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Dynamic robust optimal reorder point with uncertain lead time and changeable demand distribution*

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Abstract: In fixed order quantity systems, uncertainty in lead time is expressed as a set of scenarios with occurrence probabilities, and the mean and variance in demand distribution are supposed to be changeable according to a known pattern. A new concept of "dynamic robust optimal reorder point" is proposed in this paper and its value is calculated as a "robust optimal reorder point function with respect to reorder time". Two approaches were employed in determining the dynamic optimal reorder point. The first is a shortage rate satisfaction approach and the second is a backorder cost minimization approach. The former aims at finding the minimum value of reorder point at each reorder time which satisfies the condition that the cumulative distribution function (CDF) of shortage rate under a given set of scenarios in lead time is greater than or equal to a basic CDF of shortage rate predetermined by a decision-maker. In the latter approach, the CDF of closeness of reorder point is defined at each reorder time to express how close to the optimal reorder points under the set of scenarios, and the dynamic optimal reorder point is defined according to stochastic ordering. Some numerical examples demonstrate the features of these dynamic robust optimal reorder points.

Key words: Reorder point, Lead time, Robust optimum, Uncertainty, Scenario
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1 Introduction

In fixed order quantity systems, the economic order quantity (EOQ) is usually calculated to minimize a total annual cost consisting of ordering cost and inventory holding cost, and the safety stocks are set to absorb demand fluctuation during a given lead time, resulting in an appropriate level of reorder point. Uncertainties in both lead time and demand distribution should be considered in determining the value of reorder point (Tersine, 1982; Porteus, 1990; Mula *et al*., 2006; Leung, 2008). In this paper, the uncertainty in lead time is expressed as a set of scenarios with occurrence probabilities, and the mean and variance in demand distribution are considered changeable according to a known pattern.

If the demand distribution is stationary, a robust optimal reorder point can be set as a constant value as proposed by Tamura *et al*. (2009), but the new condition of changeable demand distribution requires a dynamic change in the value of reorder point according to the pattern of demand distribution. This paper extends this method to a case where demand in each period follows a normal distribution but is changeable according to a known pattern with respect to mean and variance. A new concept of "dynamic robust optimal reorder point" is proposed and its value is calculated as a "robust optimal reorder point function with respect to reorder time".

Two approaches are employed in determining the dynamic optimal reorder point. The first is a shortage rate satisfaction approach and the second is a

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backorder cost minimization approach. The former aims at finding the minimum value of reorder point at each reorder time to satisfy the condition that the cumulative distribution function (CDF) of shortage rate under a given set of scenarios in lead time is greater than or equal to a basic CDF of shortage rate predetermined by a decision-maker. In the latter approach, the CDF of closeness of reorder point is defined at each reorder time to express how close to the optimal reorder points under the set of scenarios, and the dynamic optimal reorder point is defined according to stochastic ordering.

Some numerical examples demonstrate the features of these dynamic robust optimal reorder points in comparison with the other promising reorder points obtained through the expected value optimization method and the min-max optimization method.

2 Model formulation

The following assumptions are made for determining order quantity and reorder point in the fixed order quantity system:

1. Demand in period t follows a normal distribution with mean μ_t and variance σ_t^2, that is, $D_t \sim N(\mu_t, \sigma_t^2)$, and is independent of demand in any other period. The pattern of changeable parameters is known in advance as (μ_t, σ_t^2), t=1, 2, …, T, where one year is divided into T periods.

2. The uncertainty of lead time is expressed by a set of lead times with occurrence probabilities, $L \equiv \{L_1, L_2, \ldots, L_n\}$ with $P \equiv \{P_1, P_2, \ldots, P_n\}$, $0 \le P_i \le 1$ and $\sum_{i=1}^{n} P_i = 1$. The pairs of (L_i, P_i) and (L, P) are called the ith scenario and a set of scenarios, respectively.

3. Stockouts are all backordered, and the additional backorder cost is proportional to the amount of stockouts in the backorder cost minimization approach.

4. Order quantity is first determined by minimizing the total annual cost under zero lead time, and reorder point is then determined either by controlling shortage rate within an allowable level or by minimizing the annual additional cost under a set of scenarios of lead times, provided that the order quantity is specified in the first phase. Therefore, the additional cost includes only the costs related to the safety stock and backordering cost.

