Journal of Zhejiang University-SCIENCE A (Applied Physics & Engineering)
ISSN 1673-565X (Print); ISSN 1862-1775 (Online)
www.jzus.zju.edu.cn; www.springerlink.com
E-mail: jzus@zju.edu.cn

Powell inversion mechanical model of foundation parameters with generalized Bayesian theory*

Jian ZHANG[†1], Chu-wei ZHOU[1], Chao JIA[2], Jing LIN[3]

([1]*Department of Mechanics and Structural Engineering, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China*)

([2]*School of Civil Engineering, Shandong University, Jinan 250061, China*)

([3]*Institute of Structural Engineering, Tongji University, Shanghai 200092, China*)

[†]E-mail: zjmech@163.com

Received June 14, 2016; Revision accepted Oct. 26, 2016; Crosschecked June 26, 2017

Abstract: The inversion mechanical model of foundation parameters based on Powell optimizing theory was studied with generalized Bayesian theory. First, the generalized Bayesian objective function for foundation parameters was deduced with maximum likelihood theory. Then, the expectation expression and the covariance expression of the foundation parameters were obtained. After selecting the Winkler foundation as representative, the governing differential equations of the typical foundation were derived. With the orthogonal series transform method, the Fourier closed form solution of a moderately-thick plate on the Winkler foundation was achieved. After the optimal step length was determined with the quadratic parabolic interpolation method, the Powell inversion mechanical model of foundation parameters was resolved, and the corresponding inversion procedure was completed. Through particular example analysis, the highlight is that the Powell inversion mechanical model of foundation parameters with generalized Bayesian theory is correct and the derived Powell inversion model has universal significance, which can be applied in other kinds of foundation parameters. Besides, the Powell inversion iterative processes of foundation parameters have excellent numerical stability and convergence. The Powell optimizing theory is unconcerned with the partial derivatives of systematic responses to foundation parameters, which undoubtedly has a satisfying iterative efficiency compared with the available Kalman filtering or conjugate gradient inversion of the foundation parameters. The generalized Bayesian objective function can synchronously take the stochastic property of systematic parameters and systematic responses into account.

Key words: Powell inversion; Mechanical model; Foundation parameter; Bayesian objective function; Stochastic property
http://dx.doi.org/10.1631/jzus.A1600440 **CLC number:** TU451

1 Introduction

In geotechnical engineering, the necessary mechanical parameters used in geotechnical analysis include typical kinds of fuzzy, indeterminate, and discrete properties (Kim *et al.*, 2015; Fathi *et al.*, 2016). How to evaluate the selected parameters efficiently becomes a fundamental task. Up to now, when the plate on the foundation or on the road surface is tackled, the general soil medium models include the Winkler foundation, the Pasternac foundation, the Hetenyi foundation, the Reissner foundation models, the elastic half sphere and finite depth elastic compression layer models, etc. (Chen *et al.*, 2015; Ong and Choo, 2016). Among these models, the Winkler analytical model has been widely used in engineering because it precisely simulates the factual working status of the foundation and is comparatively convenient in mathematical manipulation (Zhao, 2007; Ching *et al.*, 2016). In engineering, some parameters such as the thickness of the foundation plate and the concrete strength can be directly achieved through the

* Project supported by the Fundamental Research Funds for the Central Universities of China (No. NS2014003)
ORCID: Jian ZHANG, http://orcid.org/0000-0003-2457-9558; Chao JIA, http://orcid.org/0000-0002-2448-894X

spot experiment while some parameters such as the foundation parameter cannot be concisely gained. The parameters are sometimes determined by experiences which cannot take the influence of stochastic factors into account. Thus, if the parameters can be accurately inverted with systematic responses of different measuring times and spatial spots, it helps to evaluate and forecast performance of the foundation more precisely (Xin *et al.*, 2014; Zong *et al.*, 2014; Jia and Chi, 2015; Li *et al.*, 2015). The parameters might be determined through some specific methods including parameter inversion methods (Sarp Arsava *et al.*, 2016). In recent years, there has been some research emphasis on the inversion problems and some achievement in dealing with the inversion of the Winkler foundation parameter (Xie, 2011). Nevertheless, the conjugate gradient theory has been successfully used in parameter inversion (Zhao, 2007), and the Kalman filtering theory, which has the advantages of auto revision and auto optimization, has also been successfully applied in the inversion of the foundation parameter (Zhang *et al.*, 2008). However, it is unfortunately a deficiency that in both the Kalman filtering theory and the conjugate gradient theory, the partial derivatives of the systematic responses to the Winkler foundation parameters must be repeatedly computed in iterative processes, which will consequentially lead to lower computational efficiency and error accumulation. As an improvement, the Powell optimizing method is not concerned with the perplexing partial derivatives. In addition, it has been proved that the Pasternac foundation model as well as some other foundation models are undoubtedly more precise than the Winkler foundation model (Bouderba *et al.*, 2013; Bousahla *et al.*, 2014; Meziane *et al.*, 2014; Zidi *et al.*, 2014). The main point of this study is how to derive the Powell optimizing inversion mechanical model of foundation parameters with generalized Bayesian theory, and in order to make comparisons with other research results (Zhao, 2007; Zhang *et al.*, 2008) in a convenient manner, the concise Winkler foundation model is chosen the same as Zhao (2007) and Zhang *et al.* (2008). Certainly, the following Powell optimizing inversion model based on generalized Bayesian theory has universal significance for different kinds of foundation parameters and only the corresponding foundation model should be considered.

Thus, in this paper, the generalized Bayesian objective function for foundation parameters is deduced with maximum likelihood theory. Then, the expectation expression and the covariance expression of the foundation parameters are obtained. With the Fourier closed form solution for the foundation, the Powell inversion mechanical model of foundation parameters is resolved and some typical examples are analyzed in detail.

2 Generalized Bayesian objective function of foundation parameters

During the process of Powell optimizing inversion of the soil medium model, the foundation parameters can be treated as random variables, which are noted as the random vector $\boldsymbol{X}=[x_1\ x_2\ \dots\ x_m]^{\mathrm{T}}$ (m is the dimension of the vector $\boldsymbol{X}$) to carry the parameter inversion into execution. In inversion research studies such as evaluation of models performance of fracture toughness of polymeric particle nanocomposites, detection of material interfaces in piezoelectric structures and detection of flaws in piezoelectric structures using extended finite element method (FEM) (Nanthakumar *et al.*, 2013; 2016; Hamdia *et al.*, 2016), Bayesian theory is widely applied because one of its superior properties lies in taking the stochastic property of systematic parameters and systematic responses into account efficiently. However, a much more efficient Bayesian objective function is derived below. From Bayesian theory, it can be noted:

$$f(\boldsymbol{X}\,|\,\boldsymbol{W}^*)=\frac{f(\boldsymbol{W}^*\,|\,\boldsymbol{X})f(\boldsymbol{X})}{f(\boldsymbol{W}^*)}, \tag{1}$$

where $f(\boldsymbol{X})$ is the priori information distribution, $f(\boldsymbol{W}^*|\boldsymbol{X})$ is the conditional distribution of the systematic response, $f(\boldsymbol{W}^*)$ is the systematic response distribution, and $f(\boldsymbol{X}|\boldsymbol{W}^*)$ is the posterior information distribution. It is presumed that the foundation parameters $\boldsymbol{X}$ conform to a Gaussian normal distribution, and then the priori information distribution $f(\boldsymbol{X})$ is expressed:

$$f(\boldsymbol{X})=(2\pi)^{-m/2}\left|\boldsymbol{C}_X\right|^{-1/2} \cdot\exp\left[-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{X}_0)^{\mathrm{T}}\boldsymbol{C}_X^{-1}(\boldsymbol{X}-\boldsymbol{X}_0)\right], \tag{2}$$

where $\boldsymbol{X}_0$ and $\boldsymbol{C}_X$ are respectively the expectation vector and covariance matrix of foundation parameters $\boldsymbol{X}$.

