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Triangular domain extension of linear Bernstein-like trigonometric polynomial basis*

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Abstract: In computer aided geometric design (CAGD), the Bernstein-Bézier system for polynomial space including the triangular domain is an important tool for modeling free form shapes. The Bernstein-like bases for other spaces (trigonometric polynomial, hyperbolic polynomial, or blended space) has also been studied. However, none of them was extended to the triangular domain. In this paper, we extend the linear trigonometric polynomial basis to the triangular domain and obtain a new Bernstein-like basis, which is linearly independent and satisfies positivity, partition of unity, symmetry, and boundary representation. We prove some properties of the corresponding surfaces, including differentiation, subdivision, convex hull, and so forth. Some applications are shown.

Key words: Computer aided geometric design (CAGD), Free form modeling, Trigonometric polynomial, Basis function, Bernstein basis, Triangular domain
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1 Introduction

In computer aided geometric design (CAGD), the Bézier system with Bernstein basis on polynomial space is widely used for modeling free form shapes. In recent decades, some researchers considered the Bernstein-like bases for other spaces (Mainar and Peña, 2007), such as the trigonometric polynomial space (Peña, 1997; Sanchez-Reyes, 1998; 1999), the hyperbolic polynomial space (Shen and Wang, 2005), the algebraic trigonometric polynomial blended space (Zhang, 1996; 1999; Chen and Wang, 2003), the algebraic hyperbolic polynomial blended space (Li and Wang, 2005), the algebraic trigonometric hyperbolic polynomial blended space (Xu and Wang, 2006), and the algebraic ω trigonometric polynomial blended space (Fang and Wang, 2007). The properties and applications of these bases and their curves and surfaces were also studied, such as shape preserving (Mainar *et al.*, 2001), convergence (Dong and Wang, 2004), shape diagram (Yang and Wang, 2004; Hoffmann *et al.*, 2006; Juhasz, 2006; Cao and Wang, 2007b; Li *et al.*, 2008), triangular patch (Li *et al.*, 2005), conversion matrix (Fan and Wang, 2006), and highway route design (Cai and Wang, 2009). However, the basis has not been extended to the triangular domain, because of the lack of theory of bivariate trigonometric polynomial space. But, the Bernstein-Bézier surface over the triangular domain (Farin, 1986) is an important aspect in the Bézier system. Its properties, such as convexity (Chang and Davis, 1984; Chang and Feng, 1984; Wang and Liu, 1988; Chang and Zhang, 1990; Sauer, 1991; Carnicer *et al.*, 1997), evaluation (Schumaker and Volk, 1986; Mainar and Peña, 2006), and parametric extension (Cao and Wang, 2007a), have attracted some researchers. In this paper, without considering what the bivariate trigonometric polynomial space is, we simply focus

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on basis functions. We give the Bernstein-like basis over the triangular domain, which corresponds to the Bernstein-like basis for the linear trigonometric polynomial space Γ_1=span{1, sin$t$, cos$t$}. Our basis fulfills positivity, normalization, symmetry, boundary properties, and linear independence. It can be transformed to another basis that seems to be more appropriate for building the bivariate trigonometric polynomial space.

2 Linear trigonometric polynomial basis and surfaces over the triangular domain

In this section, p-Bézier basis (Sanchez-Reyes, 1998) for Γ_1 is reviewed first. Then we extend the Bernstein-like basis to the triangular domain. Finally we define the corresponding surfaces.

2.1 Review

The Bernstein-like basis on Γ_1 is called p-Bézier:

$$A_0^a(t)=\sin^2\frac{a-t}{2}\bigg/\sin^2\frac{a}{2},\ \ A_2^a(t)=\sin^2\frac{t}{2}\bigg/\sin^2\frac{a}{2},$$
$$A_1^a(t)=2\cos\frac{a}{2}\sin\frac{a-t}{2}\sin\frac{t}{2}\bigg/\sin^2\frac{a}{2},\quad t\in[0,a],$$

where $a\in(0, \pi)$ controls the domain.

2.2 Definition of basis over the triangular domain

Definition 1 Suppose that $a\in(0, \pi)$ is a real number and u, v, w are three variables that satisfy the conditions $0\le u$, v, $w\le a$ and $u+v+w=a$. Then, the function sequence $\{B_i^a(u,v,w)=B_i^a\}_{i=0}^6$ is the triangular domain extension of $\{A_i^a(t)\}_{i=0}^2$, and is defined as follows:

$$B_0^a=\frac{\sin^2\frac{u}{2}}{\sin^2\frac{a}{2}},\ \ B_1^a=\frac{\sin^2\frac{v}{2}}{\sin^2\frac{a}{2}},\ \ B_2^a=\frac{\sin^2\frac{w}{2}}{\sin^2\frac{a}{2}},$$
$$B_3^a=2\sin\frac{a}{2}\sin\frac{u}{2}\sin\frac{v}{2}\sin\frac{w}{2}\bigg/\sin^2\frac{a}{2},$$
$$B_4^a=2\cos\frac{a}{2}\sin\frac{u}{2}\sin\frac{v}{2}\cos\frac{w}{2}\bigg/\sin^2\frac{a}{2},$$
$$B_5^a=2\cos\frac{a}{2}\cos\frac{u}{2}\sin\frac{v}{2}\sin\frac{w}{2}\bigg/\sin^2\frac{a}{2},$$
$$B_6^a=2\cos\frac{a}{2}\sin\frac{u}{2}\cos\frac{v}{2}\sin\frac{w}{2}\bigg/\sin^2\frac{a}{2}.$$

Fig. 1 shows curves of some basis functions. Because of the symmetry of $\{B_i^a\}_{i=0}^6$, we give only $B_0^a(u), B_3^a(u,v,a-u-v), B_4^a(u,v,a-u-v)$. Here a=2, and since the value of B_3^a is relatively small, we use a different scaling factor along the vertical Z-axis.