Under these assumptions, the order quantity is specified as the EOQ given by

$$Q^* = \sqrt{2c_\mathrm{s} R / c_\mathrm{h}}, \tag{1}$$

where c_h is the inventory holding cost per unit per year, c_s is the ordering cost per order, and R is the average annual demand, $R = \sum_{\tau=t}^{t+T-1} \mu_\tau$.

To manage changeable demand distribution, we use a variable economic reorder quantity reflecting any change in demand distribution instead of using the above constant EOQ. A possible method is to replace $R = \sum_{\tau=t}^{t+T-1} \mu_\tau$ in Eq. (1) with a nominal changeable annual demand $R_t = (T / T_1) \sum_{\tau=t}^{t+T_1-1} \mu_\tau$, but the procurement lead time may also change corresponding to a different size of order quantity, resulting in more complicated uncertainty in lead time. Therefore, we do not use this approach but adopt other method to change the reorder point dynamically, provided that the economic reorder quantity is given by Eq. (1).

3 Shortage rate satisfaction

Under a given scenario (L_i, P_i), we define the safety stock as the expected inventory level just before a replenishment arrives such that

$$\begin{aligned} s_\mathrm{f}(t, L_i) &= \int_0^\infty \big(s(t, L_i) - D\big) f(D) \mathrm{d}D \\ &= s(t, L_i) - \sum_{\tau=t}^{t+L_i-1} \mu_\tau, \end{aligned} \tag{2}$$

where $s_\mathrm{f}(t, L_i)$ is the safety stock with lead time L_i at time t, $s(t, L_i)$ is the reorder point with lead time L_i at time t, D is the demand during lead time L_i starting at time t, $D \equiv D(t, L_i) = \sum_{\tau=t}^{t+L_i-1} D_\tau$, which follows a normal distribution with mean $\mu(t, L_i) \equiv \sum_{\tau=t}^{t+L_i-1} \mu_\tau$ and variance $\sigma^2(t, L_i) \equiv \sum_{\tau=t}^{t+L_i-1} \sigma_\tau^2$. To maintain the shortage rate at less than or equal to an allowable level α_0, we have to specify the safety stock as

$$s_{\mathrm{f}}(t, L_i) \equiv u_{\alpha_0} \sqrt{\sum_{\tau=t}^{t+L_i-1} \sigma_\tau^2}, \tag{3}$$

where α_0 is the allowable shortage rate, u_α is the upper-side 100α percent point in a standard normal distribution, $u_\alpha \equiv \Phi^{-1}(1-\alpha)$. Substituting this result into Eq. (2), we get the reorder point $s(t, L_i)$ as

$$s(t, L_i) = \sum_{\tau=t}^{t+L_i-1} \mu_\tau + u_{\alpha_0} \sqrt{\sum_{\tau=t}^{t+L_i-1} \sigma_\tau^2}. \tag{4}$$

3.1 Reorder points based on existing methods

For a set of scenarios, we can derive the following methods for setting the value of reorder point according to either the min-max method or the expected value optimization method as shown in Tamura *et al*. (2009):

1. Reorder point based on the maximum lead time, $s(t, L_{\max})$, where $L_{\max} \equiv \max_{1\le i\le n} L_i$.

2. Reorder point based on mean lead time, $s(t, \overline{L})$, where $\overline{L} \equiv \sum_{1\le i\le n} P_i L_i$.

3. Reorder point based on the expected shortage rate, $s_{\alpha_0}(t)$.

Satisfying $\alpha(s_{\alpha_0}(t), t) = \alpha_0$, or equivalently

$$1 - \sum_{i=1}^{n} P_i \Phi\left(\left(s_{\alpha_0}(t) - \sum_{\tau=t}^{t+L_i-1} \mu_\tau\right) \Bigg/ \sqrt{\sum_{\tau=t}^{t+L_i-1} \sigma_\tau^2}\right) = \alpha_0, \tag{5}$$

where the expected shortage rate $\alpha(s,t)$ and the scenario shortage rate $\alpha_i(s,t)$ are given by

$$\alpha(s,t) \equiv \sum_{i=1}^{n} P_i \alpha_i(s,t), \tag{6}$$

$$\alpha_i(s,t) \equiv 1 - \Phi\left(\left(s - \sum_{\tau=t}^{t+L_i-1} \mu_\tau\right) \Bigg/ \sqrt{\sum_{\tau=t}^{t+L_i-1} \sigma_\tau^2}\right), \quad i = 1, 2, \ldots, n. \tag{7}$$