In engineering practice, the systematic responses at the testing nodes must be measured several times and the measured systematic response data $\boldsymbol{W}_i^*$ from each measure are all extracted from the total data $\boldsymbol{W}^*$. If the ordinary Bayesian objective function is set up to inverse parameters, there is much repeated and worthless work. Thus, the generalized Bayesian objective function of the foundation parameters is deduced. The joint density function of $\boldsymbol{W}_i^*$ is $\prod_{i=1}^{n} f(\boldsymbol{W}_i^* \mid \boldsymbol{X})$, where n is the number of times of the measured systematic response data. From maximum likelihood theory, it can be obtained:

$$f(\boldsymbol{W}^* \mid \boldsymbol{X}) = (2\pi)^{-mn/2} \prod_{i=1}^{n} \left| \boldsymbol{C}_{\boldsymbol{W}_i^*} \right|^{-1/2} \cdot \exp\left[-\frac{1}{2} \sum_{i=1}^{n} (\boldsymbol{W}_i^* - \boldsymbol{W}_i)^{\mathrm{T}} \boldsymbol{C}_{\boldsymbol{W}_i^*}^{-1} (\boldsymbol{W}_i^* - \boldsymbol{W}_i) \right], \tag{3}$$

where $\boldsymbol{W}_i=\boldsymbol{W}_i(\boldsymbol{X})$ is the systematic response vector of the computational results. Substituting Eqs. (2) and (3) into Eq. (1), the generalized Bayesian objective function J can be derived as

$$J = \sum_{i=1}^{n} (\boldsymbol{W}_i^* - \boldsymbol{W}_i)^{\mathrm{T}} \boldsymbol{C}_{\boldsymbol{W}_i^*}^{-1} (\boldsymbol{W}_i^* - \boldsymbol{W}_i) + (\boldsymbol{X} - \boldsymbol{X}_0)^{\mathrm{T}} C_X^{-1} (\boldsymbol{X} - \boldsymbol{X}_0). \tag{4}$$

The generalized Bayesian objective function J in Eq. (4) is utilized in the Powell optimizing inversion. In order to attain the variance inversion result of the foundation parameters $\boldsymbol{X}$, from Eq. (4) the partial differentiations of the generalized objective function J to the foundation parameters $\boldsymbol{X}$ are expressed:

$$\frac{\partial J}{\partial \boldsymbol{X}} = \sum_{i=1}^{n} 2 \left(\frac{\partial \boldsymbol{W}_i}{\partial \boldsymbol{X}} \right)^{\mathrm{T}} C_{\boldsymbol{W}_i^*}^{-1} (\boldsymbol{W}_i - \boldsymbol{W}_i^*) + 2\boldsymbol{C}_X^{-1} (\boldsymbol{X} - \boldsymbol{X}_0). \tag{5}$$

When $\boldsymbol{W}_i(\boldsymbol{X})$ is submitted with a Taylor formula expansion at the expectation point $\boldsymbol{X}$ and only the zero- and first-order items are reserved, it can be derived as

$$\boldsymbol{W}_i(\boldsymbol{X}) = \boldsymbol{W}_i(\overline{\boldsymbol{X}}) + \boldsymbol{S}_i(\overline{\boldsymbol{X}})(\boldsymbol{X} - \overline{\boldsymbol{X}}), \tag{6}$$

where the sensitivity matrix $\boldsymbol{S}_i(\overline{\boldsymbol{X}}) = \left. \frac{\partial \boldsymbol{W}_i}{\partial \boldsymbol{X}} \right|_{\boldsymbol{X}=\overline{\boldsymbol{X}}}$. Substituting Eq. (6) into Eq. (5) gives

$$\frac{\partial J}{\partial \boldsymbol{X}} = \sum_{i=1}^{n} 2\boldsymbol{S}_i^{\mathrm{T}} \boldsymbol{C}_{W_i^*}^{-1} (\overline{\boldsymbol{W}_i} + \boldsymbol{S}_i \boldsymbol{X} - \boldsymbol{S}_i \overline{\boldsymbol{X}} - \boldsymbol{W}_i^*) + 2\boldsymbol{C}_X^{-1} (\boldsymbol{X} - \boldsymbol{X}_0), \tag{7}$$

where $\overline{\boldsymbol{W}_i} = \boldsymbol{W}_i(\overline{\boldsymbol{X}})$. When the generalized Bayesian objective function J reaches the minimum value, Eq. (7) equals zero. Then,

$$\left(\sum_{i=1}^{n} \boldsymbol{S}_i^{\mathrm{T}} \boldsymbol{C}_{W_i^*}^{-1} \boldsymbol{S}_i + \boldsymbol{C}_X^{-1} \right) \boldsymbol{X} = \sum_{i=1}^{n} \boldsymbol{S}_i^{\mathrm{T}} \boldsymbol{C}_{W_i^*}^{-1} (\boldsymbol{W}_i^* - \overline{\boldsymbol{W}_i} + \boldsymbol{S}_i \overline{\boldsymbol{X}}) + \boldsymbol{C}_X^{-1} \boldsymbol{X}_0. \tag{8}$$

Assuming $\boldsymbol{H} = \sum_{i=1}^{n} \boldsymbol{S}_i^{\mathrm{T}} \boldsymbol{C}_{W_i^*}^{-1} \boldsymbol{S}_i + \boldsymbol{C}_X^{-1}$ and $\boldsymbol{M}= \boldsymbol{H}^{-1} \left[\boldsymbol{S}_1^{\mathrm{T}} \boldsymbol{C}_{W_1^*}^{-1}, \boldsymbol{S}_2^{\mathrm{T}} \boldsymbol{C}_{W_2^*}^{-1}, \cdots, \boldsymbol{S}_n^{\mathrm{T}} \boldsymbol{C}_{W_n^*}^{-1} \right]$, from Eq. (8) the inversion value $\hat{\boldsymbol{X}}$ of the foundation parameters $\boldsymbol{X}$ can be written as

$$\hat{\boldsymbol{X}} = (\boldsymbol{I} - \boldsymbol{MS})\boldsymbol{X}_0 + \boldsymbol{MW}^* - \boldsymbol{M}(\overline{\boldsymbol{W}} - \boldsymbol{S}\overline{\boldsymbol{X}}), \tag{9}$$