2.3 Definition of surfaces over the triangular domain

The corresponding surfaces are defined in the usual way.

Definition 2 Suppose that $a\in(0, \pi)$. The Bernstein-Bézier-like surface over the triangular domain $D=\{(u,v,w)|0\le u, v, w\le a, u+v+w=a\}$ defined by $\{B_i^a\}_{i=0}^6$ can be written as

$$S(u,v,w)=\sum_{i=0}^{6}B_i^a(u,v,w)\boldsymbol{P}_i,\ \ (u,v,w)\in D,$$

where $\{\boldsymbol{P}_i\}_{i=0}^6$ are the control points.

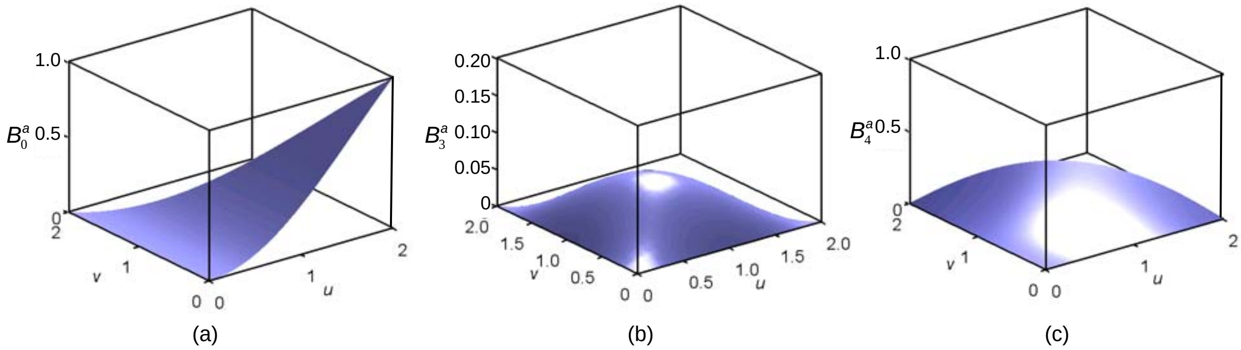
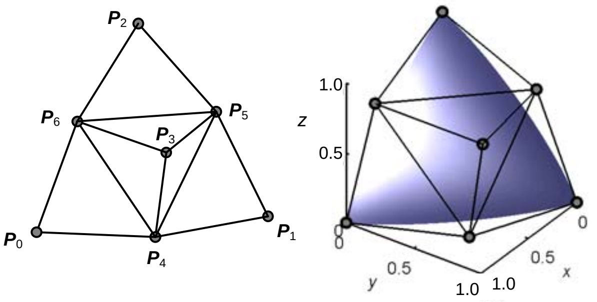


Fig. 1 Basis functions (a=2): (a) B_0^a ; (b) B_3^a ; (c) B_4^a

Control points and the control mesh are illustrated in Fig. 2a. Fig. 2b shows a triangular surface, where a=2 and control points are $\{\boldsymbol{P}_i\}_{i=0}^{6}$ ={(1,0,0), (0,1,0), (0,0,1), (0.7,0.8,0.7), (0.7,0.7,0), (0,0.7,0.7), (0.7,0,0.7)}.



(a) (b)

Fig. 2 (a) The control mesh of *S*(*u*,*v*,*w*), where $\{\boldsymbol{P}_i\}_{i=0}^{6}$ are control points; (b) The surface determined by these data

3 Properties

In this section, we show the properties of our basis and the corresponding surfaces.

3.1 Basis properties

Proposition 1 Function system $\{B_i^a(u,v,w)\}_{i=0}^{6}$ in Definition 1 has the following properties for all (u, v, w) in D:

(1) Positivity. $B_i^a(u,v,w)\geq 0,\ 0\leq i\leq 6$.

(2) Normalization. $\sum_{i=0}^{6} B_i^a(u,v,w)\equiv 1$.

(3) Symmetry. The set of functions $\{B_i^a(u,v,w)\}_{i=0}^{6}$ is symmetric with respect to all permutations of (u,v,w).

(4) Boundary property. When one of the three variables u, v, w is taken as zero, the basis $\{B_i^a\}_{i=0}^{6}$ degenerates to the univariate basis $\{A_i^a\}_{i=0}^{2}$.

(5) Linear independence. The functions B_i^a (i=0, 1, …, 6) are linearly independent.

Proof (1) and (3) are obvious.

(2) Using the constraint u+v+w=a, we can obtain the property by the following calculation:

$$B_3^a + B_4^a = \left(2\sin\frac{u}{2}\sin\frac{v}{2}\cos\frac{u+v}{2}\right)\Big/\sin^2\frac{a}{2},$$

$$B_0^a + B_1^a + (B_3^a + B_4^a) = \sin^2\frac{u+v}{2}\Big/\sin^2\frac{a}{2} = A_0^a(w),$$

$$B_5^a + B_6^a = 2\cos\frac{a}{2}\sin\frac{w}{2}\sin\frac{u+v}{2}\Big/\sin^2\frac{a}{2} = A_1^a(w),$$

$$B_2^a = A_2^a(w),\ \text{so}\ \sum_{i=0}^{6} B_i^a(u,v,w) = \sum_{i=0}^{2} A_i^a(w)\equiv 1.$$

(4) Supposing that u=0 and using v=a−w, we have $B_0^a = B_3^a = B_4^a = B_6^a = 0$ and $B_1^a = A_0^a(w)$, $B_2^a = A_2^a(w)$, $B_5^a = A_1^a(w)$, which proves the boundary property. This also applies for the conditions when v=0 or w=0.

