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Stochastic averaging of quasi partially integrable Hamiltonian systems under fractional Gaussian noise*

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Abstract: A stochastic averaging method for predicting the response of quasi partially integrable and non-resonant Hamiltonian systems to fractional Gaussian noise (fGn) with the Hurst index $1/2<H<1$ is proposed. The averaged stochastic differential equations (SDEs) for the first integrals of the associated Hamiltonian system are derived. The dimension of averaged SDEs is less than that of the original system. The stationary probability density and statistics of the original system are obtained approximately from solving the averaged SDEs numerically. Two systems are worked out to illustrate the proposed stochastic averaging method. It is shown that the results obtained by using the proposed stochastic averaging method and those from digital simulation of original system agree well, and the computational time for the former results is less than that for the latter ones.

Key words: Fractional Brownian motion (fBm); Fractional Gaussian noise (fGn); Quasi partially integrable Hamiltonian system; Stochastic averaging method; Stationary response
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1 Introduction

Compared with Gaussian white noise, the formal derivative of classical Brownian motion, which has delta correlation, fractional Gaussian noise (fGn) has strong temporal correlation. Thus, the response of a dynamical system to Gaussian white noise is a diffusive Markov process while the response of a dynamical system to fGn is not Markov process. fGn can be mathematical model of many natural, engineering, and social phenomena. In fact, fractional Brownian motion (fBm) and fGn have already been applied as models in physics (Zunino *et al*., 2008; Sliusarenko *et al*., 2010), finance (Ni *et al*., 2009; Gu *et al*., 2016), and biology (Kou and Xie, 2004; Li and Ai, 2012).

The calculus and the stochastic differential equations (SDEs) with respect to fBm are quite complicated and still in development (Biagini *et al*., 2008; Mishura, 2008). Since the response of a dynamical system to fGn is a non-Markov process and the Fokker-Planck-Kolmogorov equation cannot be applied to obtain the probability and statistics of the response, the study of a dynamical system (especially a nonlinear system) driven by fGn is very difficult and challenging. So far, the exact sample solution has been obtained only for the Ornstein-Uhlenbeck system excited by fGn (Cheridito *et al*., 2003; Kaarakka and Salminen, 2011). Therefore, it is necessary to develop some asymptotic or approximate analytical methods for predicting the response of nonlinear dynamical systems to fGn.

The stochastic averaging method, especially the stochastic averaging method for quasi Hamiltonian

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systems, is a powerful approximate analytical method in nonlinear stochastic dynamics. It has already been applied to solve a series of problems in mechanics, physics, chemistry, and biology (Zhu *et al.*, 1997; 2002; Deng and Zhu, 2012). So far, the stochastic averaging method has been developed for Gaussian white noise, Poisson white noise, wide-band stationary noise, narrow-band bounded noise, harmonic functions, and any combination of them (Zhu *et al.*, 1997; Huang *et al.*, 2002; Deng and Zhu, 2007; Zeng and Zhu, 2011; Jia and Zhu, 2014). Recently, the limit theorem for SDE with respect to fBm and the stochastic averaging principle have been established (Xu *et al.*, 2014a; 2014b). Thus, it is possible to develop a stochastic averaging method of quasi Hamiltonian systems excited by fGn. In fact, the stochastic averaging method for quasi non-integrable Hamiltonian systems excited by fGn has been successfully developed (Deng and Zhu, 2016).

In this paper, a stochastic averaging method for quasi partially integrable Hamiltonian systems excited by fGn is developed. First, the definition, correlation function, spectral density, pathwise integral, and differential rule for fBm and fGn are briefly introduced. Then, the averaged fractional SDEs for quasi partially integrable and non-resonant Hamiltonian systems excited by fGn are derived. Furthermore, two systems excited by fGn are studied by using the proposed stochastic averaging method, and the stationary probability densities and statistics are calculated and compared with those from simulation of original systems to verify the proposed stochastic averaging method.

2 Some preliminaries

2.1 fBm and fGn

The fBm $B^H(t)$ is a fractional integral with respect to standard Brownian motion $B(t)$. However, the fractional integral has different definitions, which result in different properties of fBm $B^H(t)$ (Sithi and Lim, 1995; Kou and Xie, 2004). In this paper, the Weyl type of fractional integral is used, which leads to fBm $B^H(t)$ with a stationary increment.

The Weyl type of fBm $B^H(t)$ is defined as (Mandelbrot and van Ness, 1968)

$$B^H(t)-B^H(0) = C_H\left\{\int_{-\infty}^{0}\left[(t-s)^{H-1/2}-(-s)^{H-1/2}\right]\mathrm{d}B(s) + \int_0^t (t-s)^{H-1/2}\mathrm{d}B(s)\right\}, \tag{1}$$

where t and s indicate the time; the Hurst index is $1/2<H<1$; $B^H(0)=0$; the coefficient C_H in the original definition was given by Mandelbrot and van Ness (1968):

$$C_H=\frac{1}{\Gamma(H+1/2)}, \tag{2}$$

where Γ represents gamma function. Usually, it is more convenient to use the symbol $B^H(t)$ to denote unit fBm, and then the coefficient C_H becomes

$$C_H=\frac{\sqrt{2H\Gamma(2H)\sin(H\pi)}}{\Gamma(H+1/2)}. \tag{3}$$

Some strict proofs of the normalized coefficient C_H in Eq. (3) are given by Mishura (2008).

As a rigorous definition, the unit fBm with Hurst index $0<H<1$ is defined as a centered Gaussian process with the following properties:

$$B^H(t=0)=0, \tag{4a}$$

$$E[B^H(t)]=0,\quad t\geq 0, \tag{4b}$$

$$E[B^H(t)B^H(s)] = \frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H}),\quad t,s\geq 0. \tag{4c}$$

The increment $B^H(t)-B^H(s)$ is a stationary Gaussian process with the following mean and covariance:

$$E[B^H(t)-B^H(s)]=0, \tag{5a}$$

$$E[(B^H(t)-B^H(s))^2]=|t-s|^{2H},\quad t,s\geq 0. \tag{5b}$$

Eq. (5b) leads to the useful formulas:

$$E[(B^H(t))^2]=t^{2H}, \tag{6a}$$

$$E[(\Delta B^H(t))^2]=(\Delta t)^{2H} \quad \text{or} \quad E[(\mathrm{d}B^H(t))^2]=(\mathrm{d}t)^{2H}. \tag{6b}$$

Mandelbrot and van Ness (1968) also introduced fGn to model fractional random noise. A unit fGn $W^H(t)$ is defined as the formal derivative process of unit fBm, i.e.,

$$B^H(t)=\int_0^t W^H(s)\mathrm{d}s \quad \text{or} \quad W^H(s)=\mathrm{d}B^H(t)/\mathrm{d}t. \tag{7}$$

The following auto-correlation function of fGn was given by Papoulis (1991):

$$\begin{aligned}R(\tau)&=E[W^H(t+\tau)W^H(t)]\\&=H(2H-1)|\tau|^{2H-2}+2H|\tau|^{2H-1}\,\delta(\tau).\end{aligned} \tag{8}$$

When H=1/2, the auto-correlation function $R(\tau)$ in Eq. (8) is equal to the Dirac delta function $\delta(\tau)$, the auto-correlation function of Gaussian white noise.

In this paper, only the fGn with the Hurst index 1/2<H<1 is considered. The power spectral density (PSD) $S(\omega)$ of fGn for 1/2<H<1 can be obtained from the correlation function in Eq. (8) by using the Wiener-Khintchine relation as follows:

$$\begin{aligned}S(\omega)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}R(\tau)\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau\\&=\frac{H\Gamma(2H)\sin(H\pi)}{\pi}|\omega|^{1-2H}.\end{aligned} \tag{9}$$

Figs. 1a and 1b respectively show the comparison of Eqs. (8) and (9) with the simulated results. The simulation results are obtained using Eqs. (1) and (7). It is seen that the auto-correlation function in Eq. (8) and the PSD in Eq. (9) are acceptable.

2.2 Pathwise integrals with respect to fBm and differential rule

There are several definitions of fractional integral with respect to fBm (Biagini *et al.*, 2008; Mishura, 2008). The most natural and simple one is a pathwise integral, which is defined as

$$\begin{aligned}&\int_{t_0}^t f(s)\mathrm{d}B^H(s)\\&=\lim_{n\to\infty}\sum_{i=1}^n f(t_i')[B^H(t_i)-B^H(t_{i-1})],\\&t_i=i\Delta t,\quad \Delta t=(t-t_0)/n,\end{aligned} \tag{10}$$

where $t_{i-1}\le t_i'\le t_i$. The pathwise integrals can be classified into three different types, depending on the value of t_i'. The first type is the symmetric pathwise integral when $t_i'=(t_{i-1}+t_i)/2$:

$$\begin{aligned}&\int_{t_0}^t f(s)\mathrm{d}^\circ B^H(s)\\&=\lim_{n\to\infty}\sum_{i=1}^n f\left(\frac{t_{i-1}+t_i}{2}\right)[B^H(t_i)-B^H(t_{i-1})],\end{aligned} \tag{11}$$

where $\int f(s)\mathrm{d}^\circ B^H(s)$ denotes the symmetric pathwise integral. When H=1/2, Eq. (11) is the Stratonovich integral.

