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Number estimation of controllers for pinning a complex dynamical network*

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Abstract: Number estimation of controllers is a fundamental question in pinning synchronization of complex networks. This paper studies the problem of controller number in synchronizing a complex network of coupled dynamical systems by means of pinning. For a complex network with a symmetric coupling matrix and full coupling between the nodes, we formulate network synchronization via pinning as a linear matrix inequality criterion, and provide a lower bound and an upper bound of the controller number for a given complex network with fixed architecture. Several numerical examples with Barabási-Albert network topologies are provided to verify our theoretical results.

Key words: Pinning control, Number estimation, Synchronization, Complex networks
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1 Introduction

Describing and understanding various behaviors of a complex system by abstracting the system as a network consisting of many coupled nodes is a key approach (Albert and Barabási, 2002; Newman, 2003; Boccaletti *et al*., 2006). Synchronization in complex networks, as a common phenomenon of a population of dynamically interacting individuals, is one of the most demonstrated topics (Wu and Chua, 1995; Pikovsky *et al*., 2001; Arenas *et al*., 2008; Kurths *et al*., 2009). Many specific issues and collective behaviors in physics, biological, social, and computer sciences are related to synchronization. Therefore, many researchers have devoted their attention to this subject. In this endeavor, the regulation problem is potentially of great significance for synchronization of dynamical systems on complex networks (Hu *et al*., 1998; Li and Chen, 2004; Lü and Chen, 2005; Chen and Zhou., 2006; Zhou *et al*., 2006; Xiang and Chen, 2007; Xiang *et al*., 2007; Duan *et al*., 2008; Wang *et al*., 2008a; 2009).

From the control point of view, designing controllers to all nodes is a natural idea in synchronizing a complex dynamical network. When applied, each node is forced towards the desired synchronous trajectory by its controller. However, a difficulty arises due to the impossibility of controlling each node for a large-scale network. To reduce the controller number of the considered network, pinning control is thus introduced. The general idea behind pinning control is to apply local linear feedback to some nodes to achieve network synchronization. Grigoriev *et al*. (1997) and Parekh *et al*. (1998) introduced pinning to control spatiotemporal chaos in coupled map lattices. Wang and Chen (2002) considered the problem of pinning a scale-free dynamical network to its equilibrium. Sorrentino *et al*. pointed out that the controllability of complex networks through pinning can

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be evaluated by using the master stability function approach (Pecora and Carroll, 1998; Sorrentino *et al*., 2007).

The problem of pinning controllability has been successfully solved. There then appears, however, another fundamental question; i.e., how many controllers should be designed for a given dynamical network. The answer to this question directly relates the advantages of pinning to that of the other control. Li *et al*. (2004) presented two typical pinning strategies, selective pinning and random pinning, and numerically showed the effectiveness of the selective pinning strategy. Wang *et al*. (2008b) theoretically provided several pinning strategies for some networks with several typical topologies based-on linear matrix inequality. Chen *et al*. (2007) suggested that only one controller can synchronize a general coupled complex network. However, it usually requires a sufficiently large coupling strength for the controlled network to guarantee pinning synchronization. Another significant result attained by Zhou *et al*. (2008) showed an analytical estimation of controllers for a general complex dynamical network through adaptive pinning. As shown in Zhou *et al*. (2008), the coupling matrix of the controlled network has been redefined by replacing its diagonal entries. This modification leads to a non-diffusive coupling matrix and equivalently adds controllers for each node. Therefore, it has not been completely solved for number estimation of controllers in pinning a linearly diffusively coupled complex network based on the above discussions.

In this paper, we focus on the problem of controller number estimation and attempt to give some results. For the eigenvalue criterion in pinning a complex network of a coupled dynamical system, we introduce an equivalent criterion in the form of linear matrix inequality. Furthermore, we provide a necessary condition and a sufficient condition for the holding of the linear matrix inequality, which can be used to obtain a lower bound and an upper bound of the controller number in pinning a complex network.

2 Problem descriptions

In general, the equations for a linearly coupled complex dynamical network via pinning control can be formulated as follows:

$$\dot{\boldsymbol{x}}_i = \boldsymbol{f}(\boldsymbol{x}_i) - \sigma\sum_{j=1}^{N}\boldsymbol{G}_{ij}\boldsymbol{x}_j + \boldsymbol{u}_i, \quad 1 \le i \le N, \tag{1}$$

where $\boldsymbol{x}_i=(x_{i1}, x_{i2}, \ldots, x_{in})^{\mathrm{T}}\in\mathbb{R}^n$ is the state variable of node i, smooth nonlinear vector $\boldsymbol{f}(\cdot)\in\mathbb{R}^n$ governs the dynamics of each node, σ is the overall coupling strength of the whole network, and coupling matrix $\boldsymbol{G}=(G_{ij})\in\mathbb{R}^{N\times N}$ is symmetric, describing the weighted topology. To ensure the existence of the synchronous solution for network (1), we assume that the coupling matrix $\boldsymbol{G}$ is a diffusive one satisfying