3.2 Dynamic robust satisfaction

Using scenario shortage rates $\alpha_i(s,t)$, $1 \le i \le n$, for the reorder point s in time t and the scenario occurrence probabilities P_i, $1 \le i \le n$, we define the CDF of shortage rate α for the set of scenarios as

$$\mathrm{CDF}(\alpha : s, t) \equiv \sum_{1\le i\le n} P_i \delta(\alpha - \alpha_i(s,t)), \tag{8}$$

if $z \ge 0$, $\delta(z) = 1$; otherwise, 0. From Eq. (7), it is clear that $\alpha_i(s,t)$ is increasing in L_i and decreasing in s, and that CDF(α:s, t) is also increasing in s. According to the concept of stochastic ordering, for any two reorder points s_1 and s_2, s_1 is called "stochastically better than or equal to" s_2, if and only if $\mathrm{CDF}(\alpha : s_1, t) \ge \mathrm{CDF}(\alpha : s_2, t)$ holds for any α. As the reorder point increases, the shortage rate decreases but the on-hand inventory increases especially under the scenarios with smaller lead times. If the occurrence probability for the scenario with the largest lead time is small enough to neglect, then the resulting large shortage rate under this scenario might be permitted by a decision-maker. How small the occurrence probability is enough for the decision-maker to accept such a large shortage rate depends on the preference structure of the decision-maker.

Therefore, to express the preference structure of the decision-maker, we introduce a "basic CDF $G(\alpha)$ of shortage rate α" for a given allowable shortage rate α_0 as

$$G(\alpha) \equiv \begin{cases} 0, \text{ if } 0 \le \alpha < \eta_{[1]}\alpha_0, \\ \sum_{j=1}^{t} P_{[j]}, \text{ if } \eta_{[t]}\alpha_0 \le \alpha < \eta_{[t+1]}\alpha_0, 1 \le t < n, \\ 1, \text{ if } \eta_{[n]}\alpha_0 \le \alpha, \end{cases} \tag{9}$$

$$\eta_i \equiv \begin{cases} (1-\zeta)\dfrac{1-\nu}{|I|P_i} + \zeta, i \in I_1, \\ (1-\zeta)\dfrac{1+\nu|I_1|/|I_2|}{|I|P_i} + \zeta, \ i \in I_2, \end{cases} \tag{10}$$

$$I_1 \equiv \{ i \mid P_i < 1/|I|, i \in I_1 \}, \quad I_2 \equiv I \setminus I_1, I \equiv \{1, 2, \ldots, n\}, \tag{11}$$

$$0 \le \zeta \le 1, \tag{12}$$

$$\underline{\nu} \equiv -\frac{|I_2|}{|I_1|} \le \nu \le \bar{\nu} \equiv \left(1-\frac{\max_{i\in I_1} P_i}{\min_{i\in I_2} P_i}\right) \Bigg/ \left(1+\frac{|I_1|}{|I_2|}\frac{\max_{i\in I_1} P_i}{\min_{i\in I_2} P_i}\right), \tag{13}$$

where I denotes the set of scenario indices and $|I|$ is the number of scenarios, that is, $|I|=n$. $P_{[j]}$ indicates the jth largest occurrence probability, resulting in $P_{[1]} \ge P_{[2]} \ge \cdots \ge P_{[n]}$. Parameters ζ and ν are predetermined according to the decision-maker's preference structure so that Eqs. (12) and (13) hold. The right-hand side value in Eq. (13) is derived so that $0 \le \eta_{[1]} \le \eta_{[2]} \le \cdots \le \eta_{[n]}$ holds, or equivalently, $0 \le \min_{i\in I_2} \eta_i$ and $\max_{i\in I_2} \eta_i \le \min_{j\in I_1} \eta_j$.

If $\mathrm{CDF}(\alpha : s,t) \ge G(\alpha), \forall \alpha,$ holds for some s in time t, then we can say that such a reorder point has a stochastically smaller distribution of shortage rate than the required basic distribution of shortage rate expressing the preference structure of the decision-maker. Of these reorder points, the smallest reorder point is called "dynamic robust optimal" through robust satisfaction approach.