The second type is the forward pathwise integral when $t_i'=t_{i-1}$:

$$\begin{aligned}&\int_{t_0}^t f(s)\mathrm{d}^- B^H(s)\\&=\lim_{n\to\infty}\sum_{i=1}^n f(t_{i-1})[B^H(t_i)-B^H(t_{i-1})],\end{aligned} \tag{12}$$

where $\int f(s)\mathrm{d}^- B^H(s)$ denotes the forward pathwise integral. Eq. (12) reduces to the Itô integral when H=1/2.

The third type is the backward pathwise integral when $t_i'=t_i$:

$$\begin{aligned}&\int_{t_0}^t f(s)\mathrm{d}^+ B^H(s)\\&=\lim_{n\to\infty}\sum_{i=1}^n f(t_i)[B^H(t_i)-B^H(t_{i-1})].\end{aligned} \tag{13}$$

This third type of pathwise integral is seldom used but the first and second types of pathwise integral are used widely in theoretical and practical studies. The fractional SDEs corresponding to these two integrals can be established. For instance, the SDE associated with the forward pathwise integral is of the form:

$$\mathrm{d}X(t)=\alpha(X,t)\mathrm{d}t+\beta(X,t)\mathrm{d}^- B^H(t), \tag{14}$$

which means that it is necessary to use the forward pathwise integral to perform further mathematical operation on Eq. (14), where $\alpha(X, t)$ and $\beta(X, t)$ are measurable processes. The stochastic process $X(t)$

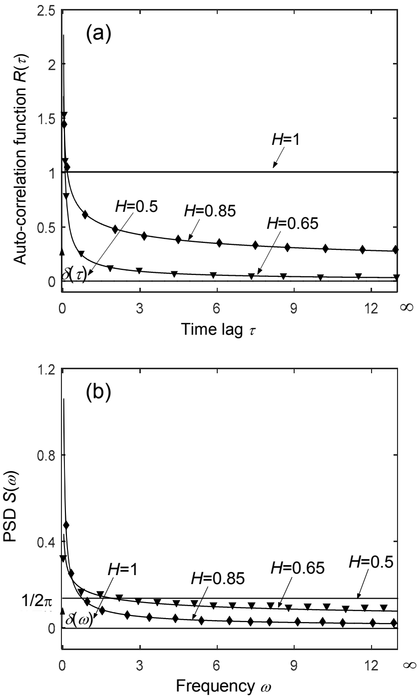

Fig. 1 Comparison of auto-correlation function $R(\tau)$ (a) and PSD $S(\omega)$ (b) of fGn $W^H(t)$ with the simulated results (▾ ◆ indicate simulated results)

produced by the SDE (Eq. (14)) is referred to as the fractional forward process (Biagini *et al.*, 2008).

In this paper, the SDEs associated with symmetric and forward pathwise integrals are used. The difference between these two types of SDEs can be simplified as

$$\beta[X(t),t]\mathrm{d}^{\circ}B^H(t)-\beta[X(t),t]\mathrm{d}^{-}B^H(t) =\sum_{m=1}^{\infty}\frac{\partial^m\beta[x(t_{i-1}),t_{i-1}]}{2^m m!\partial x^m}\beta^m[X(t_{i-1}),t_{i-1}](\mathrm{d}t)^{(m+1)H}. \quad (15)$$

When H=1/2, the right hand side of Eq. (15) indicates the Wong-Zakai correction terms. When H>1/2, these terms will vanish since they are of higher orders infinite-smaller than dt. Therefore, the following symmetric fractional SDE is equal to Eq. (14):

$$\mathrm{d}X(t)=\alpha(X,t)\mathrm{d}t+\beta(X,t)\mathrm{d}^{\circ}B^H(t),\quad H>1/2. \quad (16)$$

Suppose $X(t)$ is governed by Eq. (14) and $Y(t)=f(X(t), t)$. The differential rule for a fractional SDE established by Biagini *et al.* (2008) is of the form:

$$\mathrm{d}Y(t)=\frac{\partial f(X,t)}{\partial t}\mathrm{d}t+\frac{\partial f(X,t)}{\partial x}\mathrm{d}X(t),\quad H>1/2. \quad (17)$$

The differential rule can be extended to the n-dimensional case. Assume that $\boldsymbol{X}(t)$ is governed by the following n-dimensional forward fractional SDE:

$$\mathrm{d}\boldsymbol{X}(t)=\boldsymbol{\alpha}(\boldsymbol{X})\mathrm{d}t+\boldsymbol{\beta}(\boldsymbol{X})\mathrm{d}^{-}\boldsymbol{B}(t), \quad (18)$$

where $\boldsymbol{X}=[X_1, X_2, \ldots, X_n]^{\mathrm{T}}$, $\boldsymbol{\alpha}(\boldsymbol{X})$ is $n\times1$ vector and $\boldsymbol{\beta}(\boldsymbol{X})$ is $n\times m$ matrix, $\boldsymbol{B}(t)=[B_1^{H_1}(t), B_2^{H_2}(t), \cdots, B_m^{H_m}(t)]^{\mathrm{T}}$ is a vector of m independent unit fBm and the Hurst indexes are $H_1, H_2, \ldots, H_m$>1/2. Then, the forward fractional SDE for $Y(t)=f(\boldsymbol{X}(t))$ is

$$\mathrm{d}Y(t)=\boldsymbol{f}_{\boldsymbol{X}}^{\mathrm{T}}\alpha(\boldsymbol{X})\mathrm{d}t+\boldsymbol{f}_{\boldsymbol{X}}^{\mathrm{T}}\beta(\boldsymbol{X})\mathrm{d}^{-}\boldsymbol{B}(t),\quad H_1,H_2,\cdots,H_m>1/2, \quad (19)$$

where $\boldsymbol{f}_X=[\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n]^{\mathrm{T}}$.

3 Stochastic averaging

Consider a quasi Hamiltonian system governed by the following equations:

$$\begin{aligned}&\dot{Q}_i=\frac{\partial\mathcal{H}}{\partial P_i},\\&\dot{P}_i=-\frac{\partial\mathcal{H}}{\partial Q_i}-\varepsilon^{2H}c_{ij}\frac{\partial\mathcal{H}}{\partial P_j}+\varepsilon^H f_{ik}W_k^H(t),\\&i,j=1,2,\cdots,n,\quad k=1,2,\cdots,m,\end{aligned} \quad (20)$$

where Q_i and P_i are the generalized displacements and momenta, respectively; $\mathcal{H}=\mathcal{H}(\boldsymbol{Q}, \boldsymbol{P})$ is the Hamiltonian with continuous partial derivatives; ε is a small parameter; $\varepsilon^{2H}c_{ij}=\varepsilon^{2H}c_{ij}(\boldsymbol{Q}, \boldsymbol{P})$ are functions representing coefficients of quasi linear damping; $\varepsilon^H f_{ik}=\varepsilon^H f_{ik}(\boldsymbol{Q}, \boldsymbol{P})$ are functions representing amplitudes of excitation; $W_k^H(t)$ are the independent unit fGns with the Hurst index 1/2<H<1.