$$G_{ii} = -\sum_{i=1, i\ne j}^{N} G_{ij}. \tag{2}$$

And the pinning term $\boldsymbol{u}_i$ is assigned to be a linear feedback law (Wang and Chen, 2002; Li *et al*., 2004; Sorrentino *et al*., 2007):

$$\boldsymbol{u}_i = -\sigma k_i(\boldsymbol{x}_i - \boldsymbol{s}), \tag{3}$$

where $\boldsymbol{s}(t)$ is the desired trajectory such that $\dot{\boldsymbol{s}} = \boldsymbol{f}(\boldsymbol{s})$, and $k_i\ge 0$ is the feedback gain of node i. Obviously, $k_i=0$ means no control acting on node i.

The objective of pinning is to achieve complete synchronization of network (1) by finding l nodes and controlling them, i.e.,

$$\lim_{t\to\infty}\left\|\boldsymbol{x}_i(t) - \boldsymbol{s}(t)\right\| = 0, \tag{4}$$

where $\|\cdot\|$ is the Euclidean norm, and the value of l is the number of $k_i>0$ for $1\le i\le N$.

Note that each node in network (1) is assumed to be fully coupled with its neighbors; i.e., any two coupled nodes are connected through all corresponding components of their state variables. Based on this assumption, many investigations about pinning indicate that the pinning controllability of network (1) can be evaluated by the following results:

Lemma 1 (Li *et al*., 2004; Wang *et al*., 2008b; Wu, 2008) Assume that $\boldsymbol{f}$ satisfies a Lipschitz condition. The controlled network (1) achieves synchronization about $\boldsymbol{s}(t)$ if $\lambda_1(\boldsymbol{G}+\boldsymbol{K})>\alpha$, where λ_1 is the minimum eigenvalue of the corresponding matrix, feedback gain matrix $\boldsymbol{K}=\mathrm{diag}\{k_1, k_2, \ldots, k_N\}$ describes the number and the location of controllers in pinning, and α is a positive constant determined by the coupling strength σ and node dynamics $\boldsymbol{f}$.

Lemma 1 shows that, for a particular network, there seems to be a critical value α in pinning network (1): the desired synchronization appears as the value of $\lambda_1(\boldsymbol{G}+\boldsymbol{K})$ is larger than α. In addition, the value of α can be evaluated by constructing Lyapunov functions or by computing the largest Lyapunov exponent of $\boldsymbol{f}$. Then what we want in pinning is to determine the feedback gain matrix $\boldsymbol{K}$ to guarantee the condition in Lemma 1. Hereafter, assume that α is known a priori for a given complex network.

Lemma 2 (Schur complements, Boyd *et al*., 1994) The three inequalities below are equivalent to each other:

(1) $\begin{bmatrix} \boldsymbol{Q}(\boldsymbol{v}) & \boldsymbol{P}(\boldsymbol{v}) \\ \boldsymbol{P}^{\mathrm{T}}(\boldsymbol{v}) & \boldsymbol{R}(\boldsymbol{v}) \end{bmatrix} > \boldsymbol{0}$,

(2) $\boldsymbol{Q}(\boldsymbol{v})>\boldsymbol{0}$, $\boldsymbol{M}(\boldsymbol{v})=\boldsymbol{R}(\boldsymbol{v})-\boldsymbol{P}^{\mathrm{T}}(\boldsymbol{v})\boldsymbol{Q}^{-1}(\boldsymbol{v})\boldsymbol{P}(\boldsymbol{v})>\boldsymbol{0}$,

(3) $\boldsymbol{R}(\boldsymbol{v})>\boldsymbol{0}$, $\boldsymbol{N}(\boldsymbol{v})=\boldsymbol{Q}(\boldsymbol{v})-\boldsymbol{P}(\boldsymbol{v})\boldsymbol{R}^{-1}(\boldsymbol{v})\boldsymbol{P}^{\mathrm{T}}(\boldsymbol{v})>\boldsymbol{0}$,

where $\boldsymbol{Q}(\boldsymbol{v})=\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{v})$, $\boldsymbol{R}(\boldsymbol{v})=\boldsymbol{R}^{\mathrm{T}}(\boldsymbol{v})$, and $\boldsymbol{P}(\boldsymbol{v})$ depends affinely on $\boldsymbol{v}$.

3 Main results

Theorem 1 The inequality $\lambda_1(\boldsymbol{G}+\boldsymbol{K})>\alpha$ holds if

$$\boldsymbol{G}+\boldsymbol{K}-\lambda_{\mathrm{c}}\boldsymbol{I}\geq\boldsymbol{0}, \tag{5}$$

where λ_{c} is an arbitrary constant such that $\lambda_{\mathrm{c}}>\alpha$, and $\boldsymbol{I}$ is an identity matirx with corresponding dimension.

