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Grasp evaluation and contact points planning for polyhedral objects using a ray-shooting algorithm

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Abstract: Grasp evaluation and planning are two fundamental issues in robotic grasping and dexterous manipulation. Most traditional methods for grasp quality evaluation suffer from non-uniformity of the wrench space and a dependence on the scale and choice of the reference frame. To overcome these weaknesses, we present a grasp evaluation method based on disturbance force rejection under the assumption that the normal component of each individual contact force is less than one. The evaluation criterion is solved using an enhanced ray-shooting algorithm in which the geometry of the grasp wrench space is read by the support mapping. This evaluation procedure is very fast due to the efficiency of the ray-shooting algorithm without linearization of the friction cones. Based on a necessary condition for grasp quality improvement, a heuristic searching algorithm for polyhedral object regrasp is also proposed. It starts from an initial force-closure unit grasp configuration and iteratively improves the grasp quality to find the locally optimum contact points. The efficiency and effectiveness of the proposed algorithms are illustrated by a number of numerical examples.

Key words: Force closure, Grasp quality evaluation, Multifingered grasp, Grasping planning, Ray-shooting
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1 Introduction

Multifingered robotic hands are very powerful in object grasping and especially suitable to perform dexterous and fine manipulation tasks. They provide many grasp configurations from which we can choose those that meet our different demands. Thus, an evaluation method is necessary for us to obtain the best grasping quality.

Grasp stability is characterized by two well-known terms, 'form closure' and 'force closure' (Salisbury and Roth, 1983; Murray *et al*., 1994; Xiong, 1994; Bicchi, 2000). A grasp achieves form closure if it prevents the grasped object from slipping under the constraint of unilateral frictionless contacts, while a force closure grasp can resist arbitrary external forces and torques in consideration of the frictional grasping forces. The primary distinction between a form closure and a force closure lies in the contact model employed (Zhu and Wang, 2003). Salisbury and Roth (1983) have proved that a grasp is form closed if and only if the contact wrenches of the grasp positively span the whole wrench space. An equivalent necessary and sufficient condition for form closure is that the origin of the wrench space is in the interior of the convex hull of the contact wrenches (Xiong, 1994). Liu (1999) linearized the friction cone as a polyhedral convex cone, and then transformed the force closure test into a ray-shooting problem. Liu (1999) also presented an efficient algorithm for computing all n-finger force closure grasps on a polygonal object by recursively transferring the problem from high dimension to low dimension (Liu, 2000). Based on geometric analysis, Li *et al*. (2003) proposed a method for computing three-finger force closure grasps of 2D and 3D objects. Zhu *et al*. (2004) formulated a numerical test for closure properties of 3D grasps as a convex constrained optimization problem without linearization of the friction cones.

The qualitative tests described above are designed to check whether a grasp is closed or not. A metric index expressing the grasp quality is needed for quantitative grasp analysis and optimal grasp planning. Ferrari and Canny (1992) measured grasp quality by the radius of the largest sphere centered at the origin and fully contained in the convex hull of the primitive wrenches. This quality is a widely used criterion and is sometimes referred to as the largest ball criterion (Suárez *et al*., 2006). After that, a quality measure based on decoupled wrenches (Mirtich and Canny, 1994) was developed to remedy the non-uniformity of the wrench space. Another shortcoming of the largest ball criterion is its dependence on the coordinate system of the grasped object. Thus, Teichmann (1996) proposed a measure based on the radius of the largest origin-centered balls among all possible translations of the object coordinate system. Borst *et al*. (2004) introduced a physically motivated description of a general task wrench space (TWS) based on an object wrench space (OWS) and presented a quality measure which overcame the problems of the non-uniform wrench space. Strandberg and Wahlberg (2006) presented a method for grasp evaluation based on disturbance force rejection. Their approach not only included both task and object geometry information but also overcame the dependence on the scale and choice of the reference frame. Suárez *et al*. (2006) summarized 22 different quality measures in the grasp literature and divided them into two groups: measures associated with contact position and measures associated with hand configuration. One way to choose a global optimal grasp is to rank grasps according to each of the measures and then to combine them into a single measure (Chinellato *et al*., 2003).

As for grasp planning, most approaches have been based on iterative searching algorithms. Ding *et al*. (2001b) proposed an algorithm to find a force closure grasp by iteratively minimizing the distance between the origin and the centroid of the primitive contact wrenches along the local search direction in each step. Based on the concept of 'Q distance', the constrained optimization (Zhu *et al*., 2001), the descent searching algorithm (Zhu and Wang, 2003), and the genetic algorithm (Phoka *et al*., 2006) were designed to plan optimal grasp. Other heuristic searching algorithms can be found in the literature (Ding *et al*., 2001a; Liu *et al*., 2004; Roa and Suárez, 2009). Grasp planning can be formulated as an optimization problem, and then solved directly using a standard optimization toolbox. Mangialardi *et al*. (1996) determined the optimal grip points with minimal grasping forces by solving an optimization problem with nonlinear constraints. Mantriota (1999) then modified it to minimize the friction coefficient needed to ensure contact stability in the presence of a generic disturbing external force. Watanabe and Yoshikawa (2007) treated grasp planning as an optimization problem from the viewpoint of decreasing the magnitudes of the contact forces needed to balance all the wrenches in a required wrench set. More recently, Zheng and Qian (2009) proposed two nonlinear optimization problems to minimize the maximal predefined distance between the origin and the surface of all non-negative linear combinations of the primitive contact wrench sets.

In this paper, we extend Strandberg and Wahlberg's grasp evaluation method to a more natural and better grasp wrench space. After careful study we adapt the fast ray-shooting algorithm (Zheng *et al*., 2010) for use in grasp evaluation. A necessary condition for grasp quality improvement is also presented in this paper. Based on this condition, a heuristic planning algorithm for polyhedral object regrasp is also proposed. Our grasp evaluation method inherits all the benefits from Strandberg and Wahlberg's method with additional benefits as follows:

1. The grasp quality calculated using our evaluation method is more accurate because linearization of friction cones is not needed.

2. It can apply to a soft finger contact model because the geometry of the grasp wrench space is read by the support mapping in the ray-shooting algorithm.

Table 1 lists the notations to facilitate reading.

2 Preliminaries

2.1 Grasp mapping

Consider a rigid object fixed with an object coordinate frame grasped by m frictional point contacts. Set a local contact coordinate frame whose first principal axis orientation is the inward normal direction at