According to Section 2, Eq. (20) can be modeled as the following fractional SDEs in the sense of a symmetric pathwise integral:

$$\begin{aligned} &\mathrm{d}Q_i = \frac{\partial \mathcal{H}}{\partial P_i}\mathrm{d}t, \\ &\mathrm{d}P_i = \left(-\frac{\partial \mathcal{H}}{\partial Q_i} - \varepsilon^{2H} c_{ij}\frac{\partial \mathcal{H}}{\partial P_j}\right)\mathrm{d}t \\ &\quad + \varepsilon^H f_{ik}\mathrm{d}^\circ B_k^H(t), \\ &i, j = 1, 2, \cdots, n, \quad k = 1, 2, \cdots, m. \end{aligned} \tag{21}$$

Due to the vanishing difference between symmetric and forward pathwise differentials (Eq. (15)), SDEs (Eq. (21)) are equivalent to the following forward pathwise SDEs:

$$\begin{aligned} &\mathrm{d}Q_i = \frac{\partial \mathcal{H}}{\partial P_i}\mathrm{d}t, \\ &\mathrm{d}P_i = \left(-\frac{\partial \mathcal{H}}{\partial Q_i} - \varepsilon^{2H} c_{ij}\frac{\partial \mathcal{H}}{\partial P_j}\right)\mathrm{d}t \\ &\quad + \varepsilon^H f_{ik}\mathrm{d}^- B_k^H(t), \\ &i, j = 1, 2, \cdots, n, \quad k = 1, 2, \cdots, m. \end{aligned} \tag{22}$$

A Hamiltonian system of n degrees of freedom (DOFs) can be classified according to the number of independent first integrals (conservative quantities or motion constants) $\mathcal{H}_1(\boldsymbol{q}, \boldsymbol{p})$, $\mathcal{H}_2(\boldsymbol{q}, \boldsymbol{p})$, …, $\mathcal{H}_r(\boldsymbol{q}, \boldsymbol{p})$, which are in involution (Tabor, 1989). $\boldsymbol{q}=[q_1, q_2, \ldots, q_n]^{\mathrm{T}}$, $\boldsymbol{p}=[p_1, p_2, \ldots, p_n]^{\mathrm{T}}$. The Hamiltonian system is called non-integrable if $r=1$, integrable (or completely integrable) if $r=n$, and partially integrable if $1<r<n$. Suppose that the Hamiltonian system with Hamiltonian $\mathcal{H}$ associated with Eq. (22) is partially integrable. That is, the Hamiltonian system has r ($1<r<n$) independent first integrals $\mathcal{H}_1$, $\mathcal{H}_2$, …, $\mathcal{H}_r$ which are in involution. Specifically, the Hamiltonian is assumed to be of the form:

$$\mathcal{H}(\boldsymbol{q}, \boldsymbol{p}) = \sum_{\eta=1}^{r-1}\mathcal{H}_\eta(\boldsymbol{q}_1, \boldsymbol{p}_1) + \mathcal{H}_r(\boldsymbol{q}_2, \boldsymbol{p}_2), \tag{23}$$

where $\boldsymbol{q}_1=[q_1, q_2, \ldots, q_{r-1}]^{\mathrm{T}}$, $\boldsymbol{p}_1=[p_1, p_2, \ldots, p_{r-1}]^{\mathrm{T}}$, $\boldsymbol{q}_2=[q_r, q_{r+1}, \ldots, q_n]^{\mathrm{T}}$, $\boldsymbol{p}_2=[p_r, p_{r+1}, \ldots, p_m]^{\mathrm{T}}$; $\mathcal{H}_\eta$ and $\mathcal{H}_r$ are the r independent first integrals in involution. Eq. (23) indicates that the partially integrable Hamiltonian system consists of an integrable part and a non-integrable part. For the integrable part, action-angle variables I_η and θ_η can be introduced. Thus, Eq. (23) can be rewritten as

$$\mathcal{H}(\boldsymbol{I}, \boldsymbol{q}_2, \boldsymbol{p}_2) = \sum_{\eta=1}^{r-1}\mathcal{H}_\eta(I_\eta) + \mathcal{H}_r(\boldsymbol{q}_2, \boldsymbol{p}_2), \tag{24}$$

where $\boldsymbol{I}=[I_1, I_2, \ldots, I_{r-1}]^{\mathrm{T}}$. The integrable part of a partially integrable Hamiltonian system could be resonant and non-resonant, just as for integrable Hamiltonian system (Zhu *et al*., 1997). Here only the non-resonant case is considered.

Applying the differential rule for fBm in Eq. (19), the following SDEs for I_η, Θ_η, and $\mathcal{H}_r$ can be derived from Eq. (22):

$$\begin{aligned} &\mathrm{d}I_\eta = -\varepsilon^{2H} c_{\eta' j}\frac{\partial \mathcal{H}}{\partial P_j}\frac{\partial I_\eta}{\partial P_{\eta'}}\mathrm{d}t + \varepsilon^H\frac{\partial I_\eta}{\partial P_{\eta'}} f_{\eta' k}\mathrm{d}^- B_k^H(t), \\ &\mathrm{d}\Theta_\eta = \left(\omega_\eta - \varepsilon^{2H} c_{\eta' j}\frac{\partial \mathcal{H}}{\partial P_j}\frac{\partial \Theta_\eta}{\partial P_{\eta'}}\right)\mathrm{d}t + \varepsilon^H\frac{\partial \Theta_\eta}{\partial P_{\eta'}} f_{\eta' k}\mathrm{d}^- B_k^H(t), \\ &\mathrm{d}\mathcal{H}_r = -\varepsilon^{2H} c_{\rho j}\frac{\partial \mathcal{H}}{\partial P_j}\frac{\partial \mathcal{H}_r}{\partial P_\rho}\mathrm{d}t + \varepsilon^H\frac{\partial \mathcal{H}_r}{\partial P_\rho} f_{\rho k}\mathrm{d}^- B_k^H(t), \\ &j = 1, 2, \cdots, n, \quad k = 1, 2, \cdots, m, \\ &\eta, \eta' = 1, 2, \cdots, r-1, \quad \rho = r, r+1, \cdots, n. \end{aligned} \tag{25}$$

Now, the system is governed by Eq. (25), the last $n-r+1$ equations for Q_i and the last $n-r$ equations for P_i in Eq. (22). I_η and $\mathcal{H}_r$ are slowly varying processes while Θ_1, Θ_2, …, Θ_{r-1}, Q_r, Q_{r+1}, …, Q_n, P_{r+1}, P_{r+2}, …, P_n are rapidly varying processes. Thus, the averaging principle (Xu *et al*., 2014a; 2014b) can be applied to obtain the following averaged fractional SDEs for I_η and $\mathcal{H}_r$:

$$\begin{aligned} &\mathrm{d}I_\eta = \bar{m}_\eta(\boldsymbol{I}, \mathcal{H}_r)\mathrm{d}t + \bar{\sigma}_{\eta k}(\boldsymbol{I}, \mathcal{H}_r)\mathrm{d}^- B_k^H(t), \\ &\mathrm{d}\mathcal{H}_r = \bar{m}_r(\boldsymbol{I}, \mathcal{H}_r)\mathrm{d}t + \bar{\sigma}_{rk}(\boldsymbol{I}, \mathcal{H}_r)\mathrm{d}^- B_k^H(t), \\ &\eta = 1, 2, \cdots, r-1, \quad k = 1, 2, \cdots, m, \end{aligned} \tag{26}$$

where $\bar{m}_\eta(\boldsymbol{I}, \mathcal{H}_r)$, $\bar{m}_r(\boldsymbol{I}, \mathcal{H}_r)$, $\bar{\sigma}_{\eta k}(\boldsymbol{I}, \mathcal{H}_r)$, and $\bar{\sigma}_{rk}(\boldsymbol{I}, \mathcal{H}_r)$ can be obtained from the coefficients in Eq. (25) through time averaging. Thus, in the sense of probability and mean square, I_η and $\mathcal{H}_r$ in the original system Eq. (22) could be approximated by I_η and $\mathcal{H}_r$ in the averaged SDEs (Eq. (26)). Since the 2(r−1)-dimensional integrable sub-Hamiltonian system is ergodic on a (r−1)-dimensional torus and 2(n−r+1)-dimensional non-integrable sub-Hamiltonian system is ergodic on a [2(n−r)+1]-dimensional isoenergetic surface, the time averaging can be replaced by space averaging (Zhu *et al*., 2002). The coefficients in averaged Eq. (28) can then be obtained as follows:

$$\begin{aligned}
&\bar{m}_\eta(\boldsymbol{I},\mathcal{H})=\left\langle -\varepsilon^{2H}c_{\eta'j}\frac{\partial\mathcal{H}}{\partial P_j}\frac{\partial I_\eta}{\partial P_{\eta'}}\right\rangle,\\
&\bar{m}_r(\boldsymbol{I},\mathcal{H})=\left\langle -\varepsilon^{2H}c_{\rho j}\frac{\partial\mathcal{H}}{\partial P_j}\frac{\partial \mathcal{H}_r}{\partial P_{\rho}}\right\rangle,\\
&\bar{\sigma}_{\eta k}\bar{\sigma}_{\bar{\eta}k}(\boldsymbol{I},\mathcal{H})=\left\langle \varepsilon^{2H}f_{\eta'k}f_{\eta''k}\frac{\partial I_\eta}{\partial P_{\eta'}}\frac{\partial I_{\bar{\eta}}}{\partial P_{\eta''}}\right\rangle,\\
&\bar{\sigma}_{\eta k}\bar{\sigma}_{rk}(\boldsymbol{I},\mathcal{H})=\left\langle \varepsilon^{2H}f_{\eta'k}f_{\rho k}\frac{\partial I_\eta}{\partial P_{\eta'}}\frac{\partial \mathcal{H}_r}{\partial P_{\rho}}\right\rangle,\\
&\bar{\sigma}_{rk}\bar{\sigma}_{rk}(\boldsymbol{I},\mathcal{H})=\left\langle \varepsilon^{2H}f_{\rho k}f_{\rho'k}\frac{\partial \mathcal{H}_r}{\partial P_{\rho}}\frac{\partial \mathcal{H}_r}{\partial P_{\rho'}}\right\rangle,\\
&j=1,2,\cdots,n,\qquad \eta,\bar{\eta},\eta',\eta''=1,2,\cdots,r-1,\\
&k=1,2,\cdots,m,\qquad \rho,\rho'=r,r+1,\cdots,n,
\end{aligned}\tag{27}$$