Without loss of generality, we assume that $L_1 \le L_2 \le \ldots \le L_n$, resulting in $\alpha_1(s,t) \le \alpha_2(s,t) \le \cdots \le \alpha_n(s,t)$ for any s in period t. The $\mathrm{CDF}(\alpha : s,t)$ consists of edge points (0, 0), $(\alpha_1(s), 0)$, $(\alpha_1(s), P_1)$, $(\alpha_2(s), P_1)$, $(\alpha_2(s), P_1+P_2)$, …, $\left(\alpha_\ell(s), \sum_{j=1}^{\ell-1} P_j\right)$, $\left(\alpha_\ell(s), \sum_{j=1}^{\ell} P_j\right)$, …, $\left(\alpha_n(s), \sum_{j=1}^{n-1} P_j\right)$ and $(\alpha_n(s), 1)$. Since $G(\alpha)$ has edge points (0, 0), $(\eta_{[1]}\alpha_0, 0)$, $(\eta_{[1]}\alpha_0, P_{[1]})$, $(\eta_{[2]}\alpha_0, P_{[1]})$, …, $(\eta_{[2]}\alpha_0, P_{[1]}+P_{[2]})$, …, $\left(\eta_{[i]}\alpha_0, \sum_{j=1}^{i-1} P_{[j]}\right)$, …, $\left(\eta_{[i]}\alpha_0, \sum_{j=1}^{i} P_{[i]}\right)$, …, $\left(\eta_{[n]}\alpha_0, \sum_{j=1}^{n-1} P_{[j]}\right)$, and $(\eta_{[n]}\alpha_0, 1)$, the condition for satisfying $\mathrm{CDF}(\alpha : s,t) \ge G(\alpha)$ is that $\alpha_\ell(s) \le \eta_{i_c(\ell)}\alpha_0$ holds for any scenario ℓ, where $i_c(\ell)$ satisfies $\sum_{j=1}^{i_c(\ell)-1} P_{[j]} \le \sum_{j=1}^{\ell-1} P_j \le \sum_{j=1}^{i_c(\ell)} P_{[j]}$. Since $\alpha_\ell(s,t)$ is given by Eq. (7), this condition is equivalent to

$$1-\eta_{i_c(\ell)}\alpha_0 \le \Phi\left(\left(s - \sum_{\tau=t}^{t+L_\ell-1} \mu_\tau\right) \Bigg/ \sqrt{\sum_{\tau=t}^{t+L_\ell-1} \sigma_\tau^2}\right), \text{ or}$$

$$s \ge \sum_{\tau=t}^{t+L_\ell-1} \mu_\tau + \Phi^{-1}(1-\eta_{i_c(\ell)}\alpha_0)\sqrt{\sum_{\tau=t}^{t+L_\ell-1} \sigma_\tau^2}, \quad \forall \ell \in I.$$

Using this result, we define the "dynamic robust optimal reorder point based on an allowable shortage rate" $s^*(t)$ as

$$s^*(t) \equiv \max_{\ell\in I}\left\{ \sum_{\tau=t}^{t+L_\ell-1} \mu_\tau + \Phi^{-1}(1-\eta_{i_c(\ell)}\alpha_0)\sqrt{\sum_{\tau=t}^{t+L_\ell-1} \sigma_\tau^2} \right\}, \tag{14}$$

$$i_c(\ell) \equiv \max\left\{ i \left| \sum_{j=1}^{i-1} P_{[j]} \le P_\Sigma^{(\ell-1)},\ 1 \le i \le n \right.\right\}, \quad 2 \le \ell \le n,\ i_c(1) \equiv 1, \tag{15}$$

where $P_\Sigma^{(\ell)} \equiv \sum_{j=1}^{\ell} P_j,\ 1 \le \ell \le n,$ and $P_\Sigma^{(0)} \equiv 0.$