Table 1 Notations used in this paper

Parameter	Meaning
m	Number of contacts
$\boldsymbol{f}_i$	Contact force at contact i
μ_i	Static friction coefficient at contact i
FC_i	Friction cone for contact i
$\boldsymbol{\tau}$	Torque vector
$\boldsymbol{w}$	Total resultant wrench
d	Dimensionality of the wrench space. d=3 and d=6 for 2D and 3D grasps, respectively
$\boldsymbol{r}_i$	Position of contact i
$\boldsymbol{R}_i$	Relative orientation of the ith local contact coordinate frame with respect to (w.r.t.) the object frame
$\boldsymbol{G}_i$	Grasp matrix for the ith contact
$[\boldsymbol{r}_i]_\times$	Cross product matrix for vector $\boldsymbol{r}_i$
A	A compact set in $\mathbb{R}^d$
co$A$	Convex hull of A
R	A ray from the origin through another point $\boldsymbol{q}$
$\boldsymbol{s}$	Intersection of the boundary of co$A$ by the ray R
$h_{\mathrm{co}A}(\boldsymbol{u})$	Support function of co$A$ w.r.t. $\boldsymbol{u}\in\mathbb{R}^d$
$s_{\mathrm{co}A}(\boldsymbol{u})$	Support mapping of co$A$ w.r.t. $\boldsymbol{u}$
$\mathbf{0}$	Origin of a space or zero matrix
int()	The interior of a set
V_0	An initial set containing d+1 affinely independent points in co$A$. $V_0=\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_{d+1}\}$
coF_i	The ith facet of coV_0. $F_i=\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_{i-1}, \boldsymbol{a}_{i+1}, \cdots, \boldsymbol{a}_{d+1}\}$
W_i	Grasp wrench set for contact i
W	Total unit grasp wrench set
$\boldsymbol{w}_0$	Offset wrench generated by gravity
$\boldsymbol{w}_\mathrm{d}$	Disturbance wrench
W_d	Disturbance wrench set
$\boldsymbol{e}_i$	A fixed direction for unit disturbance force
$P(\boldsymbol{e}_i)$	A set that contains those points on the object surface with $\boldsymbol{e}_i$ on their friction cones
ρ	Magnitude of the disturbance force
$\rho^*(\boldsymbol{e}_i)$	Maximum magnitude for the disturbance force $\boldsymbol{e}_i$ the grasp can resist
ρ_m	Final grasp quality
$\boldsymbol{w}_\mathrm{T}$	A general transformation of W
$\boldsymbol{f}_\mathrm{n}$	Normal contact force. $\boldsymbol{f}_\mathrm{n}=[1\ 0]^\mathrm{T}$ and $\boldsymbol{f}_\mathrm{n}=[1\ 0\ 0]^\mathrm{T}$ for 2D and 3D grasps, respectively
$\boldsymbol{G}$	Total grasp matrix. $\boldsymbol{G}=[\boldsymbol{G}_1\ \boldsymbol{G}_2 \cdots \boldsymbol{G}_m]^\mathrm{T}$
$\boldsymbol{w}_\mathrm{c}$	Average wrench generated by all normal contact forces at each contact
$\lVert\cdot\rVert$	2-norm of a vector
ε, σ	Termination tolerances in the ray-shooting and heuristic regrasp planning algorithms, respectively
$n_{\max}$	Maximum iteration in the heuristic regrasp planning algorithm
χ	An assigned parameter in the heuristic regrasp planning algorithm

the ith contact. Then the contact force $\boldsymbol{f}_i$ can be expressed in the local contact coordinate frame as $\boldsymbol{f}_i=[f_{xi}\ f_{yi}\ f_{zi}]^\mathrm{T}$ or $\boldsymbol{f}_i=[f_{xi}\ f_{yi}]^\mathrm{T}$ for 3D and 2D grasps, respectively. We set the upper bound of normal force component f_{xi} to 1 for convenience and let μ_i denote the static friction coefficient at contact i. To avoid separation and slip at each contact, $\boldsymbol{f}_i$ must lie in the friction cone FC_i, which can be expressed by

$$\mathrm{FC}_i=\begin{cases}\left\{\boldsymbol{f}_i \mid 0\le f_{xi}\le 1, \sqrt{f_{yi}^2+f_{zi}^2}\le \mu_i f_{xi}\right\}, & \text{(3D)}\\ \left\{\boldsymbol{f}_i \mid 0\le f_{xi}\le 1, -\mu_i f_{xi}\le f_{yi}\le \mu_i f_{xi}\right\}. & \text{(2D)}\end{cases} \tag{1}$$

When combining force and torque vectors $\boldsymbol{f}$ and $\boldsymbol{\tau}$ to form a wrench of $\boldsymbol{w}=[\boldsymbol{f}^\mathrm{T}\ \boldsymbol{\tau}^\mathrm{T}]^\mathrm{T}$, the total resultant wrench applied on the object by the m frictional point contacts is

$$\boldsymbol{w}=\sum_{i=1}^{m}\boldsymbol{G}_i\boldsymbol{f}_i. \tag{2}$$

Here, $\boldsymbol{G}_i$ is the grasp matrix at contact i. Let $\boldsymbol{r}_i$ and $\boldsymbol{R}_i$ (i=1, 2, ⋯, m) be the position of contact i and the relative orientation of the ith local contact coordinate frame, respectively, with respect to (w.r.t.) the object reference frame. Let $\boldsymbol{r}_i=[x_i\ y_i]^\mathrm{T}$ and $\boldsymbol{r}_i=[x_i\ y_i\ z_i]^\mathrm{T}$ represent 2D and 3D grasps, respectively. We have

$$\boldsymbol{G}_i=\begin{bmatrix}\boldsymbol{R}_i\\ [\boldsymbol{r}_i]_\times \boldsymbol{R}_i\end{bmatrix}, \tag{3}$$

where $[\boldsymbol{r}_i]_\times$ is a cross product matrix satisfying $\boldsymbol{r}_i\times\boldsymbol{f}_i=[\boldsymbol{r}_i]_\times\boldsymbol{f}$ for any force vector $\boldsymbol{f}$. We have $[\boldsymbol{r}_i]_\times=[-x_i\ y_i]$ for 2D grasps, while for 3D grasps, we have a skewed symmetric matrix defined as

$$[\boldsymbol{r}_i]_\times=\begin{bmatrix}0 & -z_i & y_i\\ z_i & 0 & -x_i\\ -y_i & x_i & 0\end{bmatrix}. \tag{4}$$

2.2 Ray-shooting problem

Let A denote a compact set in $\mathbb{R}^d$, and co$A$ the convex hull of A. Let $R=\{\lambda\boldsymbol{q} \mid \lambda\ge 0, \boldsymbol{q}\in\mathbb{R}^d\}$ be a ray emanating from the origin through another point $\boldsymbol{q}$. The ray-shooting problem (Zheng *et al.*, 2010) is to determine the intersection of the boundary of co$A$ by this ray, denoted by $\boldsymbol{s}$. A support function of co$A$, $h_{\mathrm{co}A}$: $\mathbb{R}^d \rightarrow \mathbb{R}$, is defined by

$$h_{\mathrm{co}A}(\boldsymbol{u}) = \max_{\boldsymbol{a} \in \mathrm{co}A} \boldsymbol{u}^{\mathrm{T}} \boldsymbol{a}, \quad (5)$$

where $\boldsymbol{a} \in \mathbb{R}^d$. The support mapping $s_{\mathrm{co}A}(\boldsymbol{u})$ is a point in co$A$ satisfying $h_{\mathrm{co}A}(\boldsymbol{u}) = \boldsymbol{u}^{\mathrm{T}} s_{\mathrm{co}A}(\boldsymbol{u})$.

Zheng *et al.* (2010) presented hitherto the fastest d-dimensional algorithm for solving the ray-shooting problem. We retell their approach briefly here. Suppose $\mathbf{0} \in \mathrm{int}(\mathrm{co}A)$, where $\mathbf{0}$ is the origin and int() denotes the interior of a set (Fig. 1). Let $V_0 = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_{d+1}\}$ be an initial set containing d+1 affinely independent points in co$A$ and satisfying $\mathbf{0} \in \mathrm{int}(\mathrm{co}V_0)$. Let $F_i = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_{i-1}, \boldsymbol{a}_{i+1}, \cdots, \boldsymbol{a}_{d+1}\}$. Then co$F_i$ is the ith facet of coV_0. Let $\boldsymbol{c} = \boldsymbol{W}_i^{-1} \boldsymbol{q}$, where $\boldsymbol{W}_i = [\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_{i-1}, \boldsymbol{a}_{i+1}, \cdots, \boldsymbol{a}_{d+1}] \in \mathbb{R}^{d \times d}$. Let min($\boldsymbol{c}$) and sum($\boldsymbol{c}$) denote the minimum and the sum, respectively, of components of $\boldsymbol{c}$. Then it is easy to verify that the ray R intersects coF_i if and only if min($\boldsymbol{c}$)≥0 and the intersection point $\boldsymbol{s}_0 = \boldsymbol{q}/\mathrm{sum}(\boldsymbol{c})$. Let co$F_{\bar{i}}$ denote the intersection facet of coV_0 with the ray R. The outward normal vector of co$F_{\bar{i}}$ can be calculated by $\boldsymbol{u}_0 = \boldsymbol{W}_{\bar{i}}^{-\mathrm{T}} \boldsymbol{x}_d$, where $\boldsymbol{x}_d = [1\ 1\ \cdots\ 1]^{\mathrm{T}} \in \mathbb{R}^d$. Then a new simplex co$V_1$ can be constructed using co$F_{\bar{i}}$ and the support mapping $s_{\mathrm{co}A}(\boldsymbol{u}_0)$. A new round of iteration can be started to find a new intersection point $\boldsymbol{s}_1$ on the facet of coV_1 and the sequence $\{\boldsymbol{s}_i\}$ is convergent to the intersection point $\boldsymbol{s}$. In practice, the iterations can stop when $h_{\mathrm{co}A}(\boldsymbol{u}_i) - 1 < \varepsilon$, where ε is the termination tolerance. The last $\boldsymbol{u}_i$ can be adopted as the outward normal of co$A$ at $\boldsymbol{s}$.