and

$$\begin{aligned}
&\langle\bullet\rangle=\frac{1}{(2\pi)^{r-1}T(\mathcal{H}_r)}\\
&\quad\times\int_{\Omega_1}\int_0^{2\pi}\left[\bullet\Big/\frac{\partial\mathcal{H}_r}{\partial p_r}\right]\mathrm{d}\boldsymbol{\theta}\mathrm{d}q_r\mathrm{d}q_{r+1}\cdots\mathrm{d}q_n\mathrm{d}p_{r+1}\cdots\mathrm{d}p_n,\\
&T(\mathcal{H}_r)=\int_{\Omega_1}\left[1\Big/\frac{\partial\mathcal{H}_r}{\partial p_r}\right]\mathrm{d}q_r\mathrm{d}q_{r+1}\cdots\mathrm{d}q_n\mathrm{d}p_{r+1}\cdots\mathrm{d}p_n,\\
&\Omega_1=\{(q_r,q_{r+1},\cdots,q_n,p_{r+1},\cdots,p_n)\,|\\
&\qquad\mathcal{H}_r(q_r,q_{r+1},\cdots,q_n,0,p_{r+1},\cdots,p_n)\le\mathcal{H}_r\},\\
&\boldsymbol{I}=[I_1,I_2,\cdots,I_{r-1}]^{\mathrm{T}},\ \boldsymbol{\theta}=[\theta_1,\theta_2,\cdots,\theta_{r-1}]^{\mathrm{T}}.
\end{aligned}\tag{28}$$

The stationary probability density $p(\boldsymbol{I}, \mathcal{H}_r)$ can be obtained from the simulation of Eq. (26). Then the stationary probability density $p(\boldsymbol{q}, \boldsymbol{p})$ can be obtained from $p(\boldsymbol{I}, \mathcal{H}_r)$ as follows (Zhu *et al*., 2002):

$$p(\boldsymbol{q},\boldsymbol{p})=\left.\frac{p(\boldsymbol{I},\mathcal{H}_r)}{(2\pi)^{r-1}T(\mathcal{H}_r)}\right|_{\boldsymbol{I}=\boldsymbol{I}(\boldsymbol{q}_1,\boldsymbol{p}_1),\ \mathcal{H}_r=\mathcal{H}_r(\boldsymbol{q}_2,\boldsymbol{p}_2)}.\tag{29}$$

The marginal stationary probability density and statistics can be obtained from $p(\boldsymbol{I}, \mathcal{H}_r)$ and $p(\boldsymbol{q}, \boldsymbol{p})$ by integration. Note that the averaged Eq. (26) is much simpler than the original Eq. (22). The dimension of the former is less than a half of that of the latter, and Eq. (26) contains only slowly varying processing $\mathcal{H}_r$ and $\boldsymbol{I}$. Thus, the computer time for simulating Eq. (26) is much less than that for Eq. (22).

To verify the validity of the proposed stochastic averaging method, two examples are worked out as follows.

4 Example 1

Consider a 4-DOF quasi partially integrable Hamiltonian system subject to fGn excitation. The equations of the system are of the form:

$$\begin{aligned}
&\dot{Q}_1=P_1,\\
&\dot{P}_1=-\omega_1^2Q_1-\left[\alpha_{10}+\alpha_{11}P_1^2+\alpha_{12}P_2^2+\alpha_{13}P_3^2+\alpha_{14}P_4^2\right.\\
&\qquad\left.+(\alpha_{13}+\alpha_{14})U(Q_3,Q_4)\right]P_1+\sqrt{2D_1}W_1^H(t),\\
&\dot{Q}_2=P_2,\\
&\dot{P}_2=-\omega_2^2Q_2-\left[\alpha_{20}+\alpha_{21}P_1^2+\alpha_{22}P_2^2+\alpha_{23}P_3^2+\alpha_{24}P_4^2\right.\\
&\qquad\left.+(\alpha_{23}+\alpha_{24})U(Q_3,Q_4)\right]P_2+\sqrt{2D_2}W_2^H(t),\\
&\dot{Q}_3=P_3,\\
&\dot{P}_3=-\partial U(Q_3,Q_4)/\partial Q_3-\left[\alpha_{30}+\alpha_{31}P_1^2+\alpha_{32}P_2^2+\alpha_{33}P_3^2\right.\\
&\qquad\left.+\alpha_{34}P_4^2+(1/2)(\alpha_{34}+3\alpha_{33})U(Q_3,Q_4)\right]P_3\\
&\qquad+\sqrt{2D_3}W_3^H(t),\\
&\dot{Q}_4=P_4,\\
&\dot{P}_4=-\partial U(Q_3,Q_4)/\partial Q_4-\left[\alpha_{40}+\alpha_{41}P_1^2+\alpha_{42}P_2^2+\alpha_{43}P_3^2\right.\\
&\qquad\left.+\alpha_{44}P_4^2+(1/2)(\alpha_{43}+3\alpha_{44})U(Q_3,Q_4)\right]P_4\\
&\qquad+\sqrt{2D_4}W_4^H(t),
\end{aligned}\tag{30}$$

where

$$U(Q_3,Q_4)=k\left(\omega_3^2Q_3^2+\omega_4^2Q_4^2\right)^3\Big/6.\tag{31}$$

ω_j, α_{ij}, and k are positive constants; $W_k^H(t)$ are unit fGns with the Hurst index $1/2<H<1$; $2D_k$ are intensities of excitations; α_{ij} and D_k are assumed of the order of ε^{2H}. Eq. (30) can be recast into the form of Eq. (22). The Hamiltonian $\mathcal{H}$ associated with system Eq. (30) is

$$\mathcal{H}=\mathcal{H}_1+\mathcal{H}_2+\mathcal{H}_3=\omega_1 I_1+\omega_2 I_2+\mathcal{H}_3, \quad (32)$$

where

$$\begin{aligned} I_1&=\left(p_1^2+\omega_1^2 q_1^2\right)/(2\omega_1),\\ I_2&=\left(p_2^2+\omega_2^2 q_2^2\right)/(2\omega_2),\\ \mathcal{H}_3&=\left(p_3^2+p_4^2\right)/2+U(q_3,q_4). \end{aligned} \quad (33)$$

As $U(q_3, q_4)$ is not separable, system Eq. (30) is a quasi partially integrable Hamiltonian system with three independent first integrals I_1, I_2, and $\mathcal{H}_3$. Suppose that it is non-resonant. The following averaged fractional SDEs can be obtained:

$$\begin{aligned} \mathrm{d}I_1&=\bar{m}_1(I_1,I_2,\mathcal{H}_3)\mathrm{d}t\\ &\quad+\bar{\sigma}_{11}(I_1,I_2,\mathcal{H}_3)\mathrm{d}^-B_1^H(t),\\ \mathrm{d}I_2&=\bar{m}_2(I_1,I_2,\mathcal{H}_3)\mathrm{d}t\\ &\quad+\bar{\sigma}_{22}(I_1,I_2,\mathcal{H}_3)\mathrm{d}^-B_2^H(t),\\ \mathrm{d}\mathcal{H}_3&=\bar{m}_3(I_1,I_2,\mathcal{H}_3)\mathrm{d}t\\ &\quad+\bar{\sigma}_{33}(I_1,I_2,\mathcal{H}_3)\mathrm{d}^-B_3^H(t). \end{aligned} \quad (34)$$