4 Backorder cost minimization

4.1 Optimal reorder point

Next consider a case to minimize the backorder cost. Since mean and variance of demand distribution are changeable, we consider the individual additional cost for each reorder, denoted by $f_\mathrm{b}(s,t,L_i)$, instead of using the annual additional cost. For any reorder made at time t, demand D during lead time L_i follows a normal distribution $N(\mu(t,L_i), \sigma^2(t,L_i))$, and the individual additional cost related to safety stock $s-\mu(t,L_i)$ for a given order quantity Q under the ith scenario is given by

$$f_\mathrm{b}(s,t,L_i) = c_\mathrm{h}\frac{Q}{R}\left(s-\mu(t,L_i)\right) + c_\mathrm{b}\int_s^\infty (D-s) f(D)\mathrm{d}D, \tag{16}$$

where c_b is the backordering cost per unit, $c_\mathrm{h}Q/R$

stands for the unit inventory holding cost for the duration of Q/R year. Using the same manner as Tamura *et al.* (2009), we can reduce Eq. (16) into

$$f_{\mathrm{b}}(s,t,L_i)=\sqrt{\sigma^2(t,L_i)}\left\{c_{\mathrm{h}}\frac{Q}{R}u_i(s,t)+c_{\mathrm{b}}[\phi(u_i(s,t))-u_i(s,t)[1-\Phi(u_i(s,t))]]\right\}, \tag{17}$$

where $\phi(u)$ is the probability density function of a standard normal distribution, $u \sim N(0,1^2)$, and $u_i(s,t)=(s-\mu(t,L_i))/\sqrt{\sigma^2(t,L_i)}$. Since we have $f_{\mathrm{b}}(s, t, L_i)\cong-\mu(t, L_i)(c_{\mathrm{h}}Q/Rc_{\mathrm{b}})$, it is remarkable that if $R/Q<c_{\mathrm{h}}/c_{\mathrm{b}}$ holds, then $f_{\mathrm{b}}(s, t, L_i)$ becomes negative, and the inventory holding cost can be reduced drastically by setting the reorder point at zero, resulting in backlogging all demand before a replenishment arrives.

If $R/Q\geq c_{\mathrm{h}}/c_{\mathrm{b}}$ holds, the optimal value of reorder point $s_{i\mathrm{b}}^*(t)$ under the ith scenario exists and is derived by setting

$$\frac{\mathrm{d}f_{\mathrm{b}}(s,t,L_i)}{\mathrm{d}s}=c_{\mathrm{h}}\frac{Q}{R}-c_{\mathrm{b}}[1-\Phi(u_i(s,t))]\equiv 0, \tag{18}$$

and we can obtain

$$s_{i\mathrm{b}}^*(t)=\mu(t,L_i)+u_{\alpha_{\mathrm{b}}^*}\sqrt{\sigma^2(t,L_i)}, \tag{19}$$

$$u_{\alpha_{\mathrm{b}}^*}\equiv\Phi^{-1}(1-c_{\mathrm{h}}Q/(c_{\mathrm{b}}R)), \tag{20}$$

$$\alpha_{\mathrm{b}}^*\equiv 1-\Phi\left(u_{\alpha_{\mathrm{b}}^*}\right)=c_{\mathrm{h}}Q/(c_{\mathrm{b}}R), \tag{21}$$

$$f_{\mathrm{b}}(s_{i\mathrm{b}}^*(t),t,L_i)=\sqrt{\sigma^2(t,L_i)}\left\{c_{\mathrm{h}}Qu_{\alpha_{\mathrm{b}}^*}/R+c_{\mathrm{b}}\left[\phi\left(u_{\alpha_{\mathrm{b}}^*}\right)-u_{\alpha_{\mathrm{b}}^*}\alpha_{\mathrm{b}}^*\right]\right\}. \tag{22}$$

Since the minimum additional cost depends on lead time under each scenario, we introduce closeness $\varepsilon(s, t, L_i)$ for any reorder point s under the ith scenario with (L_i, P_i) as

$$\varepsilon(s,t,L_i)\equiv\frac{f_{\mathrm{b}}(s,t,L_i)-f_{\mathrm{b}}(s_{i\mathrm{b}}^*(t),t,L_i)}{f_{\max}^{\mathrm{b}}(t,L_i)-f_{\mathrm{b}}(s_{i\mathrm{b}}^*,t,L_i)}, \tag{23}$$

where $f_{\max}^{\mathrm{b}}(t,L_i)$ denotes the worst value of individual additional cost under the ith scenario with respect to s in $s\in[0,s_{\max}(t)]$, $s_{\max}(t)\equiv\mu(t,L_{\max})+u_{\alpha^\dagger}\sqrt{\sigma^2(t,L_{\max})}$, $\alpha^\dagger=0.0001$, that is, $f_{\max}^{\mathrm{b}}(t,L_i)=\max_{0\leq s\leq s_{\max}(t)} f_{\mathrm{b}}(s,t,L_i)$, or