Based on the results in Lemma 1 and Theorem 1, we further deduce that the pinning synchronization of network (1) can be guaranteed by Eq. (5). The subsequent discussions will focus on the holding of Eq. (5). In general, the choice of λ_{c} determines the controller number: selecting a large λ_{c} will lead to a large number of controllers. In what follows, we define $\lambda_{\mathrm{c}}=\alpha+\varepsilon$, where $\varepsilon>0$ is sufficiently small.

Theorem 2 If Eq. (5) holds, then $l\geq p$, where p is the positive index of inertia of matrix $(\lambda_{\mathrm{c}}\boldsymbol{I}-\boldsymbol{G})$.

Proof We rewrite Eq. (5) as

$$\boldsymbol{K}\geq\bar{\boldsymbol{G}}=\lambda_{\mathrm{c}}\boldsymbol{I}-\boldsymbol{G}. \tag{6}$$

Let r and p be the rank and the positive index of inertia of $\bar{\boldsymbol{G}}$, respectively. Then we have $N\geq r\geq p\geq 0$. Consider three cases of p as follows:

Case 1: If $p=0$, then $\bar{\boldsymbol{G}}$ is semi-negative definite. It is obvious that $l\geq p=0$ holds for any Eq. (6).

Case 2: If $p=r$, then $\bar{\boldsymbol{G}}$ is semi-positive definite. To ensure Eq. (6), the rank of matrix $\boldsymbol{G}$ should be at least larger than or equal to r, i.e., $l\geq r=p$.

Case 3: For a common case, i.e., $0<p<r$, we adopt reduction to absurdity to explain. Assume $0<l<p<r$. Then,

$$N-p<N-l=\mathrm{Ker}(\boldsymbol{K}), \tag{7}$$

where $\mathrm{Ker}(\cdot)$ is the kernel of the corresponding matrix.

Eq. (7) indicates that there exists a nonzero vector $\boldsymbol{y}_0\in\mathbb{R}^N$ such that $\boldsymbol{y}_0^{\mathrm{T}}\boldsymbol{K}\boldsymbol{y}_0=0$ and $\boldsymbol{y}_0^{\mathrm{T}}\bar{\boldsymbol{G}}\boldsymbol{y}_0=0$. Actually, for any nonzero vector $\boldsymbol{y}\in\mathbb{R}^N$, we always have $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{y}\geq\boldsymbol{y}^{\mathrm{T}}\bar{\boldsymbol{G}}\boldsymbol{y}$, which is in contradiction to that acquired above. Thus, we deduce $l\geq p$. The proof is completed.

A similar result can be found in Xiang and Chen (2007). Note that Theorem 2 provides a necessary condition for the holding of Eq. (5), which can also be used to estimate the necessary number of pinned nodes. That is, for a given coupling matrix $\boldsymbol{G}$ and constant λ_{c}, pinning requires at least p controllers to achieve synchronization of the network (1).

Lemma 3 For a coupling matrix $\boldsymbol{G}$, there must exist an orthogonal matrix $\boldsymbol{U}\in\mathbb{R}^{N\times N}$ such that $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{U}=\boldsymbol{\Gamma}$, where $\boldsymbol{U}$ is of block form as

$$\boldsymbol{U}=\begin{bmatrix}\boldsymbol{U}_1 & \boldsymbol{U}_2\\ \boldsymbol{U}_3 & \boldsymbol{U}_4\end{bmatrix}, \tag{8}$$

$\boldsymbol{U}_1\in\mathbb{R}^{l\times l}$ is a nonsingular matrix, $\boldsymbol{\Gamma}=\mathrm{diag}\{\mu_1,\mu_2,\ldots,\mu_N\}$ with eigenvalues $0=\mu_1\leq\mu_2\leq\mu_N$.

Proof For network (1), we can write down different coupling matrices (or Laplacian) by rearranging the order of nodes. Assume that $\boldsymbol{G}$ and $\boldsymbol{G}'$ are two coupling matrices of network (1). Then there exists a nonsingular matrix $\boldsymbol{P}$ such that

$$\boldsymbol{G}'=\boldsymbol{P}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{P}, \tag{9}$$

where $\boldsymbol{P}\in\mathbb{R}^{N\times N}$ is a permutation matrix that has one and only one entry 1 in each row and each column and 0's elsewhere.

For a given matrix $\boldsymbol{G}$, suppose that matrices $\boldsymbol{P}$ and $\boldsymbol{G}'$ are to be determined. Then there exists an orthogonal matrix $\boldsymbol{U}'\in\mathbb{R}^{N\times N}$ such that

$$\boldsymbol{U}'^{\mathrm{T}}\boldsymbol{G}'\boldsymbol{U}' = \boldsymbol{\Gamma} = \mathrm{diag}\{\mu_1, \mu_2, \cdots, \mu_N\}, \qquad (10)$$

where $\mu_1\le\mu_2\le\ldots\le\mu_N$ are the eigenvalues of matrix $\boldsymbol{G}$.