where $\boldsymbol{I}$ is a unit matrix and $\boldsymbol{W}^*=[\boldsymbol{W}_1^*, \boldsymbol{W}_2^*, \cdots, \boldsymbol{W}_n^*]^{\mathrm{T}}$, and $\boldsymbol{W}_i^*$ is the vector of the measured systematic response data of the ith time. $\overline{\boldsymbol{W}} = \left[\overline{\boldsymbol{W}_1}, \overline{\boldsymbol{W}_2}, \cdots, \overline{\boldsymbol{W}_n} \right]^{\mathrm{T}}$, where $\overline{\boldsymbol{W}_i}$ is the systematic response vector of computational data of the ith time at the expectation point $\boldsymbol{X}$. $\boldsymbol{S}=[\boldsymbol{S}_1, \boldsymbol{S}_2, \cdots, \boldsymbol{S}_n]$, where $\boldsymbol{S}_i$ is the sensitivity matrix of the measured systematic responses of the ith time. Assuming the priori information $\boldsymbol{X}_0$ of the foundation parameters $\boldsymbol{X}$ is unrelated to the measured systematic response data $\boldsymbol{W}_i^*$, from Eq. (9) the variance of $\hat{\boldsymbol{X}}$ can be written as

$$\boldsymbol{C}_{\hat{X}} = \left[\boldsymbol{I} - \boldsymbol{MS} \right] \boldsymbol{C}_X \left[\boldsymbol{I} - \boldsymbol{MS} \right]^{\mathrm{T}} + \boldsymbol{MC}_{W^*} \boldsymbol{M}^{\mathrm{T}}, \tag{10}$$

where $\boldsymbol{C}_{W^*}=\mathrm{diag}(\boldsymbol{C}_{W_1^*},\boldsymbol{C}_{W_2^*},\cdots,\boldsymbol{C}_{W_n^*})$ and $\boldsymbol{C}_{W_i^*}$ is the covariance matrix of the measured systematic response data of the ith time. Using the non-singularity property of $\boldsymbol{C}_X$ and $\boldsymbol{C}_{W^*}$, Eq. (10) can be transformed into the summation form of

$$\boldsymbol{C}_{\hat{X}}=\left[\boldsymbol{C}_X^{-1}+\sum_{i=1}^{n}\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{C}_{W_i^*}^{-1}\boldsymbol{S}_i\right]^{-1}. \tag{11}$$

3 Fourier closed form solution for the foundation model

In Eq. (4), the systematic response vector $\boldsymbol{W}_i$ of the computational results must be grasped ahead of inversion iterative analysis. The Pasternac foundation model and some other foundation models are considered to be more precise than the Winkler foundation model of moderately-thick plate supposition (Belabed *et al.*, 2014; Hebali *et al.*, 2014; Bourada *et al.*, 2015; Hamidi *et al.*, 2015; Yahia *et al.*, 2015; Bennoun *et al.*, 2016). Although in order to make comparisons with the achievements conveniently (Zhao, 2007; Zhang *et al.*, 2008), the Winkler foundation model is deliberately chosen, it will not affect the research emphasis of how to derive the Powell optimizing inversion mechanical model of foundation parameters with generalized Bayesian theory.

The main suppositions is that the normal line of the middle surface before deformation remains a straight line after deformation, but it may be not vertical to the middle surface, which means the transversal shearing deformation effect is admitted and the stress vertical to the middle surface can be ignored (Zhang *et al.*, 2008; Xie, 2011). The displacement field of the moderately-thick plate in Fig. 1 can be expressed:

$$\begin{cases} u(x,y,z)=z\theta_x(x,y), \\ v(x,y,z)=z\theta_y(x,y), \\ w(x,y,z)=w(x,y), \end{cases} \tag{12}$$

where x, y are the rectangular coordinates of the plate plane and z is the thickness coordinate. u, v, and w are the displacements along the x, y, and z directions, respectively. θ_x and θ_y are the rotational displacements of the normal lines of x-z plane and y-z plane, respectively. The equations between generalized stress and strain of the moderately-thick plate are derived as

$$\boldsymbol{\sigma}_{\mathrm{f}}=\boldsymbol{D}_{\mathrm{f}}\boldsymbol{\varepsilon}_{\mathrm{f}}, \tag{13}$$

$$\boldsymbol{\sigma}_{\mathrm{s}}=\boldsymbol{D}_{\mathrm{s}}\boldsymbol{\varepsilon}_{\mathrm{s}}, \tag{14}$$

where $\boldsymbol{\sigma}_{\mathrm{f}}=[\boldsymbol{M}_x\ \ \boldsymbol{M}_y\ \ \boldsymbol{M}_{xy}]^{\mathrm{T}}$ and $\boldsymbol{\varepsilon}_{\mathrm{f}}=\left[\frac{\partial\theta_x}{\partial x}\ \ \frac{\partial\theta_y}{\partial y}\ \ \frac{\partial\theta_x}{\partial y}+\frac{\partial\theta_y}{\partial x}\right]^{\mathrm{T}}$ are defined as generalized bending stresses and generalized bending strains, respectively. $\boldsymbol{\sigma}_{\mathrm{s}}=[\boldsymbol{Q}_x\ \ \boldsymbol{Q}_y]^{\mathrm{T}}$ and $\boldsymbol{\varepsilon}_{\mathrm{s}}=\left[\theta_x+\frac{\partial w}{\partial x}\ \ \theta_y+\frac{\partial w}{\partial y}\right]^{\mathrm{T}}$ are defined as generalized shearing stresses and generalized shearing strains, respectively. $\boldsymbol{D}_{\mathrm{f}}$ is defined as the bending elastic matrix which is connected to the elastic modulus, Poisson's ratio, and the thickness of the moderately-thick plate. $\boldsymbol{D}_{\mathrm{s}}$ is defined as the shearing elastic matrix which is connected to the shearing elastic modulus, the thickness, and the shearing correction coefficient of the moderately-thick plate. When the forces including surface tension are neglected, the governing differential equations for the moderately-thick plate on the Winkler foundation can be written as

$$\begin{cases} \frac{\partial\boldsymbol{Q}_x}{\partial x}+\frac{\partial\boldsymbol{Q}_y}{\partial y}+q-kw=0, \\ \frac{\partial\boldsymbol{M}_x}{\partial x}+\frac{\partial\boldsymbol{M}_{xy}}{\partial y}-\boldsymbol{Q}_x=0, \\ \frac{\partial\boldsymbol{M}_{xy}}{\partial x}+\frac{\partial\boldsymbol{M}_y}{\partial y}-\boldsymbol{Q}_y=0, \end{cases} \tag{15}$$

where k is the Winkler foundation parameter, and q is the load density in the z direction. The boundary conditions of the plate are considered as simply supported. Directly solving the governing differential Eq. (15) is very difficult and therefore the Fourier transform method is used. Substituting Eqs. (13) and (14) into Eq. (15), the important governing differential Eq. (15) can be turned into the differential equations with the variables named θ_x, θ_y, and w. Then, the three variables are expanded in the form of multiple Fourier orthogonal series, and they can be obtained as