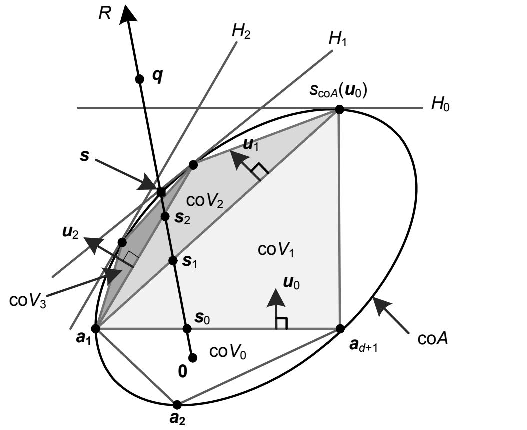

Fig. 1 Illustration of the ray-shooting problem (Zheng *et al.*, 2010)
The problem is to find the intersection point $\boldsymbol{s}$ on the boundary of co$A$ with R, which is the ray emitting from the origin and passing through a given point $\boldsymbol{q}$. $\boldsymbol{u}_0$, $\boldsymbol{u}_1$, and $\boldsymbol{u}_2$ are outward normals to the facets of simplexes coV_0, coV_1, and coV_2, respectively, hit by R

3 Grasp quality index

Strandberg and Wahlberg (2006) measured the grasp's quality by its ability to resist disturbance forces. They linearized the friction cone FC$_i$ as a k-sided pyramid and then defined the unit grasp wrench space (UGWS) as the convex combination of all the linearized primitive wrenches, for simplification. The UGWS limits the sum of all the normal force components to less than one. Strandberg and Wahlberg (2006) also stated that a more natural way to represent UGWS was as the convex combination of the Minkowski sum of primitive wrenches. This representation is also better for evaluating grasp quality (Zheng and Qian, 2006), but the heavy computation required prevented them from doing so. In this section, their grasp measure is extended to the more natural and better grasp wrench space without linearizing the friction cones.

Let W_i be the grasp wrench set for contact i, which comprises all the wrenches generated by $\boldsymbol{f}_i \in \mathrm{FC}_i$. It is the image of FC$_i$ under the mapping $\boldsymbol{G}_i$ into the wrench space $\mathbb{R}^d$ and we denote it by $W_i = \boldsymbol{G}_i(\mathrm{FC}_i)$ (Zheng and Qian, 2009). Then the total unit grasp wrench set composed of all the m grasps is

$$W = \sum_{i=1}^{m} \boldsymbol{G}_i(\mathrm{FC}_i). \quad (6)$$

It is well known that a grasp is force-closure if and only if $\mathbf{0} \in \mathrm{int}(\mathrm{co}W)$. Let $\boldsymbol{w}_0$ denote an offset wrench generated by gravitational force and $\boldsymbol{w}_{\mathrm{d}}$ be the disturbance wrench arising from unknown forces acting on the object surface. For equilibrium, we have $\boldsymbol{w}_{\mathrm{d}} + \boldsymbol{w}_0 + \sum_{i=1}^{m} \boldsymbol{G}_i \boldsymbol{f}_i = \mathbf{0}$. Let W_{d} denote the disturbance wrench set. Then a unit grasp can equilibrate all the disturbance forces if and only if

$$-W_{\mathrm{d}} \subset \mathrm{co}(W + \{\boldsymbol{w}_0\}). \quad (7)$$

Supposing the magnitudes of disturbance wrenches in W_d are infinitely small, a unit grasp can equilibrate all disturbance wrenches if and only if $\mathbf{0}\in\text{int}(\text{co}\{W+\boldsymbol{w}_0\})$. We define a grasp as a force-closure unit grasp if and only if $\mathbf{0}\in\text{int}(\text{co}\{W+\boldsymbol{w}_0\})$. Note that it is different from the commonly used term 'force-closure' because the magnitude of normal force for each contact is limited and an offset wrench $\boldsymbol{w}_0$ is associated here.

The disturbance wrench set W_d can be modeled by the entire disturbance wrenches arising from all possible pure forces acting on the surface of the grasped object. Let a unit vector $\boldsymbol{e}_i$ denote a fixed direction for the disturbance force. Then the disturbance force $\rho\boldsymbol{e}_i$ acting on all possible points on the grasped object will generate a subset of W_d. Here, ρ is a dimensionless scalar representing the magnitude of the disturbance force, and 'possible points' refers to points on the surface of the grasped object where $\boldsymbol{e}_i$ is inside each friction cone of those points. Let $P(\boldsymbol{e}_i)$ be a point set that contains all the possible points for the fixed direction $\boldsymbol{e}_i$, and $\rho^*(\boldsymbol{e}_i)$ the maximum magnitude of the disturbance force in the direction of $\boldsymbol{e}_i$ that the grasp can resist, no matter where the disturbance force is applied. From Eq. (7), we can define $\rho^*(\boldsymbol{e}_i)$ as the solution to a min-max problem formulated as

$$\rho^*(\boldsymbol{e}_i)=\min_{r\in P(\boldsymbol{e}_i)}\max\left\{\rho\,|-\rho\begin{bmatrix}\boldsymbol{e}_i\\ [\boldsymbol{r}]_\times\boldsymbol{e}_i\end{bmatrix}\in\text{co}(W+\{\boldsymbol{w}_0\})\right\}. \tag{8}$$

We adopt the minimum $\rho^*(\boldsymbol{e}_i)$ among all the disturbance force directions as the final grasp quality, i.e.,

$$\rho_m=\min_{\boldsymbol{e}_i}\rho^*(\boldsymbol{e}_i). \tag{9}$$

It is obviously impractical to calculate all the disturbance force directions; thus, we need to discretize these directions. A simple discretization could be done by uniformly sampling sufficient points on the unit circle and the unit sphere, for 2D and 3D grasps, respectively. Let n denote the number of discrete directions. For 2D grasps, we can express $\boldsymbol{e}_i=[\cos(2\pi i/n)\ \sin(2\pi i/n)]^T$ (i=1, 2, ⋯, n), explicitly. For 3D grasps, $\boldsymbol{e}_i=[\cos\varphi_i\sin\theta_i\ \sin\varphi_i\sin\theta_i\ \cos\theta_i]^T$, where $0\le\theta_i\le\pi$, $0\le\varphi_i\le 2\pi$, and i=1, 2, ⋯, n.

In general, all the points in $P(\boldsymbol{e}_i)$ need to be investigated to solve the min-max problem (8). However, when the grasped object is a polyhedron, the number of points to be investigated can be reduced based on the following theorem, similar to that presented by Strandberg and Wahlberg (2006).

Theorem 1 For polyhedral objects grasped by a force-closure unit grasp, the worst point on the object attacked by a disturbance force is a vertex.