(5) Assume that for any (u,v,w), $\sum_{i=0}^{6} b_i B_i^a(u,v,w) = 0$, where b_i (i=0, 1, …, 6) are real numbers, and let u=0. Based on the boundary property, we have $b_1 A_0^a + b_5 A_1^a + b_2 A_2^a = 0$. Since basis functions A_i^a (i=0, 1, 2) are linearly independent, b_1=b_2=b_5=0. Similarly, b_0=b_4=b_6=0 and finally b_3=0.

Proposition 2 $\{B_i^a\}_{i=0}^{6}$ and $\{C_i\}_{i=0}^{6}$ = {1, sin u, cos u, sin v, cos v, sin w, cos w} are two different bases of the same space. The transformation matrices between them (in both directions) are $\boldsymbol{M}_a$ and $\boldsymbol{M}_a^{-1}$. That is, $[C_0,C_1,\dots,C_6]=[B_0^a,B_1^a,\dots,B_6^a]\boldsymbol{M}_a$, and $[B_0^a,B_1^a,\dots,B_6^a]$ $=[C_0,C_1,\dots,C_6]\,\boldsymbol{M}_a^{-1}$, where

$$\boldsymbol{M}_a = \begin{pmatrix} 1 & \sin a & \cos a & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & \sin a & \cos a & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & \sin a & \cos a \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & \tan\frac{a}{2} & 1 & \tan\frac{a}{2} & 1 & 0 & 1 \\ 1 & 0 & 1 & \tan\frac{a}{2} & 1 & \tan\frac{a}{2} & 1 \\ 1 & \tan\frac{a}{2} & 1 & 0 & 1 & \tan\frac{a}{2} & 1 \end{pmatrix}.$$

For $\boldsymbol{M}_a^{-1}$, see the top of the next page.

Proof Calculating with triangle formulas, we have the equation as shown in the next page, where $\det\boldsymbol{M}_a=-2(1-\cos a)^3\tan^3(a/2)\neq 0$ for any $a\in(0,\pi)$, and $\boldsymbol{M}_a^{-1}=(\boldsymbol{M}_a)^{-1}$.

$$\boldsymbol{M}_a^{-1}=\begin{pmatrix} (1-\cos a)^{-1} & (1-\cos a)^{-1} & (1-\cos a)^{-1} & -1/2 & -\cot^2(a/2)/2 & -\cot^2(a/2)/2 & -\cot^2(a/2)/2 \\ 0 & 0 & 0 & -\cot(a/2)/2 & \cot(a/2)/2 & -\cot(a/2)/2 & \cot(a/2)/2 \\ -(1-\cos a)^{-1} & 0 & 0 & 1/2 & \cot^2(a/2)/2 & -\cot^2(a/2)/2 & \cot^2(a/2)/2 \\ 0 & 0 & 0 & -\cot(a/2)/2 & \cot(a/2)/2 & \cot(a/2)/2 & -\cot(a/2)/2 \\ 0 & -(1-\cos a)^{-1} & 0 & 1/2 & \cot^2(a/2)/2 & \cot^2(a/2)/2 & -\cot^2(a/2)/2 \\ 0 & 0 & 0 & -\cot(a/2)/2 & -\cot(a/2)/2 & \cot(a/2)/2 & \cot(a/2)/2 \\ 0 & 0 & -(1-\cos a)^{-1} & 1/2 & -\cot^2(a/2)/2 & \cot^2(a/2)/2 & \cot^2(a/2)/2 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ \sin u \\ \cos u \\ \sin v \\ \cos v \\ \sin w \\ \cos w \end{pmatrix}^{\mathrm{T}}=\begin{pmatrix} B_0^a \\ B_1^a \\ B_2^a \\ B_3^a \\ B_4^a \\ B_5^a \\ B_6^a \end{pmatrix}^{\mathrm{T}}\begin{pmatrix} 1 & \sin a & \cos a & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & \sin a & \cos a & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & \sin a & \cos a \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & \tan(a/2) & 1 & \tan(a/2) & 1 & 0 & 1 \\ 1 & 0 & 1 & \tan(a/2) & 1 & \tan(a/2) & 1 \\ 1 & \tan(a/2) & 1 & 0 & 1 & \tan(a/2) & 1 \end{pmatrix}=\begin{pmatrix} B_0^a \\ B_1^a \\ B_2^a \\ B_3^a \\ B_4^a \\ B_5^a \\ B_6^a \end{pmatrix}^{\mathrm{T}}\boldsymbol{M}_a$$

3.2 Surface properties

Proposition 3 Surface $S(u,v,w)$ in Definition 2 has the properties described below:

(1) Convex hull property. $S(u,v,w)$ is contained in the convex hull of its control points.

(2) Corner interpolation. $S(a,0,0)=\boldsymbol{P}_0$, $S(0,a,0)=\boldsymbol{P}_1$, $S(0,0,a)=\boldsymbol{P}_2$.

(3) Affine invariance. $S(u,v,w)$ is invariant under affine transformations of its control points.

These surface properties are easily obtained according to the properties of basis functions, so there is no need to prove them here.