$$\begin{cases} \theta_x = \sum_m \sum_n A_{mn} \cos\alpha_m x \sin\beta_n y, \\ \theta_y = \sum_m \sum_n B_{mn} \sin\alpha_m x \cos\beta_n y, \\ w = \sum_m \sum_n C_{mn} \sin\alpha_m x \sin\beta_n y, \end{cases} \quad (16)$$

where m and n are the multiple Fourier series terms; α_m and β_n are the Fourier expanded coefficients; A_{mn}, B_{mn}, and C_{mn} are the undetermined coefficients. Based on the orthogonal series transform method, substituting Eq. (16) into the achieved differential equations with the variables θ_x, θ_y, and w of the moderately-thick plate on the Winkler foundation, the discussed differential equations can be transformed into the linear algebraic equations with the variables A_{mn}, B_{mn}, and C_{mn}, which can be easily solved. After the undetermined coefficients A_{mn}, B_{mn}, and C_{mn} are determined and then back substituted into Eq. (16), the displacement field functions θ_x, θ_y, and w for the Winkler foundation are obtained, which are viewed as the necessary computational systematic responses during Powell optimizing inversion of the foundation parameters. It is convincing that the Powell optimizing inversion model based on generalized Bayesian theory has universal significance for other different kinds of foundation parameters if the discussed Eqs. (13)–(15) are properly substituted.

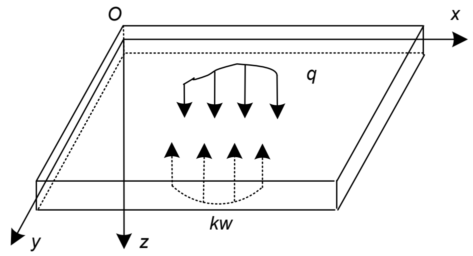

Fig. 1 Moderately-thick plate on the foundation

4 Powell stochastic inversion mechanical model for the foundation parameters

4.1 Powell theory

The available optimizing methods can be mainly assorted into two kinds: the first is the direct searching method such as the Powell method and the complex method, and the second is the gradient optimizing method such as the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method and the conjugate gradient method. The gradient optimizing method incessantly changes the matrix scale to produce new searching directions during the optimizing iterative processes. The gradient optimizing method has to determine the partial differentiations of objective function to systematic parameters, which inevitably leads to error accumulation, as in the Kalman filtering method. However, the direct searching method is independent of the partial differentiations of the objective function to systematic parameters and is especially applicable for the objective function without analytic expression shown in Eq. (4). Among the available direct searching methods, Powell theory can be regarded as an effective method (Zhang *et al.*, 2012), which uses a 1D searching method to produce the optimal orientations in the parallel directions from different initial searching points.

Combined with generalized Bayesian theory, the Powell stochastic inversion flowchart is shown in Fig. 2 and the inversion steps of the foundation parameters are presented as follows:

1. Select the initial values $\boldsymbol{X}^{0,0}$ of the foundation parameters $\boldsymbol{X}$ and the initial searching direction $\boldsymbol{d}^{0,i}$ and let $\boldsymbol{d}^{0,i}=\boldsymbol{e}_i$, where $\boldsymbol{e}_i$ is the unit coordinate vector, the variable i=1, 2, ···, m, and m is the dimension of the foundation parameters $\boldsymbol{X}$. Take the convergence criteria ε_1 and ε_2, then let the iterative variable k=0.

2. Begin with the foundation parameters $\boldsymbol{X}^{k,0}$ and complete 1D searching in turn along the optimizing direction $\boldsymbol{d}^{k,i}$ (i=1, 2, ···, m), which means that $J(\boldsymbol{X}^{k,i})=\min_h J(\boldsymbol{X}^{k,i-1}+h\boldsymbol{d}^{k,i})$, where h is the step length. Then, the foundation parameter series $\boldsymbol{X}^{k,i}$ are obtained.

3. From the Bayesian objective function Eq. (4), Eq. (17) is calculated as follows and the subscript l is subsequently recorded:

$$\Delta_l^k = \max_{1\le i\le m} \Delta_i^k = \max_{1\le i\le m}\left[J(\boldsymbol{X}^{k,i-1}) - J(\boldsymbol{X}^{k,i})\right]. \quad (17)$$

4. Begin with the foundation parameters $\boldsymbol{X}^{k,m}$ and implement a 1D search along the search direction $\boldsymbol{d}^k=\boldsymbol{X}^{k,m}-\boldsymbol{X}^{k,0}$, which means that $J(\boldsymbol{X}^{k+1,0})=\min_h J(\boldsymbol{X}^{k,m}+h\boldsymbol{d}^k)$, and then the foundation parameters $\boldsymbol{X}^{k+1,0}$ are achieved.