The averaged drift and diffusion coefficients and $T(\mathcal{H}_3)$ are

$$\begin{aligned} \bar{m}_1(I_1,I_2,\mathcal{H}_3)&=-\left[\alpha_{10}I_1+3\alpha_{11}\omega_1 I_1^2/2\right.\\ &\quad\left.+\alpha_{12}\omega_2 I_1 I_2+\left(\alpha_{13}+\alpha_{14}\right)I_1\mathcal{H}_3\right],\\ \bar{m}_2(I_1,I_2,\mathcal{H}_3)&=-\left[\alpha_{20}I_2+3\alpha_{22}\omega_2 I_2^2/2\right.\\ &\quad\left.+\alpha_{21}\omega_1 I_1 I_2+\left(\alpha_{23}+\alpha_{24}\right)I_2\mathcal{H}_3\right],\\ \bar{m}_3(I_1,I_2,\mathcal{H}_3)&=-\left[\left(\alpha_{30}+\alpha_{40}\right)\right.\\ &\quad+\left(\alpha_{31}+\alpha_{41}\right)\omega_1 I_1+\left(\alpha_{32}+\alpha_{42}\right)\omega_2 I_2\\ &\quad\left.+\left(3\alpha_{33}+3\alpha_{44}+\alpha_{34}+\alpha_{43}\right)\mathcal{H}_3/2\right]\frac{3\mathcal{H}_3}{4},\\ \bar{\sigma}_{11}^2(I_1,I_2,\mathcal{H}_3)&=2D_1 I_1/\omega_1,\\ \bar{\sigma}_{22}^2(I_1,I_2,\mathcal{H}_3)&=2D_2 I_2/\omega_2,\\ \bar{\sigma}_{33}^2(I_1,I_2,\mathcal{H}_3)&=\frac{3}{2}\left(D_3+D_4\right)\mathcal{H}_3,\\ \bar{\sigma}_{12}^2&=\bar{\sigma}_{13}^2=\bar{\sigma}_{23}^2=0,\\ T(\mathcal{H}_3)&=\frac{2\pi^2}{\omega_3\omega_4}\sqrt[3]{\frac{6\mathcal{H}_3}{k}}. \end{aligned} \quad (35)$$

The stationary probability density $p(I_1, I_2, \mathcal{H}_3)$ can be obtained from the simulation of Eq. (34). The stationary joint probability density $p(q_1, q_2, p_1, p_2)$ can be obtained from $p(I_1, I_2, \mathcal{H}_3)$ using Eq. (29) and the stationary marginal probability densities $p(q_1, q_2)$, $p(q_1, q_3)$ and other statistics, such as $E[Q_1^2]$, $E[Q_2^2]$, $E[Q_3^2]$, and $E[Q_4^2]$, can then be calculated from $p(q_1, q_2, p_1, p_2)$ as follows:

$$\begin{aligned} p(\boldsymbol{q},\boldsymbol{p})&=\left.\frac{p(I_1,I_2,\mathcal{H}_3)}{(2\pi)^2 T(\mathcal{H}_3)}\right|_{\substack{I_1=(p_1^2+\omega_1^2q_1^2)/(2\omega_1),\\ I_2=(p_2^2+\omega_2^2q_2^2)/(2\omega_2),\\ \mathcal{H}_3=(p_3^2+p_4^2)/2+U(q_3+q_4)}},\\ p(q_1,q_2)&=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}p(\boldsymbol{q},\boldsymbol{p})\mathrm{d}q_3\mathrm{d}q_4\mathrm{d}p_1\mathrm{d}p_2\cdots\mathrm{d}p_4,\\ p(q_3,q_4)&=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}p(\boldsymbol{q},\boldsymbol{p})\mathrm{d}q_1\mathrm{d}q_2\mathrm{d}p_1\mathrm{d}p_2\cdots\mathrm{d}p_4,\\ E[Q_1^2]&=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}q_1^2\,p(\boldsymbol{q},\boldsymbol{p})\mathrm{d}q_1\mathrm{d}q_2\cdots\mathrm{d}q_4\mathrm{d}p_1\mathrm{d}p_2\cdots\mathrm{d}p_4,\\ E[Q_2^2]&=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}q_2^2\,p(\boldsymbol{q},\boldsymbol{p})\mathrm{d}q_1\mathrm{d}q_2\cdots\mathrm{d}q_4\mathrm{d}p_1\mathrm{d}p_2\cdots\mathrm{d}p_4,\\ E[Q_3^2]&=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}q_3^2\,p(\boldsymbol{q},\boldsymbol{p})\mathrm{d}q_1\mathrm{d}q_2\cdots\mathrm{d}q_4\mathrm{d}p_1\mathrm{d}p_2\cdots\mathrm{d}p_4,\\ E[Q_4^2]&=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}q_4^2\,p(\boldsymbol{q},\boldsymbol{p})\mathrm{d}q_1\mathrm{d}q_2\cdots\mathrm{d}q_4\mathrm{d}p_1\mathrm{d}p_2\cdots\mathrm{d}p_4. \end{aligned} \quad (36)$$

Figs. 2–5 show some numerical results for the stationary probability densities $p(I_1)$, $p(I_2)$, $p(\mathcal{H}_3)$, $p(I_1, I_2)$, $p(I_1, \mathcal{H}_3)$, mean values $E[I_1]$, $E[I_2]$, $E[\mathcal{H}_3]$, and mean square values $E[I_1^2]$, $E[I_2^2]$, $E[\mathcal{H}_3^2]$ of action variables and sub-Hamiltonian simulated from SDE Eq. (34) and the original system Eq. (30), respectively. The probability densities $p(q_1)$, $p(q_2)$, $p(q_3)$, $p(q_4)$, $p(q_1, q_2)$, $p(q_1, q_3)$ and mean square values $E[Q_1^2]$, $E[Q_2^2]$, $E[Q_3^2]$, $E[Q_4^2]$ of displacements calculated by using Eq. (36) are shown in Figs. 6–8 (p.712). It is seen from these figures that the results simulated from the averaged SDE Eq. (34) agree well with those from the original system Eq. (30).

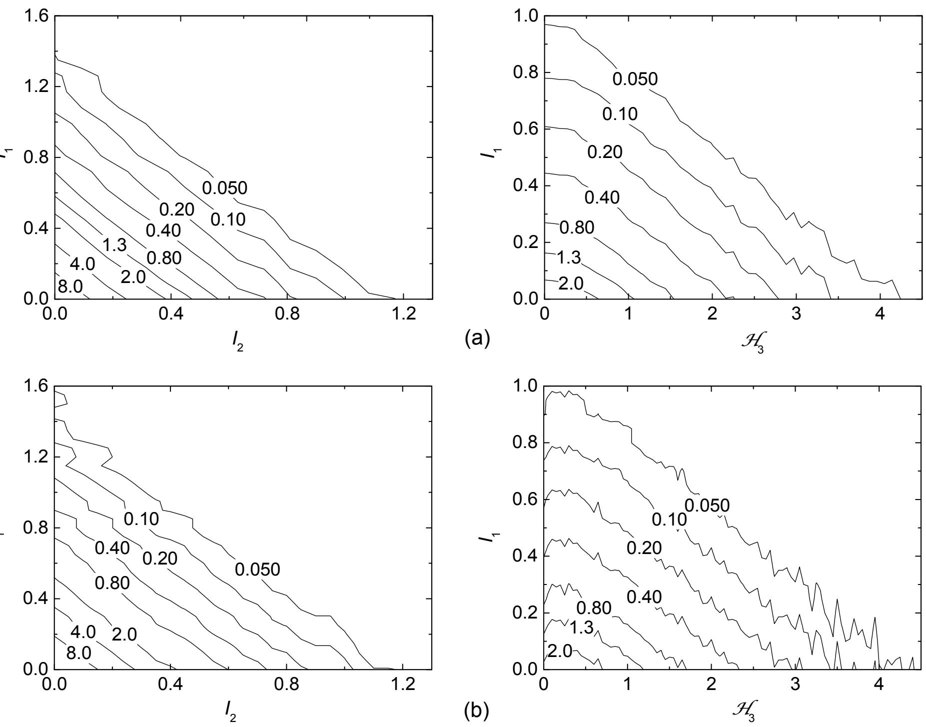

Fig. 2 Contours of stationary probability density $p(I_1, I_2)$ and $p(I_1, \mathcal{H}_3)$ of system Eq. (30)

(a) Simulated from the averaged fractional SDE Eq. (34); (b) Simulated from original system Eq. (30). The parameters are $\alpha_{10}=\alpha_{20}=0.05$, $\alpha_{30}=\alpha_{40}=0.01$, $\alpha_{13}=\alpha_{14}=\alpha_{23}=\alpha_{24}=0.01$, $\alpha_{31}=\alpha_{32}=\alpha_{41}=\alpha_{42}=0.02$, $\alpha_{11}=\alpha_{22}=\alpha_{33}=0$, $\alpha_{44}=\alpha_{34}=\alpha_{43}=0$, $\alpha_{12}=\alpha_{21}=0.01$, $\omega_1^2=3$, $\omega_2^2=4$, $\omega_3^2=5$, $\omega_4^2=6$, $D_1=D_2=D_3=D_4=0.04$, $k=3$, and $H=0.75$