$$f_{\max}^{\mathrm{b}}(t,L_i)=\begin{cases}\max\left\{f_{\mathrm{b}}(s_{\max}(t),t,L_i),\mu(t,L_i)\left(c_{\mathrm{b}}-c_{\mathrm{h}}\dfrac{Q}{R}\right)\right\}, & \text{if } \dfrac{R}{Q}\geq\dfrac{c_{\mathrm{h}}}{c_{\mathrm{b}}},\\ f_{\mathrm{b}}(s_{\max}(t),t,L_i), & \text{if } \dfrac{R}{Q}<\dfrac{c_{\mathrm{h}}}{c_{\mathrm{b}}}.\end{cases} \tag{24}$$

As mentioned above, we derive some reorder points by optimizing the related functions over $s\in[0,s_{\max}(t)]$ as follows.

1. Minimizing $f_{\mathrm{b}}(s,t,L_{\max})$ provides

$$s_{\mathrm{b}}^*(t)=\mu(t,L_{\max})+u_{\alpha_{\mathrm{b}}^*}\sqrt{\sigma^2(t,L_{\max})}.$$

2. Minimizing $f_{\mathrm{b}}(s,t,\bar{L})$ provides

$$s_{\mathrm{b}}^*(t)=\mu(t,\bar{L})+u_{\alpha_{\mathrm{b}}^*}\sqrt{\sigma^2(t,\bar{L})}.$$

3. Minimizing $\sum_{i\in I}P_i f_{\mathrm{b}}(s,t,L_i)$ provides

$$s_{\mathrm{b}}^*(t)=\sum_{i\in I}P_i\left\{\mu(t,L_i)+u_{\alpha_{\mathrm{b}}^*}\sqrt{\sigma^2(t,L_i)}\right\}.$$

4. Minimizing $\sum_{i=1}^{n}P_i\varepsilon(s,t,L_i)$ provides $s_{\mathrm{b}}^*(t)$ satisfying

$$\sum_{i=1}^{n}\frac{P_i\Phi(u_i(s_{\mathrm{b}}^*(t)))}{f_{\max}^{\mathrm{b}}(t,L_i)-f_{\mathrm{b}}(s_{i\mathrm{b}}^*(t),t,L_i)}=\sum_{i=1}^{n}\frac{P_i(1-\alpha_{\mathrm{b}}^*)}{f_{\max}^{\mathrm{b}}(t,L_i)-f_{\mathrm{b}}(s_{i\mathrm{b}}^*(t),t,L_i)}.$$

4.2 Dynamic robust optimization

We introduce the CDF of closeness ξ, denoted by $\mathrm{CDF}(\xi:s,t)$ as

$$\mathrm{CDF}(\xi:s,t)\equiv\sum_{i\in I}P_i\delta(\xi-\varepsilon(s,t,L_i)). \quad (25)$$

In this case, the $\mathrm{CDF}(\xi:s,t)$ is not monotone increasing in s, unlike the CDF of shortage rate α, $\mathrm{CDF}(\alpha:s,t)$, and the stochastic ordering cannot always hold for any two reorder points. Therefore, we introduce a rank-k basic CDF $G_k(\xi)$, and give the rank $k(s, t)$ for reorder point s in period t by comparing $\mathrm{CDF}(\xi:s,t)$ with $G_k(\xi)$ as follows:

$$k(s,t)\equiv\min\left\{k\,\middle|\,\mathrm{CDF}(\xi:s,t)\geq G_k(\xi),\forall\xi\geq 0\right\}, \quad (26)$$

$$G_k(\xi)\equiv \begin{cases}0, \text{ if } 0\leq\xi<\eta_{[1]}k\varepsilon_0,\\ \sum_{j=1}^{t}P_{[j]}, \text{ if } \eta_{[t]}k\varepsilon_0\leq\xi<\eta_{[t+1]}k\varepsilon_0, 1\leq t<n,\\ 1, \text{ if } \eta_{[n]}k\varepsilon_0\leq\xi,\end{cases} \quad (27)$$

where ε_0 is a step size to generate the rank value, and the other parameters are the same as those defined in Eqs. (10)–(13).