Proposition 4 (Differentiation property) The tangent plane of $S(u,v,w)$ can be represented as a surface determined by basis functions $\{B_i^a\}_{i=0}^6$. Specifically, the tangent planes at $(a,0,0)$, $(0,a,0)$, and $(0,0,a)$ are planes spanned by control points $\boldsymbol{P}_0\boldsymbol{P}_4\boldsymbol{P}_6$, $\boldsymbol{P}_1\boldsymbol{P}_4\boldsymbol{P}_5$, and $\boldsymbol{P}_2\boldsymbol{P}_5\boldsymbol{P}_6$, respectively.

Proof Let $w=a-u-v$. The tangent plane of $S(u,v,w)$ is ($\alpha, \beta\in\mathbb{R}$)

$$\begin{aligned} T(u,v)=&S(u,v,w)+\alpha\frac{\partial}{\partial u}S(u,v,w)+\beta\frac{\partial}{\partial v}S(u,v,w) \\ =&\left[\boldsymbol{P}_0+\alpha\left(\frac{1}{2}\csc\frac{a}{2}\boldsymbol{P}_0-\cot\frac{a}{2}\boldsymbol{P}_6\right)+\beta\cot\frac{a}{2}(\boldsymbol{P}_4-\boldsymbol{P}_6)\right]B_0^a \\ &+\left[\boldsymbol{P}_1+\alpha\cot\frac{a}{2}(\boldsymbol{P}_4-\boldsymbol{P}_5)+\beta\left(\frac{1}{2}\csc\frac{a}{2}\boldsymbol{P}_1-\cot\frac{a}{2}\boldsymbol{P}_5\right)\right]B_1^a \\ &+\left[\boldsymbol{P}_2+\alpha\left(\cot\frac{a}{2}\boldsymbol{P}_6-\frac{1}{2}\csc\frac{a}{2}\boldsymbol{P}_2\right)+\beta\left(\cot\frac{a}{2}\boldsymbol{P}_5-\frac{1}{2}\csc\frac{a}{2}\boldsymbol{P}_2\right)\right]B_2^a \\ &+\left[\boldsymbol{P}_3+\alpha\cot\frac{a}{2}(\boldsymbol{P}_4-\boldsymbol{P}_5)+\beta\cot\frac{a}{2}(\boldsymbol{P}_4-\boldsymbol{P}_6)\right]B_3^a \\ &+\left[\boldsymbol{P}_4+\alpha\cot\frac{a}{2}\left(\frac{1}{2}\sec^2\frac{a}{2}\boldsymbol{P}_0-\tan^2\frac{a}{2}\boldsymbol{P}_3+\boldsymbol{P}_4-\boldsymbol{P}_5-\boldsymbol{P}_6\right)\right. \\ &\left.+\beta\cot\frac{a}{2}\left(\frac{1}{2}\sec^2\frac{a}{2}\boldsymbol{P}_1-\tan^2\frac{a}{2}\boldsymbol{P}_3+\boldsymbol{P}_4-\boldsymbol{P}_5-\boldsymbol{P}_6\right)\right]B_4^a \\ &+\left[\boldsymbol{P}_5+\alpha\cot\frac{a}{2}\left(\tan^2\frac{a}{2}\boldsymbol{P}_3-\frac{1}{2}\sec^2\frac{a}{2}\boldsymbol{P}_2+\boldsymbol{P}_4-\boldsymbol{P}_5+\boldsymbol{P}_6\right)\right. \\ &\left.+\beta\csc a(\boldsymbol{P}_1-\boldsymbol{P}_2)\right]B_5^a \\ &+\left[\boldsymbol{P}_6+\alpha\csc a(\boldsymbol{P}_0-\boldsymbol{P}_2)\right. \\ &\left.+\beta\cot\frac{a}{2}\left(\tan^2\frac{a}{2}\boldsymbol{P}_3-\frac{1}{2}\sec^2\frac{a}{2}\boldsymbol{P}_2+\boldsymbol{P}_4+\boldsymbol{P}_5-\boldsymbol{P}_6\right)\right]B_6^a. \end{aligned}$$

Let $(u,v,w)=(a,0,0)$. Then

$$T(u,v)=\boldsymbol{P}_0+\alpha\left(\frac{1}{2}\boldsymbol{P}_0\csc\frac{a}{2}-\boldsymbol{P}_6\cot\frac{a}{2}\right)+\beta(\boldsymbol{P}_4-\boldsymbol{P}_6)\cot\frac{a}{2}.$$

The case when (u,v,w) equals $(0,a,0)$ or $(0,0,a)$ can analogously be proved.

Proposition 5 (Subdivision property) Suppose that D^* is an arbitrary triangular domain contained in D, the three sides of which are parallel to the sides of D. That is, we suppose that u^*, v^*, w^* are three real numbers satisfying the conditions $0\le u^*, v^*, w^*\le a$ and $0\le u^*+v^*, u^*+w^*, v^*+w^*\le a$. Thus we have the domain type $D^*=\{(u,v,w)|u\ge u^*, v\ge v^*, w\ge w^*, u+v+w=a\}$ (the domain in Fig. 3a) or $D^*=\{(u,v,w)|u\le u^*, v\le v^*, w\le w^*, u+v+w=a\}$ (the domain in Fig. 3b). Let $u^*+v^*+w^*=a^*$. Then there are two related properties.