Substituting Eq. (9) into Eq. (10) yields

$$(\boldsymbol{PU}')^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{PU}') = \boldsymbol{\Gamma}, \qquad (11)$$

where the orthogonal matrix $\boldsymbol{U}'$ left multiplied by $\boldsymbol{P}$ means exchange of some rows of matrix $\boldsymbol{U}'$, and $\boldsymbol{PU}'$ is still an orthogonal matrix since $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{P}=\boldsymbol{I}$.

It is easy to verify that the first l columns of matrix $\boldsymbol{U}'$ are linearly independent. For simplicity, we denote by $\boldsymbol{U}_l$ the matrix that contains the first l columns of matrix $\boldsymbol{U}'$. Then the row rank of $\boldsymbol{U}_l$ is equal to l, which also indicates that there exist l linearly independent rows. Since $\boldsymbol{P}$ is an arbitrarily chosen permutation matrix, one can find a proper matrix $\boldsymbol{P}$ to rearrange the row order of matrix $\boldsymbol{U}'$ such that the first l rows are linearly independent; i.e., $\boldsymbol{PU}'$ can be rewritten as follows:

$$\boldsymbol{PU}' = \begin{bmatrix} \boldsymbol{U}_1 & \boldsymbol{U}_2 \\ \boldsymbol{U}_3 & \boldsymbol{U}_4 \end{bmatrix}, \qquad (12)$$

where $\boldsymbol{U}_1\in\mathbb{R}^{l\times l}$ is a nonsingular matrix. Therefore, letting $\boldsymbol{PU}'=\boldsymbol{U}$, the proof is completed.

For simplicity, let the first l nodes be pinned. This can be guaranteed by rearranging the order of network nodes such that the controlled nodes are the first l nodes in the rearranged network. Then we have the following result:

Theorem 3 If $\mu_{l+1}\ge\lambda_{\mathrm{c}}/\lambda_{\mathrm{m}}$ holds for a particular l, then network (1) achieves asymptotical synchronization about $\boldsymbol{s}(t)$ via pinning the first l nodes, where λ_{m} is the minimal eigenvalue of matrix $\boldsymbol{U}_1^{\mathrm{T}}\boldsymbol{U}_1$.

Proof Writing the matrices $\boldsymbol{G}$ and $\boldsymbol{K}$ in a block form gives

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{G}_1 & \boldsymbol{G}_2 \\ \boldsymbol{G}_3 & \boldsymbol{G}_4 \end{bmatrix}, \quad \boldsymbol{K} = \begin{bmatrix} \boldsymbol{W}_l & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \qquad (13)$$

where $\boldsymbol{G}_1\in\mathbb{R}^{l\times l}$, $\boldsymbol{G}_2\in\mathbb{R}^{l\times(N-l)}$, $\boldsymbol{G}_4\in\mathbb{R}^{(N-l)\times(N-l)}$, and $\boldsymbol{W}_l\in\mathbb{R}^{l\times l}$ is a diagonal positive definite matrix.

For the result in Theorem 2, we rewrite Eq. (5) as

$$\boldsymbol{U}^{\mathrm{T}}(\boldsymbol{G}+\boldsymbol{K}-\lambda_{\mathrm{c}}\boldsymbol{I})\boldsymbol{U} \ge \boldsymbol{0}, \qquad (14)$$

where $\boldsymbol{U}$ is chosen according to Lemma 3. Furthermore, we derive

$$\begin{bmatrix} \boldsymbol{\Gamma}_1 - \lambda_{\mathrm{c}}\boldsymbol{I} + \boldsymbol{U}_1^{\mathrm{T}}\boldsymbol{W}_l\boldsymbol{U}_1 & \boldsymbol{U}_1^{\mathrm{T}}\boldsymbol{W}_l\boldsymbol{U}_2 \\ \boldsymbol{U}_2^{\mathrm{T}}\boldsymbol{W}_l\boldsymbol{U}_1 & \boldsymbol{\Gamma}_2 - \lambda_{\mathrm{c}}\boldsymbol{I} + \boldsymbol{U}_2^{\mathrm{T}}\boldsymbol{W}_l\boldsymbol{U}_2 \end{bmatrix} \ge \boldsymbol{0}, \qquad (15)$$

where $\boldsymbol{\Gamma}_1\in\mathbb{R}^{l\times l}$ and $\boldsymbol{\Gamma}_2\in\mathbb{R}^{(N-l)\times(N-l)}$ are diagonal matrices whose diagonal entries are μ_i, i=1, 2, …, N.