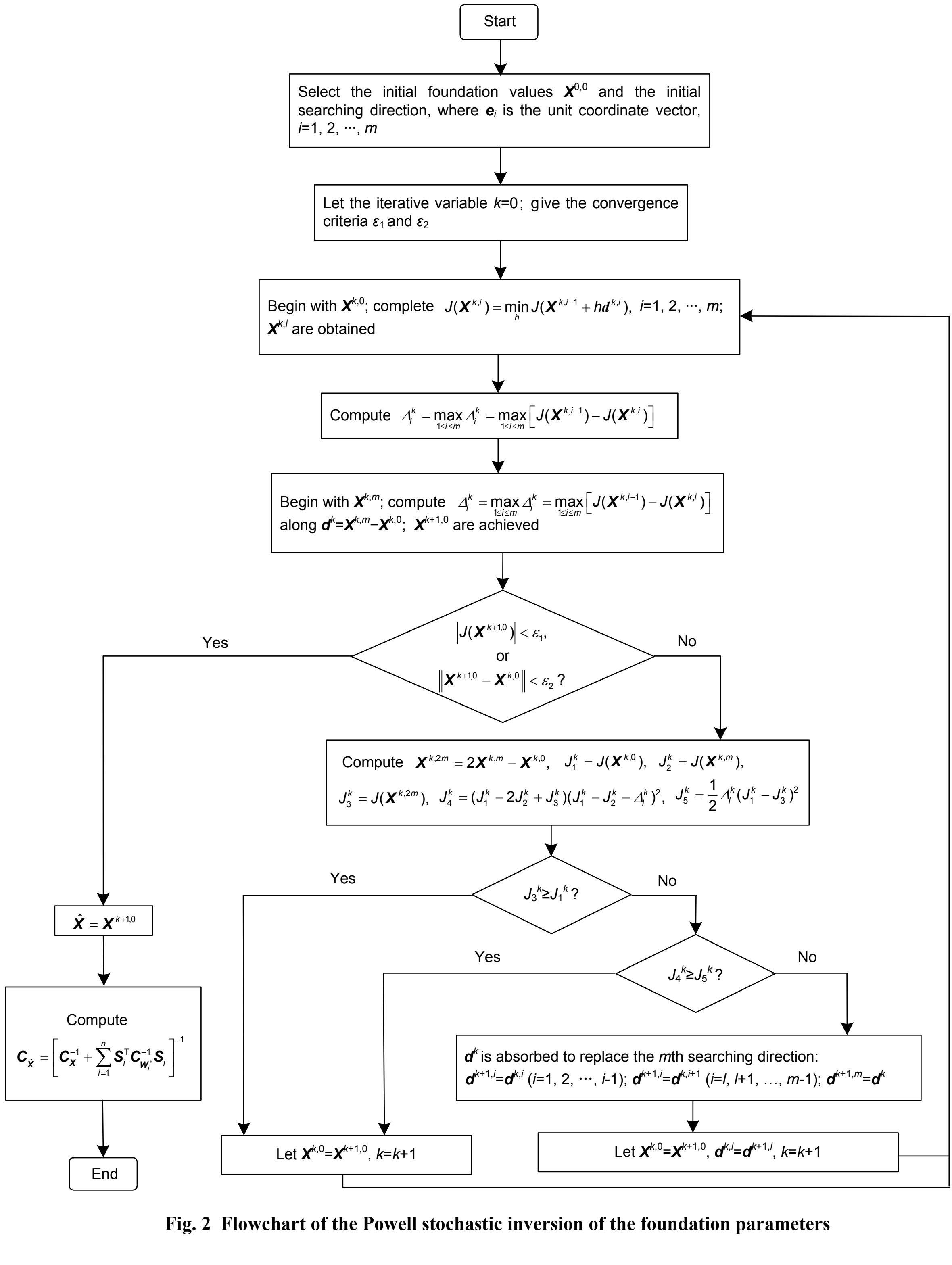

Fig. 2 Flowchart of the Powell stochastic inversion of the foundation parameters

5. The convergence judgment Eq. (18) is completed to judge whether the Powell iteration is convergent or not:

$$\left|J(X^{k+1,0})\right|<\varepsilon_1, \quad \left\|X^{k+1,0}-X^{k,0}\right\|_2<\varepsilon_2. \tag{18}$$

If ε_1 or ε_2 is satisfied, the Powell iteration is convergent and the inversion results of the foundation parameters $\boldsymbol{X}$ are $\hat{\boldsymbol{X}}=\boldsymbol{X}^{k+1,0}$. The iteration is terminated and goes to step 10. Otherwise, continue the next step.

6. This step is the judgment computation on whether the searching direction $\boldsymbol{d}^k$ is absorbed. Supposed that $\boldsymbol{X}^{k,2m}=2\boldsymbol{X}^{k,m}-\boldsymbol{X}^{k,0}$, then the following equations result from Eqs. (4) and (17):

$$\begin{aligned} &J_1^k=J(\boldsymbol{X}^{k,0}),\\ &J_2^k=J(\boldsymbol{X}^{k,m}),\\ &J_3^k=J(\boldsymbol{X}^{k,2m}),\\ &J_4^k=(J_1^k-2J_2^k+J_3^k)(J_1^k-J_2^k-\Delta_l^k)^2,\\ &J_5^k=\frac{1}{2}\Delta_l^k(J_1^k-J_3^k)^2. \end{aligned} \tag{19}$$

If $J_3^k\geq J_1^k$, it is useless for absorbing the searching direction $\boldsymbol{d}^k$. Therefore, the available searching direction is unchanged and go into step 9. Otherwise, continue the next step.

7. If $J_4^k\geq J_5^k$, the available searching direction is unchanged and go into step 9. Otherwise, the calculation defined as absorbing the searching direction $\boldsymbol{d}^k$ is completed, in which the searching direction $\boldsymbol{d}^{k,l}$ in the available searching directions is deleted and the searching direction $\boldsymbol{d}^k$ is absorbed to replace the mth searching direction:

$$\begin{aligned} &\boldsymbol{d}^{k+1,i}=\boldsymbol{d}^{k,i},\quad i=1,2,\cdots,l-1,\\ &\boldsymbol{d}^{k+1,i}=\boldsymbol{d}^{k,i+1},\quad i=l,l+1,\cdots,m-1,\\ &\boldsymbol{d}^{k+1,m}=\boldsymbol{d}^{k}. \end{aligned} \tag{20}$$

8. Let $\boldsymbol{X}^{k,0}=\boldsymbol{X}^{k+1,0}$, $\boldsymbol{d}^{k,i}=\boldsymbol{d}^{k+1,i}$, $k=k+1$, and go back to step 2 to continue iterating.

9. Let $\boldsymbol{X}^{k,0}=\boldsymbol{X}^{k+1,0}$, $k=k+1$ and go back to step 2 to continue iterating.

10. From Eq. (11), the covariance $\boldsymbol{C}_{\hat{X}}$ of the foundation parameters $\boldsymbol{X}$ is achieved.

4.2 Determination of the optimal step length

Searching of the optimal step length h is necessary in the step 2 and step 4 of the Powell stochastic inversion steps of the Winkler foundation parameters, which is a fairly complicated problem in the analysis of parameter inversion (Zhang *et al.*, 2008; 2012). In the available studies, the 1D searching method is mainly referred to as the golden section method, the quadratic parabolic interpolation method, etc. Among these methods, the quadratic parabolic interpolation method has much satisfying computational efficiency, which can automatically determine the span of the optimal step length h and then optimize the step length. The main steps include:

1. Determination of the span where the optimal step length h lies. Assume the initial step length h_1 and a step length increment h_0 and set $h_2=h_0+h_1$. If $J(h_1)\geq J(h_2)$, the step length increment is defined as $h_k=h_{k-1}+2^{k-2}h_0$, where $k\geq 3$ is calculated. The calculation does not cease until $J(h_k)\geq J(h_{k-1})$. If not, the other step length increment is defined as $h_k=h_{k-1}+2^{k-3}h_0$, where $k\geq 3$ is calculated. Similarly, the calculation does not cease until $J(h_k)\geq J(h_{k-1})$. When the iterative calculation is completed, the range of the optimal step length h is obtained and noted as $[h_a, h_d]$.

2. Interpolation of the optimal step length h. On basis of the function extremum theory of the Bayesian objective function and through pertinent mathematical deductions, the optimal step length h is achieved:

$$\begin{aligned} &h=\frac{1}{2}\left(h_a+h_d-\frac{h_b}{h_c}\right),\\ &h_b=\frac{J(h_d)-J(h_a)}{h_d-h_a},\\ &h_c=\frac{1}{h_e-h_d}\left[\frac{J(h_e)-J(h_a)}{h_e-h_a}-h_b\right], \end{aligned} \tag{21}$$

where h_a and h_d are the values of the two endpoints of the span where the optimal step length h lies. h_b and h_c are both transitional variables. h_e is the middle point of the range $[h_a, h_d]$.

5 Example analysis

In order to put the Powell stochastic inversion of the foundation parameters with generalized Bayesian objective function into action, the inversion program named POWKER.for is compiled in which the subprogram of the Fourier closed form solution is employed. The reliability of the subprogram has been validated (Zhang *et al.*, 2008), and the validation is not repeated in this paper. Three different kinds of Winkler foundations in Fig. 3, on which three different concrete bearing plates are respectively placed, are taken into account. The dimension, elastic modulus, Poisson's ratio and the shearing correction coefficient of the plates, and both the true value and the coefficient of variation of the Winkler foundations are presented in Table 1. The uniform loads q_1 and q_2, which equal 200 N/cm^2 and 120 N/cm^2, respectively, along the z coordinate direction are applied on a concrete bearing plate on the first and second Winkler foundations, respectively, and the concentrated load P, which equals 4000 kN, is applied on concrete bearing plate on the third Winkler foundation. The four selected points on the concrete bearing plate in Fig. 4 are viewed as the displacement measurement spots and the displacement of each selected point is measured five times to take into account of measurement error. The displacement measurement data and the displacement standard variance data are listed in Table 2.