(1) The subsurface over D^* can be represented as

$$S(u,v,w)=\begin{cases}\sum_{i=0}^{6}B_i^{a-a^*}(u-u^*,v-v^*,w-w^*)\boldsymbol{Q}_i,\ a^*<a,\\ \sum_{i=0}^{6}B_i^{a^*-a}(u^*-u,v^*-v,w^*-w)\boldsymbol{Q}_i,\ a^*>a,\end{cases}$$

$$(u,v,w)\in D^*,$$

where

$$[\boldsymbol{Q}_0,\boldsymbol{Q}_1,\boldsymbol{Q}_2,\boldsymbol{Q}_3,\boldsymbol{Q}_4,\boldsymbol{Q}_5,\boldsymbol{Q}_6]^{\mathrm{T}}=\boldsymbol{N}_{u^*,v^*,w^*}[\boldsymbol{P}_0,\boldsymbol{P}_1,\boldsymbol{P}_2,\boldsymbol{P}_3,\boldsymbol{P}_4,\boldsymbol{P}_5,\boldsymbol{P}_6]^{\mathrm{T}}.$$

For $\boldsymbol{N}_{u^*,v^*,w^*}$ see the bottom of this page, where

$U=a-v^*-w^*$, $V=a-u^*-w^*$, $W=a-u^*-v^*$,

$$f_0(u,v,w)=\frac{\sin\left(\frac{a}{2}-u\right)+\sin\left(\frac{a}{2}-v\right)+\sin\left(\frac{a}{2}-w\right)-\sin\frac{a}{2}}{2\sin\frac{a}{2}},$$

$$f_1(u,v,w)=\frac{\cos\frac{a}{2}\left[\cos\left(\frac{a}{2}-u\right)+\cos\left(\frac{a}{2}-v\right)-\cos\left(\frac{a}{2}-w\right)-\cos\frac{a}{2}\right]}{2\sin^2\frac{a}{2}},$$

$$f_2(u,v,w)=\left(\sin\frac{a-v-w}{2}\sin\frac{u}{2}\right)\Big/\left(\sin^2\frac{a}{2}\cos\frac{a-u-v-w}{2}\right),$$

$$f_3(u,v,w)=\frac{\left(\sin\frac{a-u-w}{2}\sin\frac{a-v-w}{2}+\sin\frac{u}{2}\sin\frac{v}{2}\right)\sin\frac{w}{2}}{\sin\frac{a}{2}\cos\frac{a-u-v-w}{2}},$$

$$f_4(u,v,w)=\frac{\cos\frac{a}{2}\left(\sin\frac{a-u-w}{2}\sin\frac{a-v-w}{2}+\sin\frac{u}{2}\sin\frac{v}{2}\right)\cos\frac{w}{2}}{\sin^2\frac{a}{2}\cos\frac{a-u-v-w}{2}},$$

$$f_5(u,v,w)=\frac{\cos\frac{a}{2}\left(\sin\frac{a-u-w}{2}\cos\frac{a-v-w}{2}+\cos\frac{u}{2}\sin\frac{v}{2}\right)\sin\frac{w}{2}}{\sin^2\frac{a}{2}\cos\frac{a-u-v-w}{2}}.$$

(2) The surface $S(u,v,w)$ over D can be subdivided into four subsurfaces S_0, S_1, S_2, S_3, where

$$S_0=\sum_{i=0}^{6}B_i^{a/2}\left(u-\frac{a}{2},v,w\right)\boldsymbol{Q}_{0i},\ (u,v,w)\in D_0,$$

$$S_1=\sum_{i=0}^{6}B_i^{a/2}\left(u,v-\frac{a}{2},w\right)\boldsymbol{Q}_{1i},\ (u,v,w)\in D_1,$$

$$S_2=\sum_{i=0}^{6}B_i^{a/2}\left(u,v,w-\frac{a}{2}\right)\boldsymbol{Q}_{2i},\ (u,v,w)\in D_2,$$

$$S_3=\sum_{i=0}^{6}B_i^{a/2}\left(\frac{a}{2}-u,\frac{a}{2}-v,\frac{a}{2}-w\right)\boldsymbol{Q}_{3i},\ (u,v,w)\in D_3,$$

For $\boldsymbol{Q}_{0i}$, $\boldsymbol{Q}_{1i}$, $\boldsymbol{Q}_{2i}$, and $\boldsymbol{Q}_{3i}$, see the next page, where $\beta=(1+\cos(a/2))^{-1}$, and domains D_0, D_1, D_2, D_3 can be seen in Fig. 3c.