According to Lemma 1, we give an equivalent condition of Eq. (15):

$$\boldsymbol{\Gamma}_1 + \boldsymbol{U}_1^{\mathrm{T}}\boldsymbol{W}_l\boldsymbol{U}_1 - \lambda_{\mathrm{c}}\boldsymbol{I} > \boldsymbol{0}, \qquad (16)$$

$$\boldsymbol{\Gamma}_2 + \boldsymbol{U}_2^{\mathrm{T}}\boldsymbol{W}\boldsymbol{U}_2 - \lambda_{\mathrm{c}}\boldsymbol{I} > \boldsymbol{0}, \qquad (17)$$

where

$$\boldsymbol{W} = \boldsymbol{W}_l - \boldsymbol{W}_l\boldsymbol{U}_1(\boldsymbol{\Gamma}_1 + \boldsymbol{U}_1^{\mathrm{T}}\boldsymbol{W}_l\boldsymbol{U}_1 - \lambda_{\mathrm{c}}\boldsymbol{I})^{-1}\boldsymbol{U}_1^{\mathrm{T}}\boldsymbol{W}_l > \boldsymbol{0}.$$

It is noted that all diagonal entries of $\boldsymbol{W}_l$ can be large enough and $\boldsymbol{U}_1$ is chosen as a nonsingular matrix by Lemma 3. Hence, no matter what $\boldsymbol{\Gamma}_1$ and λ_{c} are, Eq. (16) always holds. We then focus on Eq. (17).

For matrix $\boldsymbol{W}$, we have

$$\boldsymbol{W} = \boldsymbol{W}_l^{1/2}\left(\boldsymbol{I} - (\boldsymbol{I} + \boldsymbol{W}_l^{-1/2}\boldsymbol{\varDelta}_1\boldsymbol{W}_l^{-1/2})^{-1}\right)\boldsymbol{W}_l^{1/2}, \qquad (18)$$

where

$$\boldsymbol{\varDelta}_1 = \boldsymbol{U}_1^{-\mathrm{T}}(\boldsymbol{\Gamma}_1 - \lambda_{\mathrm{c}}\boldsymbol{I})\boldsymbol{U}_1^{-1}.$$

Since $\left\|\boldsymbol{W}_l^{-1/2}\boldsymbol{\varDelta}_1\boldsymbol{W}_l^{-1/2}\right\| < 1$, a Taylor series expansion is given by

$$(\boldsymbol{I} + \boldsymbol{W}_l^{-1/2}\boldsymbol{\varDelta}_1\boldsymbol{W}_l^{-1/2})^{-1} = \boldsymbol{I} - \boldsymbol{W}_l^{-1/2}\boldsymbol{\varDelta}_1\boldsymbol{W}_l^{-1/2} + o(\boldsymbol{W}_l^{-2}), \qquad (19)$$

where the notation $O(\cdot)$ represents an infinitesimal of the same order.

Substituting Eq. (19) into Eq. (18) yields

$$\boldsymbol{W} = \boldsymbol{\varDelta}_1 + O(\boldsymbol{W}_l^{-1}). \qquad (20)$$

Then one can write down an equivalent condition of Eq. (17) as

$$\boldsymbol{\Gamma}_2 - \lambda_{\mathrm{c}}\boldsymbol{I} + \boldsymbol{U}_2^{\mathrm{T}}\left(\boldsymbol{\varDelta}_1 + O(\boldsymbol{W}_l^{-1})\boldsymbol{U}_2\right) \ge \boldsymbol{0}. \qquad (21)$$

Apparently, Eq. (21) can be guaranteed by the following inequality:

$$\boldsymbol{\Gamma}_2^* = \boldsymbol{\Gamma}_2 - \lambda_c \boldsymbol{I} + \boldsymbol{U}_2^{\mathrm{T}} \boldsymbol{\Delta}_1 \boldsymbol{U}_2 \geq \boldsymbol{0}. \tag{22}$$

Recalling the result in Theorem 1, we have $l \geq p$. That is, $\mu_{l+1} \geq \mu_l \geq \lambda_c$. Since the smallest diagonal value of $\boldsymbol{\Gamma}_2$ is μ_{l+1}, we have

$$\boldsymbol{\Gamma}_2 - \lambda_c \boldsymbol{I} \geq (\mu_{l+1} - \lambda_c)\boldsymbol{I} \geq \boldsymbol{0}. \tag{23}$$

On the other hand, let λ_m be the minimum eigenvalue of matrix $\boldsymbol{U}_1^{\mathrm{T}}\boldsymbol{U}_1$. Then we derive that $\lambda_m>0$ is the minimum eigenvalue of matrix $\boldsymbol{U}_1\boldsymbol{U}_1^{\mathrm{T}}$. Also, notice that $\boldsymbol{U}$ is an orthogonal matrix. Then,

$$\begin{cases} \boldsymbol{U}_1\boldsymbol{U}_1^{\mathrm{T}} + \boldsymbol{U}_2\boldsymbol{U}_2^{\mathrm{T}} = \boldsymbol{I}, \\ \boldsymbol{U}_2^{\mathrm{T}}\boldsymbol{U}_2 + \boldsymbol{U}_4^{\mathrm{T}}\boldsymbol{U}_4 = \boldsymbol{I}. \end{cases} \tag{24}$$

We thus deduce that $\lambda_m<1$ is the minimum eigenvalue of matrix $\boldsymbol{U}_4^{\mathrm{T}}\boldsymbol{U}_4$.