Proof We assume all the faces of the polyhedral objects are convex because nonconvex faces can be decomposed into a number of convex polygons. A point $\boldsymbol{r}_k$ on the kth convex face can be written as a convex combination of its vertices, i.e.,

$$\boldsymbol{r}_k=\sum_{j=1}^{N_k}\beta_{kj}\boldsymbol{v}_{kj},\ \sum_{j=1}^{N_k}\beta_{kj}=1,\ \beta_{kj}\ge 0, \tag{10}$$

where $\boldsymbol{v}_{k1},\boldsymbol{v}_{k2},\cdots,\boldsymbol{v}_{kN_k}$ are the vertices of the kth convex face and N_k is the number of these vertices. Obviously, all these N_k vertices belong to the vertices of the grasped object. Let $\boldsymbol{w}_{i,j}^k$ denote the wrench generated by a unit disturbance force acting on the vertex $\boldsymbol{v}_{k,j}$ along the direction of $\boldsymbol{e}_i$, that is,

$$\boldsymbol{w}_{i,j}^k=\begin{bmatrix}\boldsymbol{e}_i\\ [\boldsymbol{v}_{k,j}]_\times\boldsymbol{e}_i\end{bmatrix}. \tag{11}$$

Let $\rho_i^k=\min_{j=1,2,\cdots,N_k}\rho_{i,j}^k$, where $\rho_{i,j}^k$ is defined by

$$\rho_{i,j}^k=\max\left\{\rho|-\rho\boldsymbol{w}_{i,j}^k\in\text{co}(W+\{\boldsymbol{w}_0\})\right\}. \tag{12}$$

Note that $\rho_{i,j}^k$ is positive because the grasp is a force-closure unit grasp, and $-\rho_i^k\boldsymbol{w}_{i,j}^k\in\text{co}(W+\{\boldsymbol{w}_0\})$, j=1, 2, ⋯, N_k. From Eqs. (10), (11), and (4), the wrench $\boldsymbol{w}_{ki}$ generated by a unit disturbance force $\boldsymbol{e}_i$ on the point $\boldsymbol{r}_k$ can be represented by

$$\boldsymbol{w}_{ki}=\begin{bmatrix}\boldsymbol{e}_i\\ [\boldsymbol{r}_k]_\times\boldsymbol{e}_i\end{bmatrix}=\sum_{j=1}^{N_k}\beta_{kj}\boldsymbol{w}_{i,j}^k,\ \sum_{j=1}^{N_k}\beta_{kj}=1,\ \beta_{kj}\ge 0. \tag{13}$$

From Eq. (13), we can easily conclude that $-\rho_i^k\boldsymbol{w}_{ki}\in\text{co}(W+\{\boldsymbol{w}_0\})$ because it is a convex combination of a point set $\{-\rho_i^k\boldsymbol{w}_{i,j}^k\}$ in the wrench space. Thus, we have $\max\{\rho\,|-\rho\boldsymbol{w}_{ki}\in\text{co}(W+\{\boldsymbol{w}_0\})\}\ge\rho_i^k$

and further conclude that only the vertices in $P(\boldsymbol{e}_i)$ can be the candidate solution to Eq. (8). Therefore, the worst point of attack by a disturbance force among all directions is a vertex of the grasped object.

4 Grasp evaluation procedure

The min-max problem (8) can be treated as a ray-shooting problem. The primary ray-shooting algorithm proposed by Zheng *et al.* (2010) is very efficient, even in high dimensions, such as the 6D wrench space of 3D grasps. However, some remedies or improvements can be made to complete the algorithm in our application described above. In this section, we propose an enhanced ray-shooting algorithm solution and present all the calculation issues.

4.1 Support function and mapping

It is necessary to calculate the support function and mapping of $W+\{\boldsymbol{w}_0\}$ to solve the min-max problem (8). The following properties for support functions and mappings are useful in reducing the computation.

Theorem 2 Assume $W, W_1, W_2, \cdots, W_m \subset \mathbb{R}^d$ and $\mathrm{FC} \subset \mathbb{R}^m$ are compact sets and $\boldsymbol{G}$ is a real $d\times m$ matrix. Then the following equations hold (Gilbert and Foo, 1990; Zheng and Chew, 2009):

(a) $h_{\mathrm{co}W}(\boldsymbol{u}) = h_W(\boldsymbol{u}),\ s_{\mathrm{co}W}(\boldsymbol{u}) = s_W(\boldsymbol{u})$.

(b) $h_{W_1+W_2+\cdots+W_m}(\boldsymbol{u}) = h_{W_1}(\boldsymbol{u}) + h_{W_2}(\boldsymbol{u}) + \cdots + h_{W_m}(\boldsymbol{u})$,

$s_{W_1+W_2+\cdots+W_m}(\boldsymbol{u}) = s_{W_1}(\boldsymbol{u}) + s_{W_2}(\boldsymbol{u}) + \cdots + s_{W_m}(\boldsymbol{u})$.

(c) $h_{\boldsymbol{G}(\mathrm{FC})}(\boldsymbol{u}) = h_{\mathrm{FC}}(\boldsymbol{G}^{\mathrm{T}}\boldsymbol{u}),\ s_{\boldsymbol{G}(\mathrm{FC})}(\boldsymbol{u}) = \boldsymbol{G}s_{\mathrm{FC}}(\boldsymbol{G}^{\mathrm{T}}\boldsymbol{u})$.

Let $W_{\mathrm{T}}=W+\{\boldsymbol{w}_{\mathrm{T}}\}$, where $\boldsymbol{w}_{\mathrm{T}}$ denotes a general transformation of W in the wrench space. From Theorem 2, we can calculate the support function and mapping of $\mathrm{co}W_{\mathrm{T}}$ as

$$h_{\mathrm{co}W_{\mathrm{T}}}(\boldsymbol{u}) = \sum_{i=1}^{m} h_{\mathrm{FC}_i}(\boldsymbol{G}_i^{\mathrm{T}}\boldsymbol{u}) + \boldsymbol{w}_{\mathrm{T}}^{\mathrm{T}}\boldsymbol{u}, \tag{14}$$

$$s_{\mathrm{co}W_{\mathrm{T}}}(\boldsymbol{u}) = \sum_{i=1}^{m} \boldsymbol{G}_i s_{\mathrm{FC}_i}(\boldsymbol{G}_i^{\mathrm{T}}\boldsymbol{u}) + \boldsymbol{w}_{\mathrm{T}}. \tag{15}$$

The support mapping is not unique in some special cases, and we select one of these support mappings for the special cases in the following calculation. Let $\boldsymbol{x}_i = \boldsymbol{G}_i^{\mathrm{T}}\boldsymbol{u}$. Then for 2D grasps, $\boldsymbol{x}_i=[x_i\ y_i]^{\mathrm{T}}$, we set

$$h_{\mathrm{FC}_i}(\boldsymbol{x}_i) = \max\{0, x_i \pm \mu_i y_i\}, \tag{16}$$

$$s_{\mathrm{FC}_i}(\boldsymbol{x}_i) = \begin{cases} [0\ \ 0]^{\mathrm{T}}, & h_{\mathrm{FC}_i}(\boldsymbol{x}_i) = 0, \\ [1\ \ \mu_i]^{\mathrm{T}}, & h_{\mathrm{FC}_i}(\boldsymbol{x}_i) > 0, y_i > 0, \\ [1\ \ -\mu_i]^{\mathrm{T}}, & \text{otherwise}. \end{cases} \tag{17}$$

For 3D grasps, $\boldsymbol{x}_i=[x_i\ y_i\ z_i]^{\mathrm{T}}$, let $\alpha_i = \sqrt{y_i^2 + z_i^2}$. We have

$$h_{\mathrm{FC}_i}(\boldsymbol{x}_i) = \max\{0, x_i + \mu_i \alpha_i\}, \tag{18}$$

$$s_{\mathrm{FC}_i}(\boldsymbol{x}_i) = \begin{cases} [0\ \ 0\ \ 0]^{\mathrm{T}}, & h_{\mathrm{FC}_i}(\boldsymbol{x}_i) = 0, \\ [1\ \ 0\ \ 0]^{\mathrm{T}}, & h_{\mathrm{FC}_i}(\boldsymbol{x}_i) > 0, \alpha_i = 0, \\ [1\ \ \mu_i y_i/\alpha_i\ \ \mu_i z_i/\alpha_i]^{\mathrm{T}}, & \text{otherwise}. \end{cases} \tag{19}$$