$$\boldsymbol{N}_{u^*,v^*,w^*}=\begin{pmatrix}
B_0^a(U,v^*,w^*) & B_1^a(U,v^*,w^*) & B_2^a(U,v^*,w^*) & B_3^a(U,v^*,w^*) & B_4^a(U,v^*,w^*) & B_5^a(U,v^*,w^*) & B_6^a(U,v^*,w^*)\\
B_0^a(u^*,V,w^*) & B_1^a(u^*,V,w^*) & B_2^a(u^*,V,w^*) & B_3^a(u^*,V,w^*) & B_4^a(u^*,V,w^*) & B_5^a(u^*,V,w^*) & B_6^a(u^*,V,w^*)\\
B_0^a(u^*,v^*,W) & B_1^a(u^*,v^*,W) & B_2^a(u^*,v^*,W) & B_3^a(u^*,v^*,W) & B_4^a(u^*,v^*,W) & B_5^a(u^*,v^*,W) & B_6^a(u^*,v^*,W)\\
B_0^a(u^*,v^*,w^*) & B_1^a(u^*,v^*,w^*) & B_2^a(u^*,v^*,w^*) & f_0(u^*,v^*,w^*) & f_1(u^*,v^*,w^*) & f_1(v^*,w^*,u^*) & f_1(u^*,w^*,v^*)\\
f_2(u^*,v^*,w^*) & f_2(v^*,u^*,w^*) & B_2^a(u^*,v^*,w^*) & f_3(u^*,v^*,w^*) & f_4(u^*,v^*,w^*) & f_5(u^*,v^*,w^*) & f_5(v^*,u^*,w^*)\\
B_0^a(u^*,v^*,w^*) & f_2(v^*,u^*,w^*) & f_2(w^*,u^*,v^*) & f_3(v^*,w^*,u^*) & f_5(w^*,v^*,u^*) & f_4(v^*,w^*,u^*) & f_5(v^*,w^*,u^*)\\
f_2(u^*,v^*,w^*) & B_1^a(u^*,v^*,w^*) & f_2(w^*,u^*,v^*) & f_3(u^*,w^*,v^*) & f_5(w^*,u^*,v^*) & f_5(u^*,w^*,v^*) & f_4(u^*,w^*,v^*)
\end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{Q}_{00} \\ \boldsymbol{Q}_{01} \\ \boldsymbol{Q}_{02} \\ \boldsymbol{Q}_{03} \\ \boldsymbol{Q}_{04} \\ \boldsymbol{Q}_{05} \\ \boldsymbol{Q}_{06} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta/2 & \beta/2 & 0 & 0 & \beta\cos(a/2) & 0 & 0 \\ \beta/2 & 0 & \beta/2 & 0 & 0 & 0 & \beta\cos(a/2) \\ \beta/2 & 0 & 0 & 1/2 & \beta/2\cdot\cos(a/2) & -\beta/2\cdot\cos(a/2) & \beta/2\cdot\cos(a/2) \\ \beta & 0 & 0 & 0 & \beta\cos(a/2) & 0 & 0 \\ \beta/2 & 0 & 0 & \beta/2(1-\cos(a/2)) & \beta/2\cdot\cos(a/2) & \beta/2\cdot\cos(a/2) & \beta/2\cdot\cos(a/2) \\ \beta & 0 & 0 & 0 & 0 & 0 & \beta\cos(a/2) \end{pmatrix} \begin{pmatrix} \boldsymbol{P}_0 \\ \boldsymbol{P}_1 \\ \boldsymbol{P}_2 \\ \boldsymbol{P}_3 \\ \boldsymbol{P}_4 \\ \boldsymbol{P}_5 \\ \boldsymbol{P}_6 \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{Q}_{10} \\ \boldsymbol{Q}_{11} \\ \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{13} \\ \boldsymbol{Q}_{14} \\ \boldsymbol{Q}_{15} \\ \boldsymbol{Q}_{16} \end{pmatrix} = \begin{pmatrix} \beta/2 & \beta/2 & 0 & 0 & \beta\cos(a/2) & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta/2 & \beta/2 & 0 & 0 & \beta\cos(a/2) & 0 \\ 0 & \beta/2 & 0 & 1/2 & \beta/2\cdot\cos(a/2) & \beta/2\cdot\cos(a/2) & -\beta/2\cdot\cos(a/2) \\ 0 & \beta & 0 & 0 & \beta\cos(a/2) & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 & \beta\cos(a/2) & 0 \\ 0 & \beta/2 & 0 & \beta/2(1-\cos(a/2)) & \beta/2\cdot\cos(a/2) & \beta/2\cdot\cos(a/2) & \beta/2\cdot\cos(a/2) \end{pmatrix} \begin{pmatrix} \boldsymbol{P}_0 \\ \boldsymbol{P}_1 \\ \boldsymbol{P}_2 \\ \boldsymbol{P}_3 \\ \boldsymbol{P}_4 \\ \boldsymbol{P}_5 \\ \boldsymbol{P}_6 \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{Q}_{20} \\ \boldsymbol{Q}_{21} \\ \boldsymbol{Q}_{22} \\ \boldsymbol{Q}_{23} \\ \boldsymbol{Q}_{24} \\ \boldsymbol{Q}_{25} \\ \boldsymbol{Q}_{26} \end{pmatrix} = \begin{pmatrix} \beta/2 & 0 & \beta/2 & 0 & 0 & 0 & \beta\cos(a/2) \\ 0 & \beta/2 & \beta/2 & 0 & 0 & \beta\cos(a/2) & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta/2 & 1/2 & -\beta/2\cdot\cos(a/2) & \beta/2\cdot\cos(a/2) & \beta/2\cdot\cos(a/2) \\ 0 & 0 & \beta/2 & \beta/2(1-\cos(a/2)) & \beta/2\cdot\cos(a/2) & \beta/2\cdot\cos(a/2) & \beta/2\cdot\cos(a/2) \\ 0 & 0 & \beta & 0 & 0 & \beta\cos(a/2) & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 & \beta\cos(a/2) \end{pmatrix} \begin{pmatrix} \boldsymbol{P}_0 \\ \boldsymbol{P}_1 \\ \boldsymbol{P}_2 \\ \boldsymbol{P}_3 \\ \boldsymbol{P}_4 \\ \boldsymbol{P}_5 \\ \boldsymbol{P}_6 \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{Q}_{30} \\ \boldsymbol{Q}_{31} \\ \boldsymbol{Q}_{32} \\ \boldsymbol{Q}_{33} \\ \boldsymbol{Q}_{34} \\ \boldsymbol{Q}_{35} \\ \boldsymbol{Q}_{36} \end{pmatrix} = \begin{pmatrix} 0 & \beta/2 & \beta/2 & 0 & 0 & \beta\cos(a/2) & 0 \\ \beta/2 & 0 & \beta/2 & 0 & 0 & 0 & \beta\cos(a/2) \\ \beta/2 & \beta/2 & 0 & 0 & \beta\cos(a/2) & 0 & 0 \\ \beta/2 & \beta/2 & \beta/2 & -1/2 & \beta/2\cdot\cos(a/2) & \beta/2\cdot\cos(a/2) & \beta/2\cdot\cos(a/2) \\ 0 & 0 & \beta/2 & \beta/2(1-\cos(a/2)) & \beta/2\cdot\cos(a/2) & \beta/2\cdot\cos(a/2) & \beta/2\cdot\cos(a/2) \\ \beta/2 & 0 & 0 & \beta/2(1-\cos(a/2)) & \beta/2\cdot\cos(a/2) & \beta/2\cdot\cos(a/2) & \beta/2\cdot\cos(a/2) \\ 0 & \beta/2 & 0 & \beta/2(1-\cos(a/2)) & \beta/2\cdot\cos(a/2) & \beta/2\cdot\cos(a/2) & \beta/2\cdot\cos(a/2) \end{pmatrix} \begin{pmatrix} \boldsymbol{P}_0 \\ \boldsymbol{P}_1 \\ \boldsymbol{P}_2 \\ \boldsymbol{P}_3 \\ \boldsymbol{P}_4 \\ \boldsymbol{P}_5 \\ \boldsymbol{P}_6 \end{pmatrix}$$