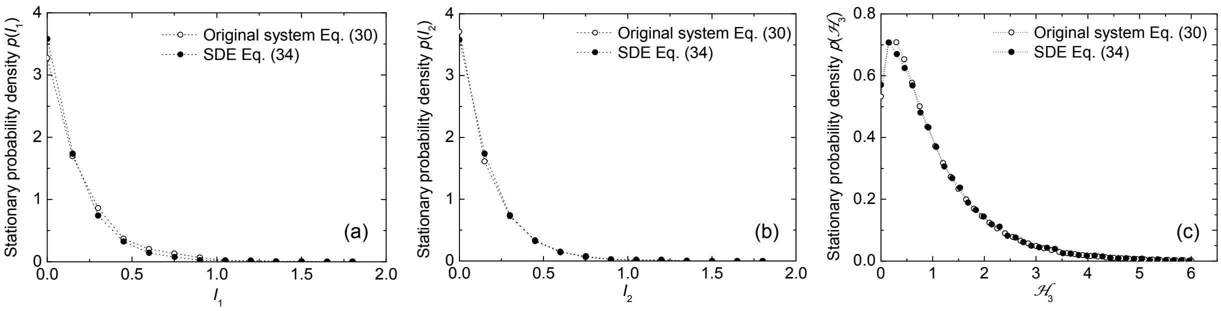

Fig. 3 Stationary probability density $p(I_1)$ (a), $p(I_2)$ (b), and $p(\mathcal{H}_3)$ (c) calculated from the averaged SDE Eq. (34) and from the original system Eq. (30), respectively (the parameters are the same as those in Fig. 2)

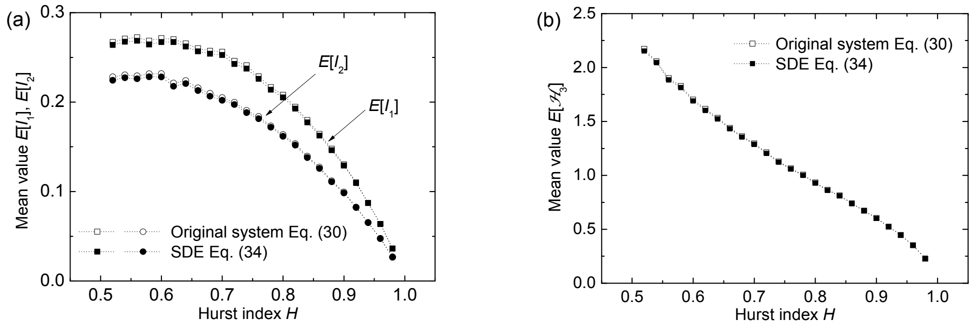

Fig. 4 Mean values $E[I_1]$, $E[I_2]$, and $E[\mathcal{H}_3]$ simulated from the averaged fractional SDE Eq. (34) and the original system Eq. (30), respectively (the parameters are the same as those in Fig. 2)

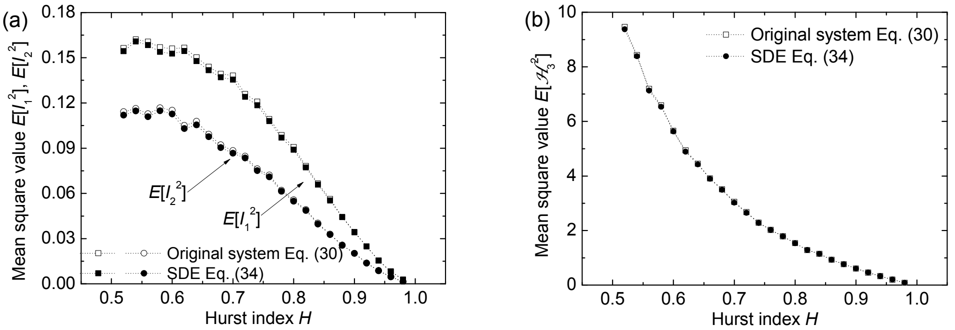

Fig. 5 Mean square values $E[I_1^2]$, $E[I_2^2]$ (a), and $E[\mathcal{H}_3^2]$ (b) simulated from the averaged Eq. (34) and the original system Eq. (30), respectively (the parameters are the same as those in Fig. 2)

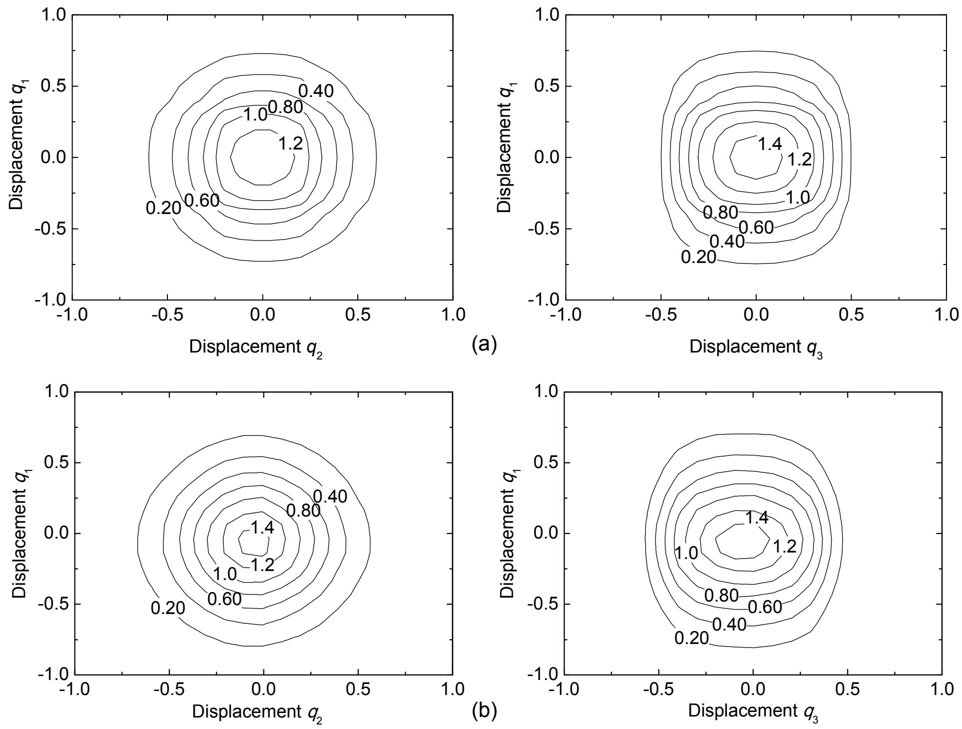

Fig. 6 Contours of stationary probability densities $p(q_1, q_2)$ and $p(q_1, q_3)$ of system Eq. (30)

(a) Simulated from the averaged fractional SDE Eq. (34); (b) Simulated from the original system Eq. (30). The parameters are the same as those in Fig. 2

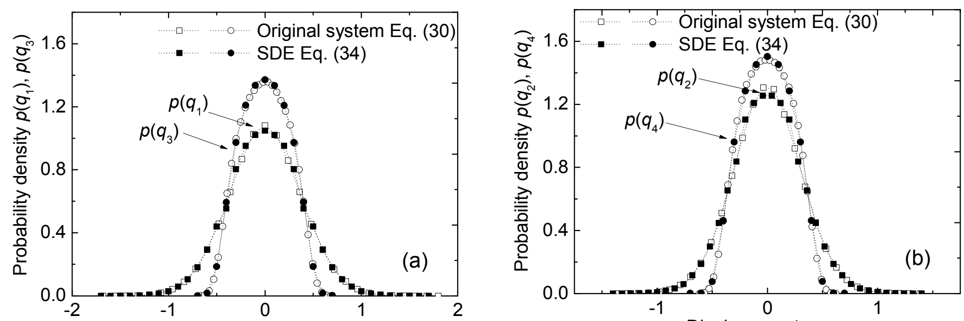

Fig. 7 Stationary probability densities $p(q_1)$, $p(q_3)$ (a), and $p(q_2)$, $p(q_4)$ (b) simulated from the averaged fractional SDE Eq. (34) and the original system Eq. (30), respectively (the parameters are the same as those in Fig. 2)

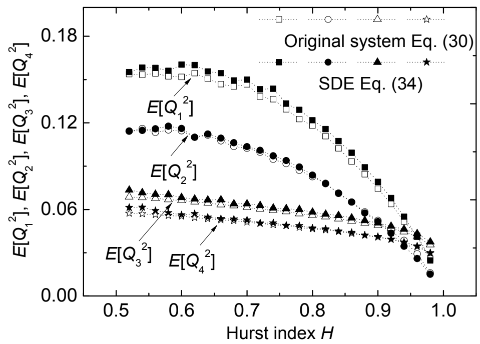

Fig. 8 Mean square values $E[Q_1^2]$, $E[Q_2^2]$, $E[Q_3^2]$, $E[Q_4^2]$ simulated from the averaged fractional SDE Eq. (34) and the original system Eq. (30), respectively (the parameters are the same as those in Fig. 2)

For 10000 samples, the simulation time of the original system Eq. (30) is 81.452 s on the i5-2400 CPU @ 3.10 GHz computer, while it takes 34.625 s to simulate SDE Eq. (34).