Substituting Eq. (24) into Eq. (23), we have

$$\begin{aligned} \boldsymbol{\Gamma}_2 - \lambda_c \boldsymbol{I} &\geq (\mu_{l+1} - \lambda_c)(\boldsymbol{U}_2^{\mathrm{T}}\boldsymbol{U}_2 + \boldsymbol{U}_4^{\mathrm{T}}\boldsymbol{U}_4) \\ &\geq (\mu_{l+1} - \lambda_c)\left[\boldsymbol{U}_2^{\mathrm{T}}\boldsymbol{U}_2 + \lambda_m(\boldsymbol{U}_2^{\mathrm{T}}\boldsymbol{U}_2 + \boldsymbol{U}_4^{\mathrm{T}}\boldsymbol{U}_4)\right] \\ &\geq (\mu_{l+1} - \lambda_c)/(1-\lambda_m)\cdot\boldsymbol{U}_2^{\mathrm{T}}\boldsymbol{U}_2. \end{aligned} \tag{25}$$

Moreover, the matrix $\boldsymbol{\Gamma}_2^*$ satisfies

$$\begin{aligned} \boldsymbol{\Gamma}_2^* &\geq \boldsymbol{U}_2^{\mathrm{T}}\left[\frac{\mu_{l+1}-\lambda_c}{1-\lambda_m}\boldsymbol{I} + \boldsymbol{U}_1^{-\mathrm{T}}(\boldsymbol{\Gamma}_1 - \lambda_c\boldsymbol{I})\boldsymbol{U}_1^{-1}\right]\boldsymbol{U}_2 \\ &\geq \boldsymbol{U}_2^{\mathrm{T}}\left[\frac{\mu_{l+1}-\lambda_c}{1-\lambda_m}\boldsymbol{I} + (\mu_1 - \lambda_c)\boldsymbol{U}_1^{-\mathrm{T}}\boldsymbol{U}_1^{-1}\right]\boldsymbol{U}_2 \\ &\geq \frac{\mu_{l+1}\lambda_m - \lambda_c}{\lambda_m(1-\lambda_m)}\boldsymbol{U}_2^{\mathrm{T}}\boldsymbol{U}_2. \end{aligned} \tag{26}$$

If $\mu_{l+1} \geq \lambda_c/\lambda_m$, we deduce $\boldsymbol{\Gamma}_2^* \geq \boldsymbol{0}$. The result further ensures linear matrix inequalities (16) and (17). The proof is thus completed.

From Theorem 2, we observe that the minimum eigenvalue λ_m of $\boldsymbol{U}_1^{\mathrm{T}}\boldsymbol{U}_1$ can be used to evaluate the number of pinned nodes. Obviously, a large λ_m leads to a small μ_{l+1}, and hence a small number of controllers. Also, notice that the pinning strategy determines the matrix $\boldsymbol{U}_1$. If the pinning strategy is unknown or uncertain, then there is difficulty in computing the value of λ_m. Here, we simply provide an upper bound of λ_m. Recalling the unitary diagonalization of matrix $\boldsymbol{G}$ in Eq. (10), we have

$$\boldsymbol{U}_3\boldsymbol{\Gamma}_1\boldsymbol{U}_3^{\mathrm{T}} + \boldsymbol{U}_4\boldsymbol{\Gamma}_2\boldsymbol{U}_4^{\mathrm{T}} = \boldsymbol{G}_3, \tag{27}$$

where $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ are selected the same as in Theorem 4. Moreover,

$$\mu_1\boldsymbol{U}_3\boldsymbol{U}_3^{\mathrm{T}} + \mu_{l+1}\boldsymbol{U}_4\boldsymbol{U}_4^{\mathrm{T}} \leq \boldsymbol{G}_3. \tag{28}$$

Since the minimum eigenvalue of $\boldsymbol{U}_4\boldsymbol{U}_4^{\mathrm{T}}$ is equal to λ_m, we then have

$$\lambda_m(\mu_{l+1} - \mu_1) \leq \lambda_1(\boldsymbol{G}_3) - \mu_1, \tag{29}$$

where $\lambda_1(\boldsymbol{G}_3)$ is the minimum eigenvalue of $\boldsymbol{G}_3$. From Eq. (29), the upper bound of λ_m is constrained by the matrix $\boldsymbol{G}_3$ consisting of unpinned nodes and coupling relations among them. Actually, the result can be deduced by the similar matrix block approach as shown in Theorem 2. We have the following result:

Corollary 1 If $\lambda_1(\boldsymbol{G}_3) \geq \lambda_c - \mu_1$, then the synchronous solution $\boldsymbol{s}(t)$ of network (1) is asymptotically stable via pinning the first l nodes.

Obviously, once finding a matrix $\boldsymbol{G}_3$ satisfying the condition in Corollary 1, we then derive the number and the location of controllers of a particular pinning strategy corresponding to $\boldsymbol{G}_3$. That is, the number of pinned nodes, no matter random pinning or selective pinning, can both be estimated using Corollary 1.