Letting $\varepsilon^i\equiv\varepsilon(s,t,L_i)$ and assuming $\varepsilon^{\langle 1\rangle}\leq\varepsilon^{\langle 2\rangle}\leq\cdots\leq\varepsilon^{\langle n\rangle}$ without loss of generality, we can reduce Eq. (26) into

$$k(s,t)=\min\left\{k\,\middle|\,\varepsilon^{\langle\ell\rangle}\leq\eta_{i_c(\langle\ell\rangle)}k\varepsilon_0, 1\leq\ell\leq n, k\geq 0\right\}$$
$$=\max_{1\leq\ell\leq n}\left\{\frac{\varepsilon^{\langle\ell\rangle}}{\eta_{i_c(\langle\ell\rangle)}\varepsilon_0}\right\}, \quad (28)$$

$$i_c(\ell)\equiv\max\left\{i\,\middle|\,\sum_{j=1}^{i-1}P_{[j]}\leq P_{\Sigma}^{(\ell-1)}, 1\leq i\leq n\right\},$$
$$2\leq\ell\leq n, i_c(1)\equiv 1, \quad (29)$$

where $P_{\Sigma}^{(\ell)}\equiv\sum_{j=1}^{\ell}P_j$, $1\leq\ell\leq n$, and $P_{\Sigma}^{(0)}\equiv 0$.

Considering a set of feasible reorder points, $0\leq s\leq s_{\max}(t)$, we define the "dynamic robust optimal reorder point $s_{\mathrm{b}}^*(t)$" as the reorder point with the minimum rank given by

$$k(s_{\mathrm{b}}^*(t),t)=\min\{k(s,t)\,|\,0\leq s\leq s_{\max}(t)\}. \quad (30)$$

5 Numerical examples

To show some features of reorder points derived by various methods including the proposed dynamic robust optimal reorder point, we apply these methods to some numerical examples. Consider the nine sets of scenarios, $(L^{(k)}, P^{(i)})$, k=1, 2, 3, i=1, 2, 3, where $L^{(1)}$= {3, 4, 5, 6, 7}, $L^{(2)}$={8, 9, 10, 11, 12}, $L^{(3)}$={18, 19, 20, 21, 22}, $P^{(1)}$={0.05, 0.20, 0.50, 0.20, 0.05}, $P^{(2)}$={0.70, 0.20, 0.05, 0.03, 0.02}, $P^{(3)}$={0.10, 0.50, 0.30, 0.05, 0.05}. Demand in each period follows a normal distribution with μ=100 and σ=30, and the total annual demand is given as R=10 000. The cost coefficients are given as c_{h}=5 and c_{s}=10, resulting in the EOQ Q^*=200. The allowable shortage level is set as $\alpha_0=0.05$. Parameters in the changeable demand distribution are assumed to be (μ_t,σ_t^2)=(100, 30^2) for the first and third quarters, (130, 40^2) for the second quarter, and (70, 20^2) for the fourth quarter.

In shortage rate satisfaction approach, Fig. 1 shows the robust optimal reorder point $s^*(t)$ with respect to time t under scenarios ($L^{(k)}$, $P^{(i)}$), k=1, 2, 3, i=1, 2, 3, where the value of parameters in $G(\alpha)$ and $G_k(\xi)$ are set as $\nu=0.0$ and ζ=0.0. From Fig. 2, it is obvious that the robust optimal reorder point $s^*(t)$ is smaller than the reorder point obtained through the maximum lead time method (Max L), and is larger than the other methods (Exp L, Exp α). In the Max L method, shortage rate is small enough to neglect in any scenario as shown in Figs. 3 and 4, but the excess on-hand inventory caused by the largest reorder point may lead to dead stock. On the other hand, the other methods result in too large shortage rate to be allowed under scenarios with large lead times as shown in Figs. 3 and 4, although the inventory level is curbed to a lower level. The robust optimal reorder point provides sufficiently small shortage rate controlled by the basic CDF and the inventory level is also kept to a moderate level. These features are observed in all sets of scenarios ($L^{(k)}$, $P^{(i)}$), k=1, 2, 3, i=1, 2, 3, especially in case of ($L^{(k)}$, $P^{(2)}$), k=1, 2, 3, where the difference among these methods becomes the largest.