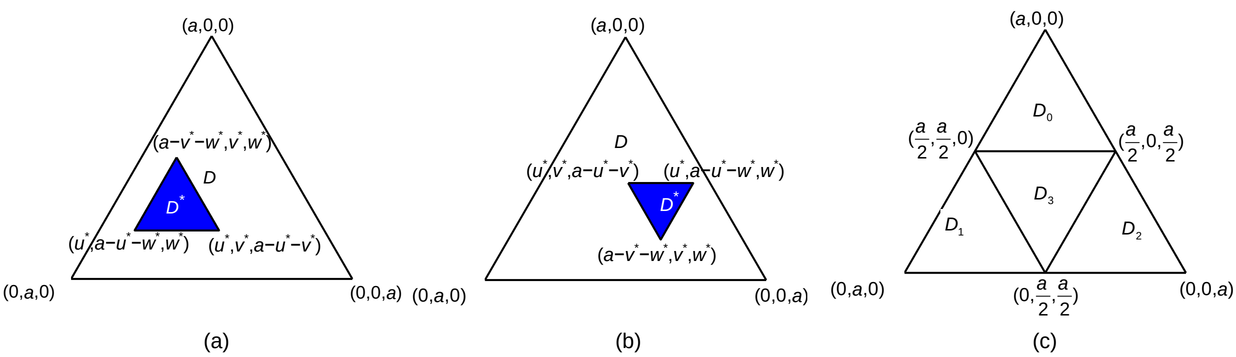


Fig. 3 The triangular domain $D=\{(u,v,w)|0\le u, v, w\le a, u+v+w=a\}$: (a) The triangular domain $D^*=\{(u,v,w)|u\ge u^*, v\ge v^*, w\ge w^*, u+v+w=a\}$; (b) The triangular domain $D^*=\{(u,v,w)|u\le u^*, v\le v^*, w\le w^*, u+v+w=a\}$; (c) Domain D is subdivided into four triangular domains D_0, D_1, D_2, D_3

Proof (1) We make use of Proposition 2. When $a^*<a$, there exist conditions $u-u^*$, $v-v^*$, $w-w^*\geq 0$ and $u-u^*+v-v^*+w-w^*=a-a^*$ on domain D^*. If $\{E_i\}_{i=0}^6=\{1, \sin(u-u^*), \cos(u-u^*), \sin(v-v^*), \cos(v-v^*), \sin(w-w^*), \cos(w-w^*)\}$ then

$$\boldsymbol{G}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos u^* & -\sin u^* & 0 & 0 & 0 & 0 \\ 0 & \sin u^* & \cos u^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos v^* & -\sin v^* & 0 & 0 \\ 0 & 0 & 0 & \sin v^* & \cos v^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos w^* & -\sin w^* \\ 0 & 0 & 0 & 0 & 0 & \sin w^* & \cos w^* \end{pmatrix}$$

is the transformation matrix between $\{E_i\}_{i=0}^6$ and $\{C_i\}_{i=0}^6$ ($[C_0, C_1, \ldots, C_6]=[E_0, E_1, \ldots, E_6]\boldsymbol{G}$). Thus,

$$\begin{aligned}&[B_0^a(u,v,w), B_1^a(u,v,w), \cdots, B_6^a(u,v,w)]\\ &=[C_0, C_1, \cdots, C_6]\boldsymbol{M}_a^{-1}=[E_0, E_1, \cdots, E_6]\boldsymbol{G}\boldsymbol{M}_a^{-1}\\ &=[B_0^{a-a^*}(u-u^*, v-v^*, w-w^*), B_1^{a-a^*}(u-u^*, v-v^*,\\ &\quad w-w^*), \cdots, B_6^{a-a^*}(u-u^*, v-v^*, w-w^*)]\boldsymbol{M}_{a^*-a}\boldsymbol{G}\boldsymbol{M}_a^{-1}.\end{aligned}$$

Let $\boldsymbol{N}_{u^*,v^*,w^*}=\boldsymbol{M}_{a^*-a}\boldsymbol{G}\boldsymbol{M}_a^{-1}$. Hence, we have $S(u,v,w)$ $=\sum_{i=0}^6 B_i^{a-a^*}(u-u^*, v-v^*, w-w^*)\boldsymbol{Q}_i,\quad (u,v,w)\in D^*$, and

$$\begin{aligned}&[\boldsymbol{Q}_0, \boldsymbol{Q}_1, \boldsymbol{Q}_2, \boldsymbol{Q}_3, \boldsymbol{Q}_4, \boldsymbol{Q}_5, \boldsymbol{Q}_6]^{\mathrm{T}}\\ &\quad=\boldsymbol{N}_{u^*,v^*,w^*}[\boldsymbol{P}_0, \boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3, \boldsymbol{P}_4, \boldsymbol{P}_5, \boldsymbol{P}_6]^{\mathrm{T}}.\end{aligned}$$

This also applies to the condition when $a^*>a$.