Sometimes, a clear and simple approximation of the number of controllers is more useful than the conditions acquired above. A good solution is to estimate the number of pinned nodes just according to Theorem 2 and an eigenvector of matrix $\boldsymbol{G}$. This is particularly suitable for the controlled network with a symmetric coupling matrix $\boldsymbol{G}$. In detail, if $\boldsymbol{G}$ is symmetric, 0 must be the minimum eigenvalue with corresponding eigenvector $\boldsymbol{\eta} = (1,\ 1,\ \cdots,\ 1)^{\mathrm{T}}/\sqrt{N}$. It is easy to verify that eigenvector $\boldsymbol{\eta}$ can be always located in the first l rows in Lemma 2. That is to say, $\boldsymbol{\eta}$ belongs to the matrix $\boldsymbol{U}_1$. Then l/N must be a diagonal entry of matrix $\boldsymbol{U}_1^{\mathrm{T}}\boldsymbol{U}_1$. For any symmetric matrix, its minimum eigenvalue must be equal to or smaller than any one of its diagonal entries. As a result, a possible minimum eigenvalue is l/N, i.e.,

$$\lambda_m \le l/N. \tag{30}$$

Note that, in the previous explanation, we always consider each node of the controlled network to be equivalent without any specification. Thus, the value l/N can be as per the approximation of random pinning, where $\delta=l/N$ is the probability of random pinning. Then Eq. (30) can be approximately used to compute the random pinning probability for a fixed complex dynamical network.

4 Numerical simulations

This section presents several simple examples to explain the results above. In simulations, we consider network (1) consists of 500 identical Lü oscillators (Lü and Chen, 2002); i.e.,

$$\dot{\boldsymbol{x}}_i = \boldsymbol{f}(\boldsymbol{x}_i) - \sigma\sum_{j=1}^{N} G_{ij}\boldsymbol{x}_j - \sigma k_i(\boldsymbol{x}_i - \boldsymbol{s}), \tag{31}$$

where $1\le i\le 500$, $\boldsymbol{G}=(G_{ij})$ is the Laplacian matrix with $G_{ij}=0$ or -1 ($i\ne j$), and the coupling strength $\sigma=10^3$. The node dynamics are given by

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -36(x-y) \\ 20y - xz \\ -3z + xy \end{bmatrix}. \tag{32}$$

The theoretical results show that the Lü attractor is bounded and that the value of α in Lemma 1 is approximately equal to 117.6 (Li *et al*., 2006; Zhou *et al*., 2008). Thus, according to Lemma 1, the critical eigenvalue for synchronization is $\lambda_c=\alpha/(\sigma\rho_m)=0.1176$.

Here, the network topology is assumed to obey the power-law distribution of the BA network model (Barabási and Albert, 1999), where the parameters are as follows: $m_0=m=5$, $N=500$. The second smallest eigenvalue of $\boldsymbol{G}$ is calculated as $\mu_2=2.9521>\lambda_c$. According to Theorem 1, there must be at least one node to be pinned. It was found, by searching the whole complex dynamical network, that there indeed exists a case where pinning only one node (the one with the largest degree) achieves the control goal. Fig. 1 shows the numerical simulation by pinning the one node, where index $Q(t)=\left(\sum_{i=1}^{N}\|\boldsymbol{x}_i(t)-\boldsymbol{s}(t)\|\right)/N$, $\boldsymbol{x}_i(0)$ with $1\le i\le 500$ are random values in the region (3, 7), $\boldsymbol{s}(0)=(4, 5, 6)^T$. Clearly, the curve $Q(t)$ approaches a constant as time evolves. It means that the controlled network (31) is asymptotically synchronizable about $\boldsymbol{s}(t)$ under one controller.

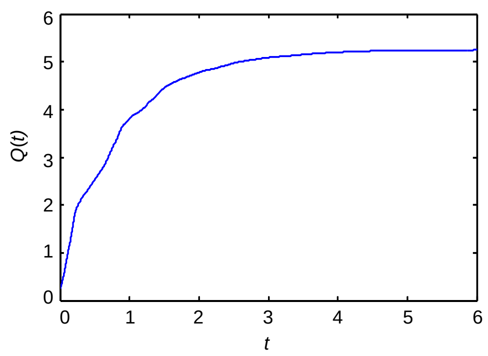

Fig. 1 Index Q of network (31) by pinning only one node

When the coupling strength σ decreases to 200, the critical eigenvalue $\lambda_c=0.5880<\mu_2$. In this case, the controlled network (31) cannot achieve synchronization by pinning only one node. In other words, the necessary condition in Theorem 1 cannot sufficiently guarantee the pinning criterion of synchronization. One solution is to apply the result in Corollary 1 by choosing a certain pinning strategy. For example, to satisfy $\lambda_1(\boldsymbol{G}_3)>\lambda_c$, one easily obtains $l=5$ by the selective pinning strategy and $l=66$ by random pinning. Fig. 2 shows the corresponding numerical simulation. From the figure, the synchronous solution $\boldsymbol{s}(t)$ of the controlled network (31) is asymptotically stable via selective pinning and random pinning.