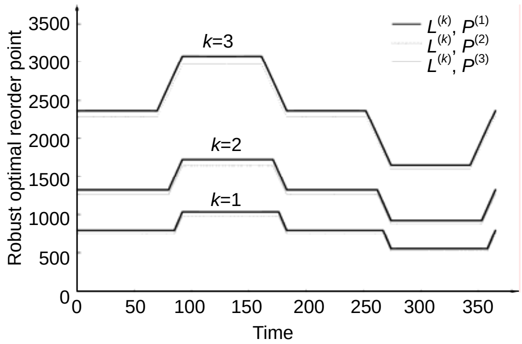

Fig. 1 Robust optimal reorder point $s^*(t)$

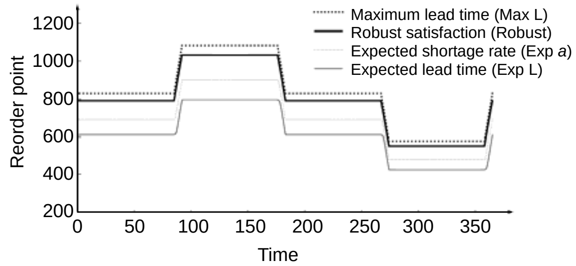

Fig. 2 Comparison of reorder points in case of ($L^{(1)}$, $P^{(1)}$) in the shortage rate satisfaction approach

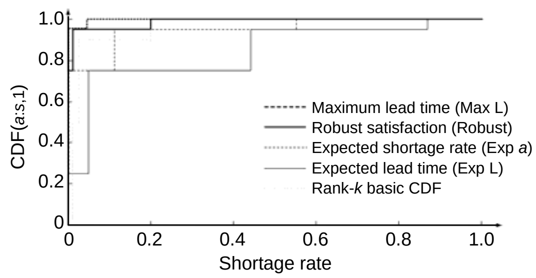

Fig. 3 CDF(α:s,1) for each reorder point s derived through each method in case of ($L^{(1)}$, $P^{(1)}$)

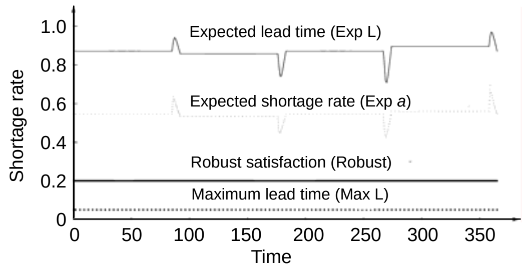

Fig. 4 Shortage rate $\alpha_5(s, t)$ in case of ($L^{(1)}$, $P^{(1)}$)

In the backorder cost minimization approach, we can find a property that the robust optimal reorder point is very close to reorder point obtained through the expected closeness minimization method (Exp ε) as shown in Fig. 5 with ν=0.15 and ζ=0.4, but it depends on parameters of ν and ζ and on a set of scenarios. In any event, the robust optimal reorder point exists between reorder points obtained by the min-max and expected value minimization methods.

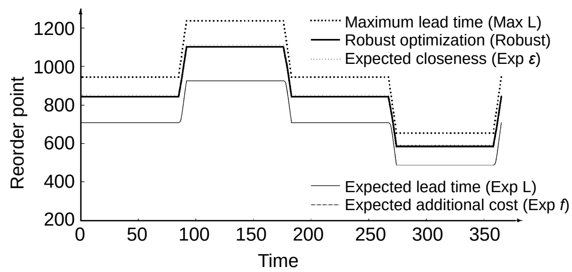

Fig. 5 Comparison of reorder points in case of ($L^{(1)}$, $P^{(1)}$) in the cost minimization approach

6 Conclusions

Uncertainties of lead times and demand distribution in fixed order quantity systems were expressed as a set of scenarios with occurrence probabilities and a pattern of changeable mean and variance, respectively. To manage these uncertainties, a "dynamic robust optimal reorder point function based on allowable shortage rate" was introduced according to stochastic ordering. This concept was extended to the case of minimizing the additional backorder cost. Some numerical examples demonstrated some features of these dynamic robust reorder point functions compared to that of the reorder points obtained through existing methods based on either a min-max optimization method or an expected value optimization method.

Since the proposed dynamic robust optimal reorder point functions depend on the preference structure of the decision-maker, sensitivity with respect to the preference structure should be analyzed for determining the final reorder point function, which is for future work.

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