4.2 Grasp evaluation algorithm

It should be checked whether $\boldsymbol{0} \in \mathrm{int}(\mathrm{co}W_{\mathrm{T}})$ before implementing the ray-shooting algorithm to the min-max problem (8). Let $\boldsymbol{f}_{\mathrm{n}}=[1\ 0\ 0]^{\mathrm{T}}$ and $\boldsymbol{f}_{\mathrm{n}}=[1\ 0]^{\mathrm{T}}$ for 3D and 2D grasps, respectively. It can be proved that $\boldsymbol{w}_{\mathrm{c}}$ given below lies in the interior of $\mathrm{co}W$ if and only if $\boldsymbol{G}=[\boldsymbol{G}_1\ \boldsymbol{G}_2 \cdots \boldsymbol{G}_m]$ is of full row rank (Zhu *et al.*, 2004):

$$\boldsymbol{w}_{\mathrm{c}} = \frac{1}{m}\sum_{i=1}^{m} \boldsymbol{G}_i \boldsymbol{f}_{\mathrm{n}}. \tag{20}$$

If $\boldsymbol{w}_{\mathrm{c}}=-\boldsymbol{w}_0$, we can directly conclude $-\boldsymbol{w}_0 \in \mathrm{int}(\mathrm{co}W)$ if $\boldsymbol{G}$ is of full row rank. For the nontrivial case of $\boldsymbol{w}_{\mathrm{c}} \neq -\boldsymbol{w}_0$, we have the following straightforward theorem:

Theorem 3 Let $\boldsymbol{w}_{\mathrm{c}} \in \mathrm{int}(\mathrm{co}W)$ and $\boldsymbol{w}_{\mathrm{s}} \in \mathrm{co}W$. Then $(1-\lambda)\boldsymbol{w}_{\mathrm{c}}+\lambda\boldsymbol{w}_{\mathrm{s}} \in \mathrm{int}(\mathrm{co}W)$ for $0<\lambda<1$.

The following corollary of Theorem 3 is also straightforward:

Corollary 1 If $\boldsymbol{G}$ is of full row rank and $\boldsymbol{w}_{\mathrm{c}} \neq -\boldsymbol{w}_0$, letting $\boldsymbol{w}_{\mathrm{s}}$ be the intersection of the boundary of $\mathrm{co}W$ by a ray emanating from $\boldsymbol{w}_{\mathrm{c}}$ passing $-\boldsymbol{w}_0$, then $-\boldsymbol{w}_0 \in \mathrm{int}(\mathrm{co}W)$ if and only if $\|\boldsymbol{w}_{\mathrm{c}} - \boldsymbol{w}_{\mathrm{s}}\| > \|\boldsymbol{w}_{\mathrm{c}} + \boldsymbol{w}_0\|$.

Fig. 2a shows a force-closure unit grasp checked by Corollary 1. Let $\boldsymbol{q}=-\boldsymbol{w}_0-\boldsymbol{w}_{\mathrm{c}}$. The calculation of $\boldsymbol{w}_{\mathrm{s}}$ in Fig. 2a can be transformed to the standard ray-shooting problem. Thus, if $\|\boldsymbol{s}\| > \|\boldsymbol{q}\|$ as in Fig. 2b, then $-\boldsymbol{w}_0 \in \text{int}(\text{co}W)$. Let $\lambda = \|\boldsymbol{q}\|/\|\boldsymbol{s}\|$. Then $\lambda<1$ indicates a force-closure unit grasp.

Two shortcomings exist for the primary ray-shooting algorithm. Firstly, it does not handle the singular cases when calculating $\boldsymbol{c} = \boldsymbol{W}_i^{-1}\boldsymbol{q}$. A simple singular case is illustrated in Fig. 3, where $s_{\text{co}A}(\boldsymbol{u}_0)$ lies in a ray emanating from the origin and passing through a vertex of coV_0. Thus, we should make sure $\boldsymbol{W}_i$ is of full rank before calculating its inverse in our enhanced algorithm.

The ray-shooting algorithm needs to start from an initial set V_0 such that coV_0 contains the origin in its interior and is contained in co$A$. The lack of method to obtain this initial set V_0 is another shortcoming of the primary ray-shooting algorithm and we try to fix it here. First, we make the assumption that the friction coefficients $\mu_i>0$ for all the m ($m>1$) contact points. Let r_i be the largest radius of the sphere centered at $\boldsymbol{f}_\text{n}/m$ in the force space and fully contained in the friction cone FC$_i$ (Fig. 4a). Obviously, r_i can be computed by

$$r_i = \frac{\mu_i}{m\sqrt{1+\mu_i^2}}, \quad i=1,2,\cdots,m. \tag{21}$$

If $\boldsymbol{G}$ has full row rank, then for an arbitrary nonzero point $\boldsymbol{p}_j$ in the wrench space $\mathbb{R}^d$, let $\varepsilon_{i,j}$ be formulated by

$$\varepsilon_{i,j} = \frac{r_i}{\left\|\boldsymbol{G}_i^\text{T}(\boldsymbol{G}\boldsymbol{G}^\text{T})^{-1}\boldsymbol{p}_j\right\|}, \quad i=1,2,\cdots,m. \tag{22}$$

Let $\varepsilon_{i,j}=+\infty$ when the denominator encounters zero in Eq. (22). Then, we have the following theorem:

Theorem 4 For any positive ε_j, if $\varepsilon_j \le \min\{\varepsilon_{i,j}\}$ for all i, we have $\boldsymbol{w}_\text{c}+\varepsilon_j\boldsymbol{p}_j \in \text{co}W$.

Proof Let $\sum_{i=1}^m \boldsymbol{G}_i\boldsymbol{f}_i = \boldsymbol{w}_\text{c} + \varepsilon_j\boldsymbol{p}_j$, where $\boldsymbol{f}_i=\boldsymbol{f}_\text{n}/m+\Delta\boldsymbol{f}_i$. From Eq. (20), we have $\sum_{i=1}^m \boldsymbol{G}_i\Delta\boldsymbol{f}_i = \varepsilon_j\boldsymbol{p}_j$. The special solution is $\Delta\boldsymbol{f}_i = \boldsymbol{G}_i^\text{T}(\boldsymbol{G}\boldsymbol{G}^\text{T})^{-1}\varepsilon_j\boldsymbol{p}_j$ since $\boldsymbol{G}$ has full row rank. From $0<\varepsilon_j\le\varepsilon_{i,j}$, it can be deduced that $\|\Delta\boldsymbol{f}_i\| \le r_i$. We further conclude that $\boldsymbol{f}_i \in \text{FC}_i$ for all the m contact points (Fig. 4a). Then from the definition of the grasp wrench set W in Eq. (6), $\sum_{i=1}^m \boldsymbol{G}_i\boldsymbol{f}_i \in W$ is affirmed. Therefore, $\boldsymbol{w}_\text{c}+\varepsilon_j\boldsymbol{p}_j \in \text{co}W$.

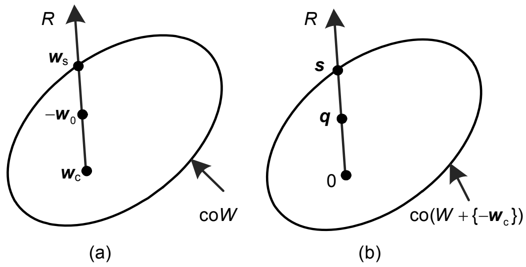

Fig. 2 Illustration of Corollary 1
(a) A ray R emanating from $\boldsymbol{w}_\text{c}$ passing $-\boldsymbol{w}_0$ intersects the boundary of co$W$ at $\boldsymbol{w}_\text{s}$; (b) Adding an offset of $-\boldsymbol{w}_\text{c}$ to co$W$, the ray-shooting problem of (a) transforms to the standard form

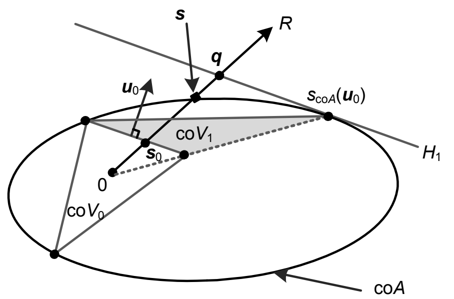

Fig. 3 A singular case in the ray-shooting algorithm
This differs from the singular case depicted by Zheng *et al.* (2010)