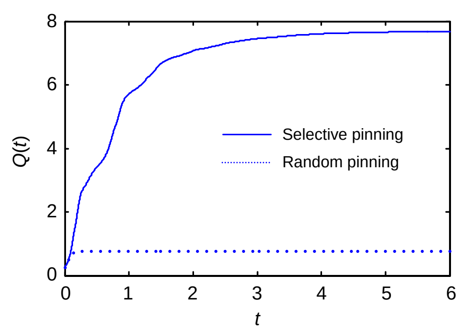

Fig. 2 Index Q of the controlled network by selective pinning and random pinning

Also, we can estimate the number of randomly pinned nodes by Theorem 2 and Eq. (30). It is calculated that $l\lambda_{l+1}\geq N\lambda_c$ holds when l=73, which is very close to that obtained by random pinning. Fig. 3 shows the value of $\lambda_1(\boldsymbol{G}+\boldsymbol{K})$ through randomly pinning 73 nodes, where k_i=10 for all pinned nodes. From the figure, one can see that almost all random pinning strategies can guarantee $\lambda_1(\boldsymbol{G}+\boldsymbol{K})>\lambda_c$ and further guarantee synchronization of network (31). Thus, Eq. (30) can be simply used for the approximation of random pinning.

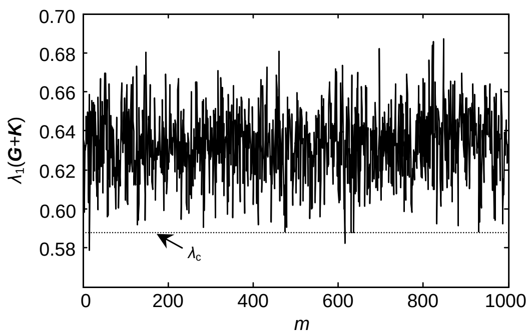

Fig. 3 The value of $\lambda_1(G+K)$ by random pinning 14.6% nodes of the whole network (*m* is the random times)

From previous discussions, a larger $\lambda_1(\boldsymbol{G}+\boldsymbol{K})$ means synchronizing network (31) more easily under the same pinning probability. Based on it, we further explore how conservative the theoretical results are by numerical simulations. There is no doubt that, Theorem 2 sometimes is too conservative to estimate the controller number; e.g., in Fig. 1, for any λ_c<2.9521, there needs pinning of at least one node according to Theorem 1. However, as shown in Fig. 4, network (31) needs to pin more nodes to maintain synchronization, where the network topology is the same as above, δ_r is the random approximation according to Eq. (30), δ_n is numerically calculated by the random pinning strategy, δ_s is the selective pinning probability obtained using Corollary 1, and δ_o is the smallest pinning probability among all possible pinning strategies. From Fig. 4, δ_s agrees quite well with δ_o when $\lambda_1(\boldsymbol{G}+\boldsymbol{K})$ is small; but the difference becomes larger and larger with the increase of $\lambda_1(\boldsymbol{G}+\boldsymbol{K})$. Obtaining δ_o by enumerating all pinning strategies is an NP-hard problem; δ_s is acceptable for analytical estimation of controllers. Also, notice the approximation $\delta_r>\delta_n$, which sufficiently guarantees network synchronization via random pinning.

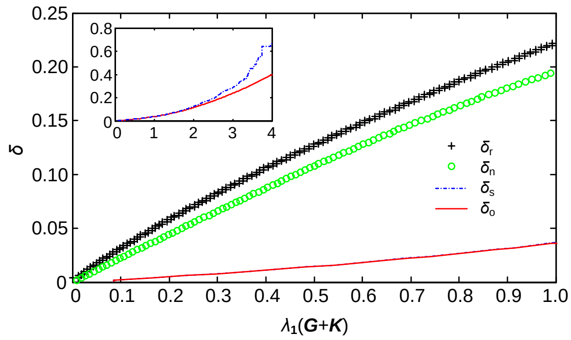

Fig. 4 Pinning probability as a function of the minimum eigenvalue

δ_r: the random approximation according to Eq. (30); δ_n: the numerical result by random pinning; δ_s: the selective pinning probability obtained using Corollary 1; δ_o: the smallest pinning probability among all possible pinning strategies

5 Conclusions

Pinning control is an important fitting model to reduce the number of controllers for complex networks of linearly coupled dynamical systems. In this paper, the number estimation of controllers has been theoretically investigated for pinning a complex dynamical network. For a given network model with fixed network architecture, we show that network synchronization via pinning can be evaluated using a linear matrix inequality, and subsequently provide a necessary condition and a sufficient condition for the holding of the linear matrix inequality. The two conditions indicate a lower bound and an upper bound of number estimation of pinned nodes, respectively, which can be used for theoretical analysis in pinning control. In addition, an approximation of the number of randomly pinned nodes is given by the trivial vector $\boldsymbol{\eta}$. This result is quite simple and shows great effectiveness in numerical simulations. This work provides an understanding of the mechanisms that govern complex networks and may be useful in applications of real-world complex systems.

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