5 Example 2

Consider the following 3-DOF quasi Hamiltonian system:

$$\begin{aligned}
&\dot{Q}_1 = P_1,\\
&\dot{P}_1 = -\omega_1^2 Q_1 - P_1(\alpha_{10} + \alpha_{11}P_1^2 + \alpha_{12}P_2^2 + \alpha_{13}P_3^2)\\
&\qquad + \sqrt{2D_1}W_1^H(t),\\
&\dot{Q}_2 = P_2,\\
&\dot{P}_2 = -\frac{\partial U(Q_2,Q_3)}{\partial Q_2} - P_2(\alpha_{20} + \alpha_{21}P_1^2 + \alpha_{22}P_2^2 + \alpha_{23}P_3^2)\\
&\qquad + \sqrt{2D_2}W_2^H(t),\\
&\dot{Q}_3 = P_3,\\
&\dot{P}_3 = -\frac{\partial U(Q_2,Q_3)}{\partial Q_3} - P_3(\alpha_{30} + \alpha_{31}P_1^2 + \alpha_{32}P_2^2 + \alpha_{33}P_3^2)\\
&\qquad + \sqrt{2D_3}W_3^H(t),
\end{aligned} \tag{37}$$

where

$$U(Q_2,Q_3) = \frac{\left(\omega_2^2Q_2^2 + \omega_3^2Q_3^2\right)^3}{2} + \frac{b\left(\omega_2^2Q_2^2 + \omega_3^2Q_3^2\right)^3}{4}. \tag{38}$$

α_{ij} and b>0 are constants; $W_k^H(t)$ are unit fGns with the Hurst index 1/2<H<1. The Hamiltonian associated with system Eq. (37) is

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 = \omega_1 I_1 + \mathcal{H}_2, \tag{39}$$

where

$$\begin{aligned}
&I_1 = \left(p_1^2 + \omega_1^2 q_1^2\right)/(2\omega_1),\\
&\mathcal{H}_2 = \left(p_2^2 + p_3^2\right)/2 + U(Q_2,Q_3).
\end{aligned} \tag{40}$$

System Eq. (37) is a quasi partially integrable Hamiltonian system if α_{ij} and D_k are of the order of ε^{2H}. The averaged fractional SDEs are of the form:

$$\begin{aligned}
&\mathrm{d}I_1 = \bar{m}_1(I_1,\mathcal{H}_2)\mathrm{d}t + \bar{\sigma}_{11}(I_1,\mathcal{H}_2)\mathrm{d}^- B_1^H(t),\\
&\mathrm{d}\mathcal{H}_2 = \bar{m}_2(I_1,\mathcal{H}_2)\mathrm{d}t + \bar{\sigma}_{22}(I_1,\mathcal{H}_2)\mathrm{d}^- B_2^H(t),
\end{aligned} \tag{41}$$

where the drift and diffusion coefficients can be obtained as follows:

$$\begin{aligned}
&\bar{m}_1(I_1,\mathcal{H}_2) = -\Big[\alpha_{10}I_1 + 3\alpha_{11}\omega_1 I_1^2/2\\
&\qquad + (\alpha_{12}+\alpha_{13})\left(\mathcal{H}_2 - \frac{1}{4}R^2 - \frac{b}{12}R^4\right)I_1\Big],\\
&\bar{m}_2(I_1,\mathcal{H}_2) = -(\alpha_{20} + \alpha_{21}\omega_1 I_1 + \alpha_{30}\\
&\qquad + \alpha_{31}\omega_1 I_1)\left(\mathcal{H}_2 - \frac{1}{4}R^2 - \frac{b}{12}R^4\right)\\
&\qquad - \frac{1}{8}(3\alpha_{22} + \alpha_{23} + \alpha_{32} + 3\alpha_{33})\Big[4\mathcal{H}_2^2\\
&\qquad + 8\mathcal{H}_2R^2\left(\frac{1}{4} + \frac{b}{12}R^2\right) + R^4\left(\frac{1}{3} + \frac{b^2}{20}R^4 + \frac{b}{4}R^2\right)\Big],\\
&\bar{\sigma}_{11}^2(I_1,\mathcal{H}_2) = 2D_1 I_1/\omega_1,\\
&\bar{\sigma}_{22}^2(I_1,\mathcal{H}_2) = 2(D_2+D_3)\left(\mathcal{H}_2 - \frac{1}{4}R^2 - \frac{b}{12}R^4\right),\\
&\bar{\sigma}_{12}^2 = \bar{\sigma}_{21}^2 = 0,
\end{aligned} \tag{42}$$

where

$$R = \mathrm{sqrt}\left(\frac{\sqrt{1+4b\mathcal{H}_2} - 1}{b}\right). \tag{43}$$

$p(I_1, \mathcal{H}_2)$ is directly simulated from the original system Eq. (37) and averaged fractional SDE Eq. (41), respectively. $E[I_1]$, $E[\mathcal{H}_2]$ and $E[I_1^2]$, $E[\mathcal{H}_2^2]$ can be obtained by integration. The joint stationary probability density of generalized displacements and momenta is calculated from $p(I_1, \mathcal{H}_2)$ as follows:

$$p(\boldsymbol{q}, \boldsymbol{p}) = \frac{p(I_1, \mathcal{H}_2)}{2\pi T(\mathcal{H}_2)}\Bigg|_{\substack{I_1=(p_1^2+\omega_1^2 q_1^2)/(2\omega_1), \\ \mathcal{H}_2=(p_2^2+p_3^2)/2+U(q_2+q_3)}}. \quad (44)$$

Then, $p(q_1, q_2)$, $p(q_1, q_3)$ and $E[Q_1^2]$, $E[Q_2^2]$, $E[Q_3^2]$ can be calculated from Eq. (44) by integration. Some numerical results for $p(I_1, \mathcal{H}_2)$, $p(I_1)$, $p(\mathcal{H}_2)$, $E[I_1]$, $E[\mathcal{H}_2]$, and $E[I_1^2]$, $E[\mathcal{H}_2^2]$ are shown in Figs. 9–12, and those for $p(q_1, q_2)$, $p(q_2, q_3)$, $p(q_1)$, $p(q_2)$, $p(q_3)$, and $E[Q_1^2]$, $E[Q_2^2]$, $E[Q_3^2]$ are shown in Figs. 13–15. It is seen that the error between the results calculated from averaged fractional SDEs and those from original system is acceptable.

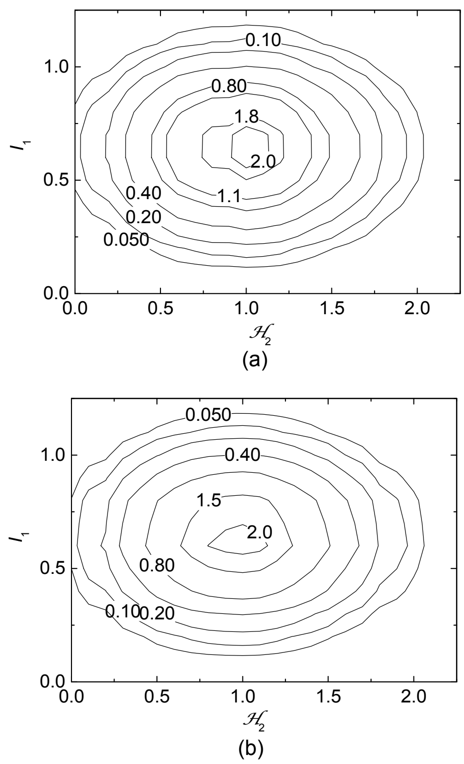

Fig. 9 Contours of stationary probability density $p(I_1, \mathcal{H}_2)$ of system Eq. (37)

(a) Simulated from the averaged fractional SDE Eq. (41); (b) Simulated from the original system Eq. (37). The parameters are $\alpha_{10}=\alpha_{20}=\alpha_{30}=-0.08$, $\alpha_{12}=\alpha_{13}=0.01$, $\alpha_{21}=\alpha_{31}=0.02$, $\alpha_{11}=\alpha_{22}=\alpha_{33}=0.01$, $\alpha_{23}=\alpha_{32}=0.04$, $\omega_1=1.414$, $\omega_2=1$, $\omega_3=1.732$, $b=1$, $D_1=0.005$, $D_2=0.01$, $D_3=0.015$, and $H=0.75$

For 10000 samples, the simulation time of the original system Eq. (37) is 61.984 s on the i5-2400 CPU @ 3.10 GHz computer, while simulation of SDE Eq. (41) requires 23.341 s.