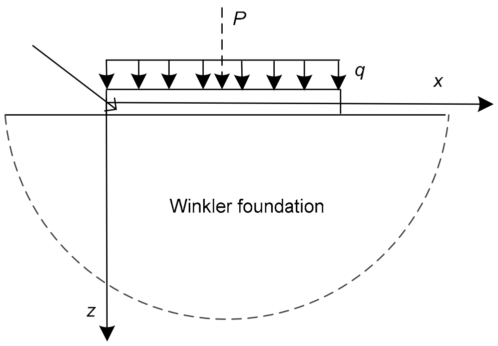

Fig. 3 Concrete bearing plate on Winkler foundation

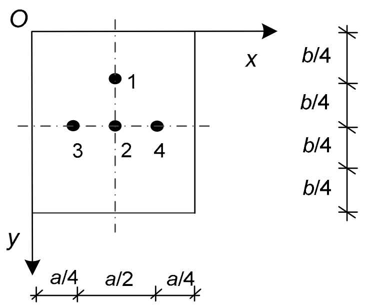

Fig. 4 The four selected points on a concrete bearing plate
a is the length dimension; b is the width dimension

Circumstance 1 In order to validate the reliability of this Powell optimizing inversion theory and the correctness of the completed analytical program, the priori information of the Winkler foundation parameter is firstly supposed to satisfy a precise condition. The priori information data of the three Winkler foundation parameters are 105, 80, and 65 N/cm^3, respectively. The two groups of the initial parameter data k_{10} and k_{20} are set as 150 and 30 N/cm^3, respectively. The convergence tolerances of ε_1 and ε_2 are both equal to 0.001. Combined with the displacement measurement and the displacement standard variance shown in Table 2 and from the developed inversion program, the Powell optimizing results are achieved in Fig. 5 and Table 3. From the results in Fig. 5 and Table 3, although different initial parameter data are set, the relative errors of the Winkler foundation parameters are far less than 5% and the Powell iterative inversion processes are steadily convergent to the true values, which indicates that this Powell optimizing inversion mechanical model is correct and the completed program is reliable. From the inversion analysis, unlike the Kalman filtering theory and the conjugate gradient theory (Zhao, 2007; Zhang *et al.*, 2008), the partial derivatives of the systematic responses to the Winkler foundation parameters are not involved in the Powell inversion processes, which

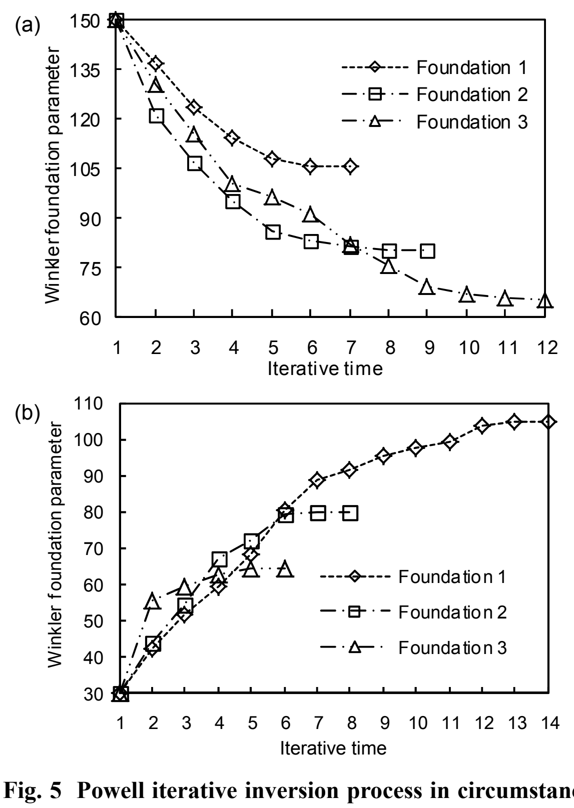

Fig. 5 Powell iterative inversion process in circumstance 1: (a) k_{10} is selected; (b) k_{20} is selected

Table 1 Dimensions of concrete bearing plate and other parameters

Foundation	Plate length, a (cm)	Plate width, b (cm)	Plate thickness, t (cm)	Elastic modulus, E (N/cm^2)	Poisson's ratio, μ	Shearing correction coefficient, γ	Winkler parameter, k (N/cm^3)	Variation coefficient, η
I	120	160	10	3.0×10^6	0.16	0.83	105	0.1
II	100	80	8	3.2×10^6	0.17	0.83	80	0.1
III	60	120	10	3.0×10^6	0.17	0.83	65	0.1

Table 2 Displacement measurement (w_1–w_5) and displacement standard variance (σ_{w1}–σ_{w5}) (unit: cm)

Foundation number	Selected points number	w_1	w_2	w_3	w_4	w_5	σ_{w1}	σ_{w2}	σ_{w3}	σ_{w4}	σ_{w5}
I	1	0.591	0.597	0.594	0.598	0.585	0.015	0.013	0.018	0.012	0.017
	2	0.793	0.798	0.797	0.788	0.792	0.022	0.026	0.028	0.020	0.025
	3	0.573	0.575	0.578	0.568	0.569	0.015	0.018	0.012	0.016	0.019
	4	0.571	0.574	0.567	0.575	0.577	0.013	0.016	0.015	0.011	0.018
II	1	0.143	0.141	0.139	0.146	0.138	0.011	0.008	0.009	0.006	0.012
	2	0.199	0.192	0.195	0.203	0.206	0.012	0.005	0.007	0.010	0.009
	3	0.146	0.144	0.151	0.150	0.142	0.010	0.006	0.008	0.004	0.007
	4	0.141	0.153	0.143	0.142	0.147	0.007	0.011	0.012	0.009	0.008
III	1	0.449	0.452	0.447	0.451	0.448	0.006	0.010	0.009	0.008	0.011
	2	1.156	1.159	1.153	1.161	1.153	1.041	1.045	1.036	1.039	1.043
	3	0.635	0.633	0.631	0.632	0.638	0.013	0.006	0.007	0.011	0.008
	4	0.640	0.632	0.635	0.637	0.636	0.011	0.008	0.009	0.008	0.010

Table 3 Powell optimizing results in circumstance 1

Group	Number of foundation	Final parameter (N/cm^3)	Iterative time	Relative error (%)	Convergent
k_{10}	1	105.895	7	0.852	ε_1
	2	80.305	9	0.381	ε_2
	3	65.448	12	0.689	ε_2
k_{20}	1	104.877	14	0.117	ε_2
	2	79.877	8	0.154	ε_2
	3	64.579	6	0.648	ε_1

consequentially leads to higher computational efficiency.