(2) Let $(u^*,v^*,w^*)=(a/2,0,0)$, $(0,a/2,0)$, $(0,0,a/2)$, $(a/2,a/2,a/2)$ separately. Using (1), we can obtain the control points of subsurfaces S_0, S_1, S_2, S_3.

4 Applications

In this section we show some applications of linear triangular trigonometric surfaces.

Example 1 By means of our basis one can represent some classical surfaces without rational forms. Fig. 4 shows the exact description of a triangular patch of a cylinder of revolution, the radius of which is 1 and the axis is the vertical Z-axis of the coordinate system. Here $a=2\pi/3$ and control points are $\{\boldsymbol{P}_i\}_{i=0}^6=\{(\sin a, \cos a, 1), (0, 1, \cos a), (0,1,1,), (0,1,1,), (\tan(a/2), 1,1), (0, 1,1), (\tan(a/2),1,1)\}$.

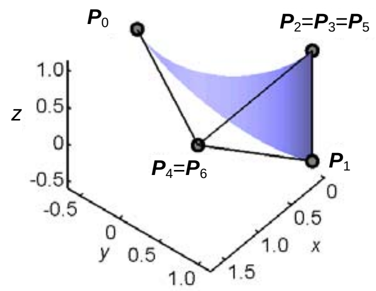
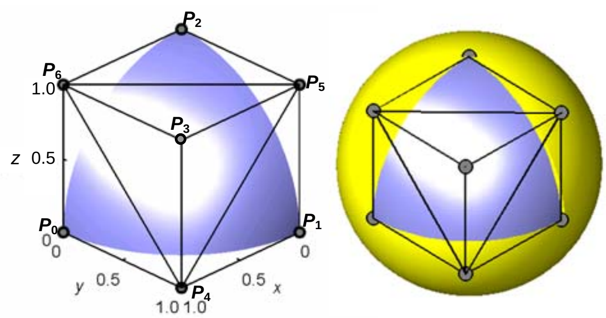


Fig. 4 The exact representation of a triangular patch of a cylinder of revolution, using our proposed basis

Example 2 Using our basis one can construct surfaces whose boundaries are circular arcs. Fig. 5 is a surface patch whose three boundaries are circular arcs which coincide with the boundaries of an octant of the unit sphere centered at the origin (Fig. 5a). Its control points are $\{\boldsymbol{P}_i\}_{i=0}^6=\{(1,0,0), (0,1,0), (0,0,1), (1,1,1), (1,1,0), (0,1,1), (1,0,1)\}$ and $a=\pi/2$. In Fig. 5b the boundaries of the previous patch are fitted onto the unit sphere. The inner points of the patch are not on this sphere, but the two surfaces are of C^0 continuity at the boundaries. However, it is impossible to adjust control point $\boldsymbol{P}_3$ to describe an octant of the sphere, because our basis cannot represent the trigonometric function and the product term of two trigonometric functions at one time, which is essential for the parameterization of a sphere.



(a) (b)

Fig. 5 (a) A triangular surface whose three sides are all circular arcs; (b) The boundaries of the patch in (a) are fitted onto the unit sphere

Example 3 The three boundaries of a surface are totally determined by $\boldsymbol{P}_0$, $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, $\boldsymbol{P}_4$, $\boldsymbol{P}_5$, $\boldsymbol{P}_6$, and $\boldsymbol{P}_3$ makes no contribution. Compared with quadratic Bernstein-Bézier surfaces over the triangular domain, our surfaces are more convenient for deformations when the boundaries should be fixed. Fig. 6 shows some phases $\boldsymbol{P}_3$ from (−2,−2,−2) to (2,2,2), and the rest of the control points, which control the three boundaries of the patch, $\{\boldsymbol{P}_0, \boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_4, \boldsymbol{P}_5, \boldsymbol{P}_6\}$= {(1,0,0), (0,1,0), (0,0,1), (1,1,0), (0,1,1), (1,0,1)} are fixed, and $a=2\pi/3$.

Fig. 6 An example for a surface deformation when the boundaries remain unchanged

5 Conclusion

In this paper, we extend basis $\{A_i^a\}_{i=0}^2$ of the linear trigonometric polynomial space Γ_1=span{1, sin t, cos t} to obtain the Bernstein-like basis $\{B_i^a\}_{i=0}^6$ over the triangular domain. This basis is linearly independent and satisfies positivity, normalization, symmetry, and boundary properties. The corresponding surfaces fulfill the convex hull property, corner interpolation, affine invariance and have differentiation, subdivision properties.

Since our basis is just an extension for linear trigonometric polynomial space and without a rational form, its ability for constructing surfaces may be limited. However, it is a core for extending the Bernstein-like basis to the triangular domain for other spaces—for trigonometric polynomial space, hyperbolic polynomial space, and blending spaces, the basis may be studied from the basis for Γ_1.

Our work is an outset of Bernstein-like systems over the triangular domain for other spaces. Thus, more work can be considered. The examination of the extended basis of higher orders, other properties and applications, and the triangular domain basis for blended spaces remain for future research.

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