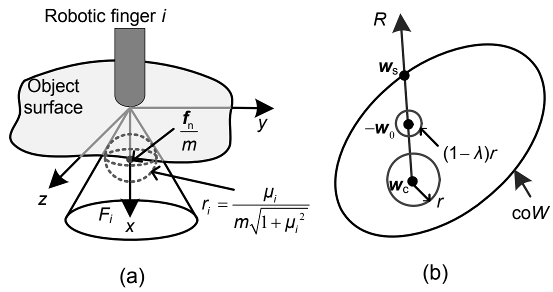

Fig. 4 Construction for the proof of Theorem 4
(a) A sphere with a radius of r_i centered at $\boldsymbol{f}_\text{n}/m$ in the force space is contained in the friction cone FC$_i$; (b) Two spheres with radii of r and $(1-\lambda)r$, respectively, and centered at $\boldsymbol{w}_\text{c}$ and $-\boldsymbol{w}_0$ in the wrench space are both included by co$W$

For 3D grasps, choose $\boldsymbol{p}_j$ (j=1, 2, ⋯, 7) as the vertices of a 6D regular simplex inscribed on the unit sphere centered at the origin of the wrench space $\mathbb{R}^6$, and let $r=\min\{\varepsilon_{i,j}\}/6$. With the aid of Theorem 4, it can be easily proven that the sphere with a radius of r centered at $\boldsymbol{w}_\text{c}$ (Fig. 4b) is fully contained in co$W$. If the grasp is affirmed to be a force-closure unit grasp, i.e., obtaining a non-negative $\lambda<1$ from Corollary 1, the sphere with a radius of $(1-\lambda)r$ centered at $-\boldsymbol{w}_0$ is also included by co$W$. For 2D grasps, we reach the same conclusions except that the selection of $\boldsymbol{p}_j$ (j=1, 2, 3, 4) comes from a 3D regular simplex and r is changed to $\min(\varepsilon_{i,j})/3$. Therefore, for 3D (or 2D) grasps, a simple initial set V_0 can be the vertices of a 6D (or 3D) regular simplex inscribed on the sphere centered at the origin with a radius of r and $(1-\lambda)r$ for $A=W+\{-\boldsymbol{w}_\mathrm{c}\}$ and $A=W+\{\boldsymbol{w}_0\}$, respectively.

The enhanced ray-shooting algorithm is almost the same as its original form proposed by Zheng *et al.* (2010) except for the two improvements described above. Thus, we ignore its explicit expression and implement it in our grasp evaluation algorithm as follows:

Algorithm 1 Grasp evaluation

Input: The offset wrench $\boldsymbol{w}_0$, the friction coefficients μ_i and the positions $\boldsymbol{r}_i$ for m contact points, the vertices $\boldsymbol{v}_j$, j=1, 2, ⋯, of the grasped polyhedral object, and discretized force direction $\boldsymbol{e}_i$, i=1, 2, ⋯, n.

Output: If the grasp is a force-closure unit grasp, return the grasp quality ρ_m dedicated by Eq. (9).

1 Compute $\boldsymbol{G}_i$, $\boldsymbol{w}_\mathrm{c}$, and W by Eqs. (3), (20), and (6), respectively. Let $\boldsymbol{G}\leftarrow[\boldsymbol{G}_1\ \boldsymbol{G}_2\ \cdots\ \boldsymbol{G}_m]$
2 **if** $\boldsymbol{G}\boldsymbol{G}^\mathrm{T}$ is not of full rank **then**
3 abort {it is not a force-closure unit grasp}
4 **end if**
5 **if** $\boldsymbol{w}_\mathrm{c}=-\boldsymbol{w}_0$ **then**
6 $\lambda\leftarrow 0$
7 **else**
8 $A\leftarrow W+\{-\boldsymbol{w}_\mathrm{c}\}$, $\boldsymbol{q}\leftarrow-\boldsymbol{w}_0-\boldsymbol{w}_\mathrm{c}$. Compute $\boldsymbol{s}$ by the enhanced ray-shooting algorithm
9 **if** $\|\boldsymbol{s}\|>\|\boldsymbol{q}\|$ **then**
10 $\lambda\leftarrow\|\boldsymbol{q}\|/\|\boldsymbol{s}\|$
11 **else**
12 abort {it is not a force-closure unit grasp}
13 **end if**
14 **end if**
15 **for all** discretized force directions $\boldsymbol{e}_i$ **do**
16 select vertices $\boldsymbol{v}_j$ such that $\boldsymbol{e}_i$ lies in their friction cones $A\leftarrow W+\{\boldsymbol{w}_0\}$, $\boldsymbol{w}_{i,j}\leftarrow\left[\boldsymbol{e}_i^\mathrm{T}\ \boldsymbol{e}_i^\mathrm{T}[\boldsymbol{v}_j]_\times^\mathrm{T}\right]^\mathrm{T}$
17 **for all** vertices $\boldsymbol{v}_j$ **do**
18 $\boldsymbol{q}\leftarrow-\boldsymbol{w}_{i,j}$, calculate $\boldsymbol{s}$ by the enhanced ray-shooting algorithm. $\rho_{i,j}\leftarrow\|\boldsymbol{s}\|/\|\boldsymbol{q}\|$
19 **end for**
20 $\rho^*(\boldsymbol{e}_i)\leftarrow\min\{\rho_{i,j}\}$
21 **end for**
22 **return** $\rho_\mathrm{m}\leftarrow\min\{\rho^*(\boldsymbol{e}_i)\}$

4.3 Discussions on the grasp evaluation algorithm

In the above algorithm, the force-closure unit grasp is checked in lines 1 to 14 according to Corollary 1, and the grasp quality is calculated in lines 15 to 22 by the enhanced ray-shooting algorithm. The above grasp evaluation algorithm is very fast because linearization of the friction cones is not needed. However, a large number of callings for the enhanced ray-shooting algorithm are needed in line 18 when the number of force directions and the number of vertices of the grasped polyhedral object are very large. Suppose $\boldsymbol{w}_j$ (j=1, 2, ⋯, M) are the disturbance wrenches in W_d (Fig. 5a). We first calculate a simplex coV_0 containing the origin of the wrench space in its interior and with all the d+1 vertices on the boundary of co($W+\{\boldsymbol{w}_0\}$). When calculating the intersection point of the boundary of co($W+\{\boldsymbol{w}_0\}$) by the ray in the direction of $\boldsymbol{w}_1$ by means of the enhanced ray-shooting algorithm, a sequence of simplexes will be generated. We can record some of these simplexes to speed up the computation of the ray-shooting problem in the direction of $\boldsymbol{w}_2$. A recursive problem decomposition strategy is recommended to avoid reduplicate simplex calculations:

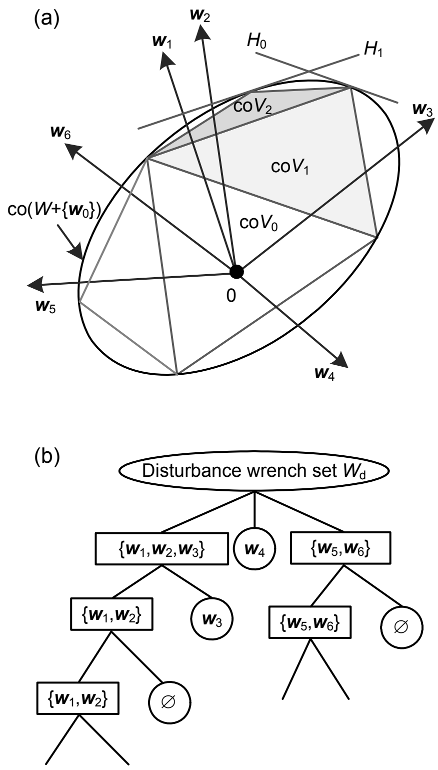

Fig. 5 Recursive problem decomposition
(a) The grasp evaluation algorithm needs to determine all the intersection points; (b) A searching tree generated by recursive problem decomposition

1. Divide the disturbance wrench set W_d into d+1 subsets such that all the rays shooting in the direction $\boldsymbol{w}_j$ from the same subset intersect the same facet of coV_0. Attach the corresponding intersection facet to each subset.