Circumstance 2 The priori information of the Winkler foundation parameter is supposed not to satisfy the precise condition, which means that the deviation degree of the priori information from the true parameter value exceeds 10%. From large quantities of trial computations, it is a pity that the Powell iterative processes are always divergent because the second item in Eq. (4) cannot converge in accordance with the set criteria in Eq. (18). In engineering practice, the priori information cannot be always precisely grasped and in this circumstance, the second item in Eq. (4) should be omitted and the discussed generalized Bayesian objective function is changed into the generalized Markov form (Zhang *et al.*, 2012). The rest of the data are the same as in circumstance 1 and from the developed inversion program, the Powell optimizing results are shown in Fig. 6 and Table 4. From the results in Fig. 6 and Table 4, the Powell iterative inversion processes are also steadily convergent to the true values. Compared

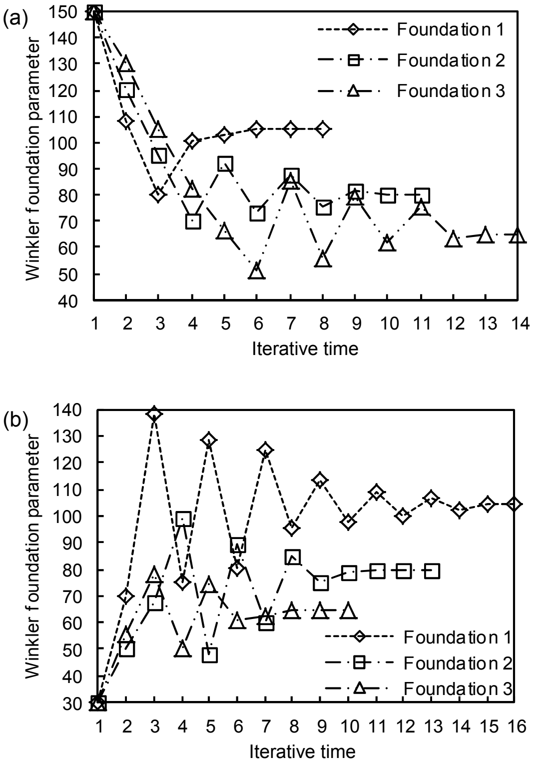

Fig. 6 Powell iterative inversion process in circumstance 2: (a) k_{10} is selected; (b) k_{20} is selected

with the results of generalized Bayesian objective function in circumstance 1, the iterative inversion times are larger, which indicates that the precise priori information of the Winkler foundation parameter accelerates the Powell iteration. In this circumstance, it is an improvement that the partial derivatives of the systematic responses to the Winkler foundation parameters are still not needed or considered.

Circumstance 3 The displacement measurement data are supposed to dissatisfy the precise condition, which means that the deviation degree of the measurement data from the true data in Table 2 is set as 10%. The priori information of the Winkler foundation parameter is assumed to satisfy the precise condition and the other data are similar to those in circumstance 1. From the developed inversion program, the Powell optimizing results are shown in Table 5. From the results in Table 5, when the displacement measurement data are imprecise, the Powell iterative processes of the Winkler foundation parameter are divergent and the relative errors are far more than 5% even though the number of the iterations reaches 60. The computational results indicate that the systematic responses must be accurately measured and provided, otherwise the Powell

Table 4 Powell optimizing results in circumstance 2

Group	Number of foundation	Final parameter (N/cm^3)	Iterative time	Relative error (%)	Convergent
k_{10}	1	104.880	8	0.114	ε_2
	2	80.120	11	0.150	ε_1
	3	64.679	14	0.494	ε_2
k_{20}	1	104.655	16	0.329	ε_1
	2	79.558	13	0.552	ε_2
	3	64.788	10	0.326	ε_2

Table 5 Powell optimizing results in circumstance 3

Group	Number of foundation	Final parameter (N/cm^3)	Iterative time	Relative error (%)	Convergent
k_{10}	1	129.324	60	23.165	No
	2	96.547	60	20.683	No
	3	84.338	60	29.751	No
k_{20}	1	86.373	60	17.740	No
	2	63.124	60	21.095	No
	3	50.281	60	22.645	No

optimizing inversion cannot be efficiently completed. Thus, the precision of the measured systematic response is of great importance.

6 Conclusions

The research emphasis in this paper is how to derive the Powell optimizing inversion mechanical model of foundation parameters with generalized Bayesian theory. From inversion model deduction and typical example analysis and compared with valuable references, the main conclusions can be achieved:

1. The Powell optimizing inversion theory of the foundation parameter based on generalized Bayesian theory is deduced and the Powell iterative inversion processes are steadily convergent to the true values, which indicates that the Powell inversion mechanical model is correct and reliable.

2. The derived Powell optimizing inversion model based on generalized Bayesian theory has universal significance for different kinds of foundation parameters and only the corresponding foundation model should be taken into consideration.

3. Unlike the Kalman filtering theory and the conjugate gradient theory, the partial derivatives of the systematic responses to the foundation parameters are not relevant to the Powell inversion process, which indicates that the Powell inversion theory is of higher computational efficiency.

4. Searching of the optimal step length is a fairly complicated problem in parameter inversion. The quadratic parabolic interpolation method derived in this paper can automatically determine the span of the optimal step length and then achieve the step length.

References

Belabed, Z., Houari, M.S.A., Tounsi, A., *et al*., 2014. An efficient and simple higher order shear and normal deformation theory for functionally graded material (FGM) plates. *Composites Part B: Engineering*, **60**:274-283. http://dx.doi.org/10.1016/j.compositesb.2013.12.057

Bennoun, M., Houari, M.S.A., Tounsi, A., 2016. A novel five variable refined plate theory for vibration analysis of functionally graded sandwich plates. *Mechanics of Advanced Materials and Structures*, **23**(4):423-431. http://dx.doi.org/10.1080/15376494.2014.984088

Bouderba, B., Houari, M.S.A., Tounsi, A., 2013. Thermomechanical bending response of FGM thick plates resting on Winkler-Pasternak elastic foundations. *Steel and Composite Structures*, **14**(1):85-104. http://dx.doi.org/10.12989/scs.2013.14.1.085

Bourada, M., Kaci, A., Houari, M.S.A., *et al*., 2015. A new simple shear and normal deformations theory for functionally graded beams. *Steel and Composite Structures*, **18**(2):409-423. http://dx.doi.org/10.12989/scs.2015.18.2.409

Bousahla, A.A., Houari, M.S.A., Tounsi, A., *et al*., 2014. A novel higher order shear and normal deformation theory based on neutral surface position for bending analysis of advanced composite plates. *International Journal of Computational Methods*, **11**(06):1350082. http://dx.doi.org/10.1142/S0219876213500825

Chen, B.H., Zhao, H.B., Ru, Z.L., *et al*., 2015. Probabilistic back analysis for geotechnical engineering based on Bayesian and support vector machine. *Journal of Central South University*, **22**(12):4778-4786. http://dx.doi.org/10.1007/s11771-015-3029-1

Ching, J., Phoon, K.K., Wu, S.H., 2016. Impact of statistical uncertainty on geotechnical reliability estimation. *Journal of Engineering Mechanics*, **142**(6):04016027. http://dx.doi.org/10.1061/(ASCE)EM.1943-7889.0001075