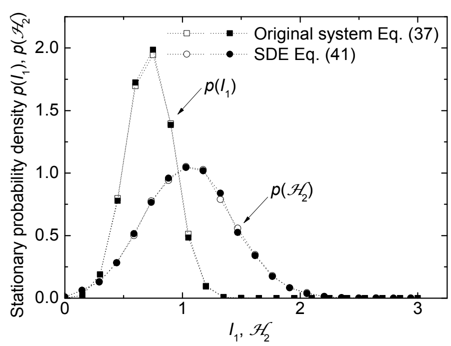

Fig. 10 Stationary probability densities $p(I_1)$ and $p(\mathcal{H}_2)$ calculated from the averaged Eq. (41) and from the original system Eq. (37), respectively (the parameters are the same as those in Fig. 9)

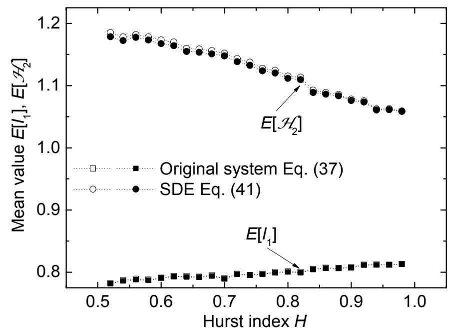

Fig. 11 Mean values $E[I_1]$ and $E[\mathcal{H}_2]$ simulated from the averaged fractional SDE Eq. (37) and the original system Eq. (41), respectively (the parameters are the same as those in Fig. 9)

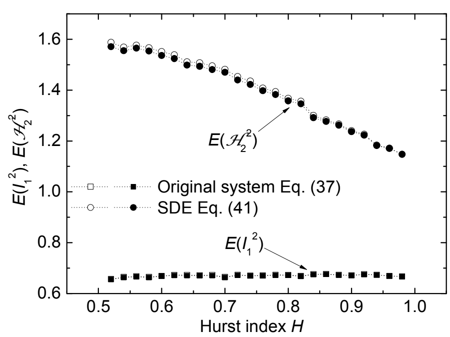

Fig. 12 Mean square values $E[I_1^2]$ and $E[\mathcal{H}_2^2]$ simulated from the averaged fractional SDE Eq. (41) and the original system Eq. (37), respectively (the parameters are the same as those in Fig. 9)

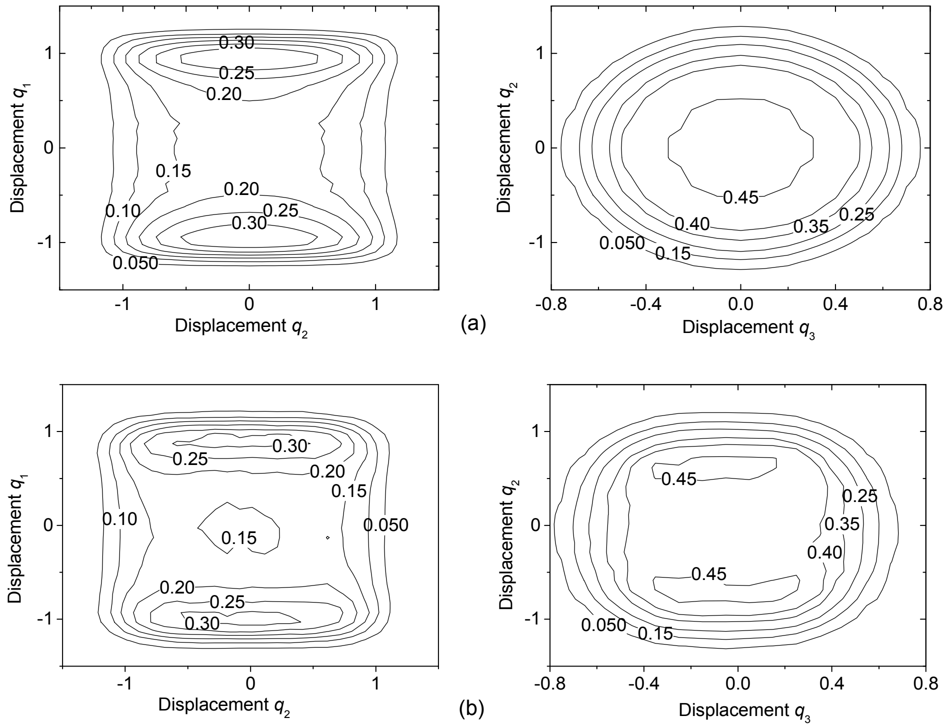

Fig. 13 Contours of stationary probability densities $p(q_1, q_2)$ and $p(q_2, q_3)$ of system Eq. (37)
(a) Simulated from the averaged fractional SDE Eq. (41); (b) Simulated from the original system Eq. (37). The parameters are the same as those in Fig. 9

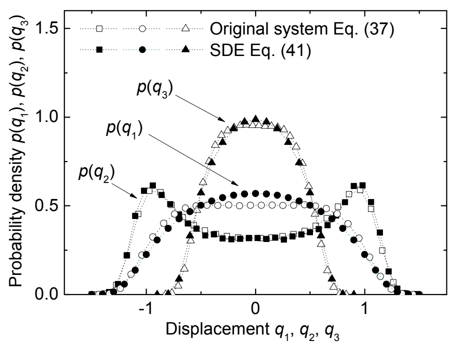

Fig. 14 Stationary probability densities $p(q_1)$, $p(q_2)$, $p(q_3)$ simulated from the averaged fractional SDE Eq. (41) and the original system Eq. (37), respectively (the parameters are the same as those in Fig. 9)

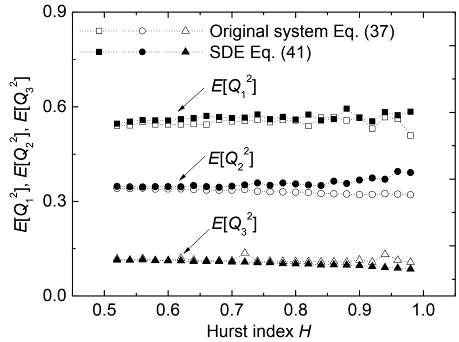

Fig. 15 Mean square values $E[Q_1^2]$, $E[Q_2^2]$, $E[Q_3^2]$ simulated from the averaged fractional SDE Eq. (41) and the original system Eq. (37), respectively (the parameters are the same as those in Fig. 9)

6 Conclusions

The response of a dynamical system to fGn is not a Markov process and the classical diffusion process theory cannot be applied to predict the response. That makes the study of nonlinear dynamics with fGn excitation very difficult. In this paper, a stochastic averaging method for predicting the response of quasi partially integrable and non-resonant Hamiltonian systems to fGn with $1/2<H<1$ has been proposed. The prominent advantage of this method is that the dimension of the averaged fractional SDEs is less than a half of that of the original system and the averaged fractional SDEs involve only slowly varying processes. Thus, the computation time for simulating averaged fractional SDEs is much less than that for original system. The results of two examples have shown that the probability density and statistics of first integrals simulated from averaged fractional SDEs and those from the original system agree well while the error between probability density and the statistics of displacements calculated from averaged fractional SDEs and those from the original system is acceptable. Therefore, the proposed stochastic averaging method is quite promising.

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中文概要

题　目： 分数阶高斯噪声激励下拟部分可积哈密顿系统的随机平均法

目　的： 提出预测分数阶高斯噪声激励下拟部分可积非共振哈密顿系统的稳态响应的方法。

创新点： 现有文献中，对于分数阶高斯噪声激励下动态系统响应的研究，多为单自由度或二自由度线性系统，而本文的方法针对的是多自由度强非线性系统，可预测分数阶高斯噪声激励下的多自由度强非线性系统的稳态响应。

方　法： 1. 根据分数阶布朗运动的顺式积分原理及其随机微分规则，将分数阶高斯噪声激励下的多自由度强非线性系统模型化为分数阶高斯噪声激励下的拟部分可积哈密顿系统。2. 运用随机平均原理进行降维，得到维数更低的分数阶随机微分方程组，由此，原系统可被这组方程近似代替。3. 运用数值方法求解分数阶随机微分方程组，得到原系统的近似稳态响应。

结　论： 1. 从平均后的分数阶随机微分方程组模拟得到的近似稳态响应与原系统方程模拟得到的稳态响应吻合度较高，说明了此方法的有效性。2. 模拟平均后的分数阶随机微分方程组的时间比模拟原系统方程的时间短很多，说明此方法效率高。

关键词： 分数布朗运动；分数高斯噪声；拟部分可积哈密顿系统；随机平均法；稳态响应