2. For each subset with more than one element, calculate the support mapping s_A of co(W+{$\boldsymbol{w}_0$}) in the outward normal direction of the corresponding intersection facet co$F$. If the distance between s_A and co$F$ is less than the termination tolerance ε, then output the intersection points on the facet co$F$ by all the rays shooting in the direction $\boldsymbol{w}_j$ chosen from the subset; otherwise, construct a new simplex co$V$ using the face co$F$ and the support mapping s_A. Then further divide the disturbance wrenches in the subset into d low-layer subsets such that all the rays in the direction $\boldsymbol{w}_j$ in the same low-layer subset intersect the same new facet of co$V$. Attach the new corresponding intersection facet to each low-layer subset.

3. For each subset with only one element $\boldsymbol{w}_j$, we can resort to the enhanced ray-shooting algorithm to calculate the intersection point on the boundary of co(W+{$\boldsymbol{w}_0$}). Note that we can skip some of the steps in the enhanced ray-shooting algorithm since some of the simplexes have already been calculated.

4. Recursively repeat step 2 or 3 for each new subset until no new subsets are generated in step 2.

The recursive problem decomposition strategy described above can be represented by a search tree structure. The root node represents all elements in the disturbance wrench set W_d. All the elements in a non-leaf node are divided and assigned to its child nodes depending on the face intersected by the ray. A simple search tree is illustrated in Fig. 5b, where d=2, $\varnothing$ is the empty set, and there are only six disturbance wrenches in W_d. Note that the searching tree strategy will not accelerate the grasp evaluation algorithm when the number of force directions and the number of vertices of a grasped polyhedral object are relatively small.

Another method to speed up the grasp evaluation algorithm is a short-circuiting technique. In the iteration of the ray-shooting algorithm shown in Fig. 1, we obtain a sequence of {$\boldsymbol{s}_i$} which is convergent to the intersection point $\boldsymbol{s}$. Obviously, $\|\boldsymbol{s}_i\|\leq\|\boldsymbol{s}\|$. Let ρ_m be the minimum grasp quality among all the checked disturbance wrenches. The iteration of the enhanced ray-shooting algorithm in line 18 can be terminated immediately once $\|\boldsymbol{s}_i\|>\rho_m\|\boldsymbol{q}\|$ occurs; i.e., the calculation of $\boldsymbol{s}$ is not required.

The ray-shooting algorithm proposed by Zheng *et al.* (2010) can cope with arbitrary object grasping. However, our grasp evaluation algorithm is limited to polyhedral object grasping because the disturbance wrench set W_d is difficult to calculate for non-polyhedral objects. A possible approach to modeling the disturbance wrench may be as follows. First, approximate the surface of an arbitrary object by a point set and their corresponding normal direction (Roa and Suárez, 2009). Second, calculate the candidate wrench set that contains all the wrenches generated by those discretized forces $\boldsymbol{e}_i$ in the friction cone for each point. Finally, the disturbance wrench set W_d can be selected as those wrenches on the boundary of the convex hull of the candidate wrench set.

5 Contact points regrasp planning

When allowing the contact points to move on the faces of the grasped polyhedral object, the grasp quality measured by Eq. (9) will change accordingly. In this section, we discuss how to place the contact points appropriately to achieve better grasp quality for 3D grasps. The following assumptions are made first:

1. The initial contact points form a force-closure unit grasp.

2. The ith contact position $\boldsymbol{r}_i$ is restricted to a face denoted by C_i.

5.1 Necessary condition for grasp quality improvement

Considering the ith contact position $\boldsymbol{r}_i$ moving to a new position $\boldsymbol{r}_i'$ on the contact face C_i (Fig. 6a), the amount of the change $\delta\boldsymbol{r}_i=\boldsymbol{r}_i'-\boldsymbol{r}_i$ can be expressed in the initial local contact coordinate frame by $[0\ \delta y_i\ \delta z_i]^{\mathrm{T}}$. Note that $\boldsymbol{R}_i$ in Eq. (3) is a constant matrix when $\boldsymbol{r}_i$ is restricted to the movement on the face C_i. Therefore, $\boldsymbol{r}_i'$ can be expressed in the object coordinate frame as

$$\boldsymbol{r}_i'=\boldsymbol{r}_i+\boldsymbol{R}_i[0\ \ \delta y_i\ \ \delta z_i]^{\mathrm{T}}. \tag{23}$$

The grasp matrix $\boldsymbol{G}_i$ changes to $\boldsymbol{G}_i'$ accordingly,

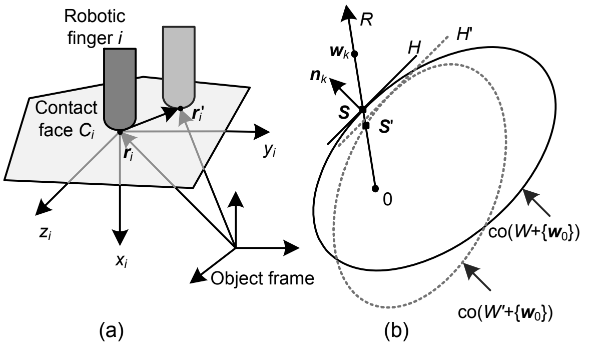

Fig. 6 The moving of the contact points
(a) The ith contact point moving on a face; (b) The variation of the total grasp wrench set resulting from the moving of all the contact points

$$\boldsymbol{G}_i' = \boldsymbol{G}_i + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{R}_i \boldsymbol{S}_i \end{bmatrix}, \tag{24}$$

where $\boldsymbol{0}$ is a 3×3 zero matrix and $\boldsymbol{S}_i$ is a skewed symmetric matrix in the following form:

$$\boldsymbol{S}_i = \begin{bmatrix} 0 & -\delta z_i & \delta y_i \\ \delta z_i & 0 & 0 \\ -\delta y_i & 0 & 0 \end{bmatrix}. \tag{25}$$

Suppose after moving by $\delta \boldsymbol{r}_i$ for all the contact points on each contact face, the total grasp wrench set W calculated using Eq. (6) changes to W'. Let $A=W+\{\boldsymbol{w}_0\}$ and $A'=W'+\{\boldsymbol{w}_0\}$ (Fig. 6b). Suppose both co$A$ and coA' contain the origin of the wrench space in their interiors. Let $\rho_j=\max\{\rho|-\rho \boldsymbol{w}_j \in \text{co}A\}$ for all the disturbance wrenches $\boldsymbol{w}_j$ contained in W_d, and k be the index that satisfies $\rho_k \le \rho_j$ for all j=1, 2, ⋯, M. Obviously, ρ_k is the grasp quality measured by Eq. (9). In Fig. 6b, the hyperplane H with normal $\boldsymbol{n}_k$ supports co$A$ at $\boldsymbol{s}$, which is the intersection point on the boundary of co$A$ by the ray R. Suppose both $\boldsymbol{s}$ and $\boldsymbol{n}_k$ are calculated by the ray-shooting algorithm beforehand (the short-circuiting technique is not implemented). We have $h_A(\boldsymbol{n}_k)=\boldsymbol{n}_k^\text{T}\boldsymbol{s}=1$ when ignoring the tolerance ε. Therefore, it can be concluded that

$$\rho_k = \frac{\|\boldsymbol{s}\|}{\|\boldsymbol{w}_k\|} = \frac{1}{\boldsymbol{n}_k^\text{T}\boldsymbol{w}_k}. \tag{26}$$

When the total grasp wrench set is changed to W' (Fig. 6b, dotted line) we have $h_{A'}(\boldsymbol{n}_k) \ge \boldsymbol{n}_k^\text{T}\boldsymbol{s}'$, where $\boldsymbol{s}'$ is the intersection point on the boundary of coA' by the ray R. All three points $\boldsymbol{s}'$, $\boldsymbol{s}$, and $\boldsymbol{w}_k$ lie on the same ray R; thus, the solution to $\rho_k' = \max\{\rho \mid -\rho \boldsymbol{w}_k \in \text{co}A'\}$ is bounded by

$$\rho_k' = \frac{\|\boldsymbol{s}'\|}{\|\boldsymbol{w}_k\|} \le h_{A'}(\boldsymbol{n}_k)\rho_k. \tag{27}$$

Therefore, $h_{A'}(\boldsymbol{n}_k)>1$ is a necessary condition for the grasp quality improvement when W is changed to W'. We use this necessary condition to guide the movement of contact points, and it works well in our numerical examples demonstrated in Section 6.

