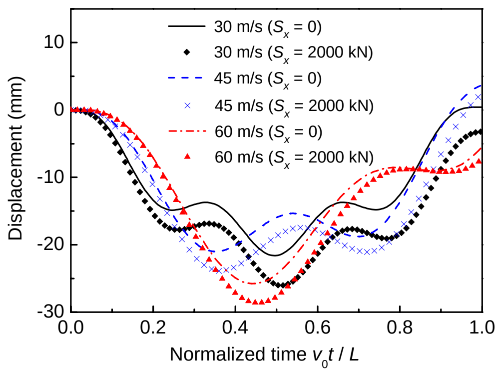

Fig. 5 Displacement response at x=L/2 under different moving speeds v_0=30, 45, and 60 m/s

5 Conclusions

In this paper, we investigate the free and forced dynamic responses of an unbonded prestressed Timoshenko beam by an alternative and effective method, the MRRM. The general solution of problem of the beam subject to arbitrary excitation is obtained in the frequency domain, and responses are finally analysed with joint continuity conditions and compatibility conditions. The natural frequencies determined by the MRRM are compared to those using the FEM. It is found that much finer mesh is required for FEM to obtain the results with the same accuracy as that for the MRRM. Numerical results also show that the frequencies decrease with the prestress due to the commonly known "compression softening" effect. The early time and medium time transient responses for the beam under a single fixed or moving load are also investigated, and the displacement is found to be the most sensitive one for the prestress.

The present analysis can be easily applied to a beam with various boundary conditions or to a beam with non-uniform properties by employing the piece-wise uniform model, without changing the uniformity of the formulations in the MRRM.

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