Fathi, A., Poursartip, B., Stokoe II, K.H., *et al*., 2016. Three-dimensional P- and S-wave velocity profiling of geotechnical sites using full-waveform inversion driven by field data. *Soil Dynamics and Earthquake Engineering*, **87**(8):63-81. http://dx.doi.org/10.1016/j.soildyn.2016.04.010

Hamdia, K.M., Zhuang, X.Y., He, P.F., *et al*., 2016. Fracture toughness of polymeric particle nanocomposites: evaluation of models performance using Bayesian method. *Composites Science and Technology*, **126**:122-129. http://dx.doi.org/10.1016/j.compscitech.2016.02.012

Hamidi, A., Houari, M.S.A., Mahmoud, S.R., *et al*., 2015. A sinusoidal plate theory with 5-unknowns and stretching effect for thermomechanical bending of functionally graded sandwich plates. *Steel and Composite Structures*, **18**(1):235-253. http://dx.doi.org/10.12989/scs.2015.18.1.235

Hebali, H., Tounsi, A., Houari, M.S.A., *et al*., 2014. New quasi-3D hyperbolic shear deformation theory for the static and free vibration analysis of functionally graded plates. *Journal of Engineering Mechanics*, **140**(2):374-383. http://dx.doi.org/10.1061/(ASCE)EM.1943-7889.0000665

Jia, Y.F., Chi, S.C., 2015. Back-analysis of soil parameters of the Malutang II concrete face rockfill dam using parallel mutation particle swarm optimization. *Computers and Geotechnics*, **65**:87-96. http://dx.doi.org/10.1016/j.compgeo.2014.11.013

Kim, J., Kim, J., Kim, M., *et al*., 2015. Prediction of ground load by performing back analysis using composite support model in concrete lining design. *KSCE Journal of Civil Engineering*, **19**(6):1697-1706. http://dx.doi.org/10.1007/s12205-015-1514-6

Li, T.C., Lyu, L.X., Zhang, S.L., *et al*., 2015. Development and application of a statistical constitutive model of damaged rock affected by the load-bearing capacity of damaged elements. *Journal of Zhejiang University-SCIENCE A (Applied Physics & Engineering)*, **16**(8):644-655. http://dx.doi.org/10.1631/jzus.A1500034

Meziane, M.A.A., Abdelaziz, H.H., Tounsi, A., 2014. An efficient and simple refined theory for buckling and free vibration of exponentially graded sandwich plates under various boundary conditions. *Journal of Sandwich Structures and Materials*, **16**(3):293-318. http://dx.doi.org/10.1177/1099636214526852

Nanthakumar, S.S., Lahmer, T., Rabczuk, T., 2013. Detection of flaws in piezoelectric structures using XFEM. *International Journal for Numerical Methods in Engineering*, **96**(6):373-389. http://dx.doi.org/10.1002/nme.4565

Nanthakumar, S.S., Lahmer, T., Zhuang, X., *et al*., 2016. Detection of material interfaces using a regularized level set method in piezoelectric structures. *Inverse Problems in Science and Engineering*, **24**(1):153-176. http://dx.doi.org/10.1080/17415977.2015.1017485

Ong, D.E.L., Choo, C.S., 2016. Back-analysis and finite element modeling of jacking forces in weathered rocks. *Tunnelling and Underground Space Technology*, **51**:1-10. http://dx.doi.org/10.1016/j.tust.2015.10.014

Sarp Arsava, K., Nam, Y.Y., Kim, Y., 2016. Nonlinear system identification of smart reinforced concrete structures under impact loads. *Journal of Vibration and Control*, **22**(16):3576-3600. http://dx.doi.org/10.1177/1077546314563966

Xie, Y.F., 2011. Least square inversed analysis of soil parameter for foundation with two-order gradient theoretic method. *Advanced Materials Research*, **243-249**:2294-2299. http://dx.doi.org/10.4028/www.scientific.net/AMR.243-249.2294

Xin, Y., Zhang, J., Han, X.D., *et al*., 2014. Research on ultimate load of highway prestressed concrete U-shaped continuous rigid frame bridge based on nonlinear finite method. *Applied Mechanics and Materials*, **501-504**: 1398-1402. http://dx.doi.org/10.4028/www.scientific.net/AMM.501-504.1398

Yahia, S.A., Atmane, H.A., Houari, M.S.A., *et al*., 2015. Wave propagation in functionally graded plates with porosities using various higher-order shear deformation plate theories. *Structural Engineering and Mechanics*, **53**(6):1143-1165. http://dx.doi.org/10.12989/sem.2015.53.6.1143

Zhang, J., Ye, J.S., Tang, X.S., 2008. Kalman filtering identification of Winkler foundation's parameter based on Mindlin theory. *Rock and Soil Mechanics*, **29**(2):425-430 (in Chinese).

Zhang, J., Zhou, C.W., Lin, J., 2012. Dynamic Bayesian identification of mechanical parameters of multi-cell curve box girder based on conjugate gradient theory. *Science China Technological Sciences*, **55**(4):1057-1065. http://dx.doi.org/10.1007/s11431-011-4741-1

Zhao, X.M., 2007. Dynamic Bayesian identification of Winkler subgrade parameter based on Mindlin theory. *Engineering Mechanics*, **24**(10):57-63 (in Chinese).

Zidi, M., Tounsi, A., Houari, M.S.A., *et al*., 2014. Bending analysis of FGM plates under hygro-thermo-mechanical loading using a four variable refined plate theory. *Aerospace Science and Technology*, **34**:24-34. http://dx.doi.org/10.1016/j.ast.2014.02.001

Zong, Z.H., Zhou, R., Huang, X.Y., *et al*., 2014. Seismic response study on a multi-span cable-stayed bridge scale model under multi-support excitations. Part I: shaking table tests. *Journal of Zhejiang University-SCIENCE A (Applied Physics & Engineering)*, **15**(5):351-363. http://dx.doi.org/10.1631/jzus.A1300339

中文概要

题　目：基于广义 Bayes 理论地基参数的 Powell 反演力学模型

目　的： 通过 Powell 优化反演方法建立 Winkler 地基参数的反演力学模型，获得地基参数的稳定数值解。

创新点： 根据 Bayes 理论，推导广义 Bayes 目标函数；利用 Fourier 变换，推求 Winkler 地基上简支板的 Fourier 闭式解，建立地基参数的反演力学模型。

方　法： 1. 根据 Bayes 理论，推导广义 Bayes 目标函数（公式(4)）及地基参数的广义 Bayes 均值和方差表达式（公式(9)和(11)）；2. 引入 Mindlin 理论，推导 Winkler 地基上板的控制微分方程，推求 Winkler 地基上简支板的 Fourier 闭式解；3. 提出步长的一维自动寻优方案，结合 Powell 优化方法建立 Winkler 地基参数的广义 Bayes 反演力学模型。

结　论： 1. 地基参数的反演迭代过程稳定收敛于参数真值；2. 与 Kalman 滤波方法和共轭梯度法不同，Powell 优化方法的迭代过程不涉及目标函数的偏导数计算；3. 广义 Bayes 目标函数能同时考虑不同测量点和不同测量次数的位移实测资料，计算效率更高。

关键词： Powell 反演；力学模型；地基参数；Bayes 目标函数；随机性