5.2 Heuristic regrasp planning algorithm

We begin with a discussion about the influence of the movement of $\delta \boldsymbol{r}_i$ on $h_A(\boldsymbol{n}_k)$. Choose $\delta \boldsymbol{r}_i=[0\ \delta y_i\ \delta z_i]^\text{T}$ with respect to each local contact coordinate frame for i=1, 2, ⋯, m. Let $\boldsymbol{d}_i = \boldsymbol{G}_i^\text{T}\boldsymbol{n}_k$. Then from Eq. (24), the variation $\delta \boldsymbol{d}_i$ is formulated by

$$\delta \boldsymbol{d}_i = \begin{bmatrix} -\boldsymbol{n}_{k\text{R}}^\text{T}\boldsymbol{R}_{\text{V}i3} & \boldsymbol{n}_{k\text{R}}^\text{T}\boldsymbol{R}_{\text{V}i2} \\ 0 & -\boldsymbol{n}_{k\text{R}}^\text{T}\boldsymbol{R}_{\text{V}i1} \\ \boldsymbol{n}_{k\text{R}}^\text{T}\boldsymbol{R}_{\text{V}i1} & 0 \end{bmatrix} \begin{bmatrix} \delta y_i \\ \delta z_i \end{bmatrix}, \tag{28}$$

where $\boldsymbol{R}_{\text{V}i1}$, $\boldsymbol{R}_{\text{V}i2}$, and $\boldsymbol{R}_{\text{V}i3}$ are the first, second, and third column vectors, respectively, of the rotation matrix $\boldsymbol{R}_i$, and $\boldsymbol{n}_{k\text{R}} \in \mathbb{R}^3$ is a column vector formulated by the last three elements in $\boldsymbol{n}_k$. It is easy to prove that $h_{\text{FC}_i}(\boldsymbol{d}_i)$ satisfies the Lipschitz continuity from Eq. (18), which implies its differentiability almost everywhere w.r.t. $\boldsymbol{d}_i$. From Eq. (28), $\boldsymbol{d}_i$ is differentiable w.r.t. y_i and z_i. Let $\boldsymbol{d}_i=[\text{d}x_i\ \text{d}y_i\ \text{d}z_i]^\text{T}$. Then the partial derivatives of $h_{\text{FC}_i}(\boldsymbol{d}_i)$ w.r.t. y_i and z_i are determined by

$$\frac{\partial h_{\text{FC}_i}(\boldsymbol{d}_i)}{\partial y_i} = \begin{cases} -\boldsymbol{n}_{k\text{R}}^\text{T} R_{\text{V}i3} + \dfrac{\mu_i \text{d}z_i}{\sqrt{\text{d}y_i^2 + \text{d}z_i^2}} \boldsymbol{n}_{k\text{R}}^\text{T} R_{\text{V}i1}, \\ \qquad \text{if } \mu_i\sqrt{\text{d}y_i^2 + \text{d}z_i^2} > \max\{0, -\text{d}x_i\}, \\ 0, \\ \qquad \text{if } \mu_i\sqrt{\text{d}y_i^2 + \text{d}z_i^2} < \max\{0, -\text{d}x_i\}, \end{cases} \tag{29}$$

$$\frac{\partial h_{\mathrm{FC}_i}(\boldsymbol{d}_i)}{\partial z_i}=\begin{cases}\boldsymbol{n}_{k\mathrm{R}}^{\mathrm{T}}R_{\mathrm{V}i2}-\dfrac{\mu_i \mathrm{d}y_i}{\sqrt{\mathrm{d}y_i^{\,2}+\mathrm{d}z_i^{\,2}}}\boldsymbol{n}_{k\mathrm{R}}^{\mathrm{T}}R_{\mathrm{V}i1}, \\ \qquad \text{if } \mu_i\sqrt{\mathrm{d}y_i^{\,2}+\mathrm{d}z_i^{\,2}}>\max\{0,-\mathrm{d}x_i\}, \\ 0, \\ \qquad \text{if } \mu_i\sqrt{\mathrm{d}y_i^{\,2}+\mathrm{d}z_i^{\,2}}<\max\{0,-\mathrm{d}x_i\}.\end{cases} \tag{30}$$

Note that $h_{\mathrm{FC}_i}(\boldsymbol{d}_i)$ is not differentiable when $\mu_i\sqrt{\mathrm{d}y_i^{\,2}+\mathrm{d}z_i^{\,2}}=\max\{0,-\mathrm{d}x_i\}$. Luckily, the irregular $\boldsymbol{d}_i$ is rarely encountered in numerical examples. Besides, we can perform a small perturbation to escape this irregular point even if a non-differentiable case is encountered.

Let $\boldsymbol{y}=[y_1\ z_1\ y_2\ z_2\ \cdots\ y_m\ z_m]^{\mathrm{T}}$ be a $2m$-dimensional column vector and $\delta\boldsymbol{y}$ combine the variations of all the contact points. From Eq. (14), $\delta h_A(\boldsymbol{n}_k)=\boldsymbol{L}_k\delta\boldsymbol{y}$, where $\boldsymbol{L}_k$ is formulated by

$$\boldsymbol{L}_k=\left[\frac{\partial h_{\mathrm{FC}_1}(\boldsymbol{d}_1)}{\partial y_1}\ \frac{\partial h_{\mathrm{FC}_1}(\boldsymbol{d}_1)}{\partial z_1}\ \frac{\partial h_{\mathrm{FC}_2}(\boldsymbol{d}_2)}{\partial y_2}\ \frac{\partial h_{\mathrm{FC}_2}(\boldsymbol{d}_2)}{\partial z_2}\ \cdots\ \frac{\partial h_{\mathrm{FC}_m}(\boldsymbol{d}_m)}{\partial y_m}\ \frac{\partial h_{\mathrm{FC}_m}(\boldsymbol{d}_m)}{\partial z_m}\right]. \tag{31}$$

Note that we use a subscript k in Eq. (31) because $\boldsymbol{n}_k$ is hidden in $\boldsymbol{d}_i=\boldsymbol{G}_i^{\mathrm{T}}\boldsymbol{n}_k$. For all the other disturbance wrenches $\boldsymbol{w}_j$ $(j\neq k)$ contained in W_{d}, we can calculate $\boldsymbol{L}_j$ by Eq. (31) in the same way. Therefore, the variation δy satisfying $\boldsymbol{L}_k\delta\boldsymbol{y}>\boldsymbol{0}$ is a possible direction for grasp quality improvement. Based on this observation, we formulate the following linear programming problem for the estimation of the optimum amount of change $\Delta\boldsymbol{y}$ in the initial local contact coordinate frame:

$$\begin{gathered}\max \boldsymbol{L}_k\Delta\boldsymbol{y}\\ \text{s.t. } \boldsymbol{A}_1\Delta\boldsymbol{y}\geq\boldsymbol{b}_1,\ \boldsymbol{A}_2\Delta\boldsymbol{y}\geq\boldsymbol{b}_2,\end{gathered} \tag{32}$$

where $\boldsymbol{A}_1=[\boldsymbol{L}_1^{\mathrm{T}}\ \boldsymbol{L}_2^{\mathrm{T}}\ \cdots\ \boldsymbol{L}_{k-1}^{\mathrm{T}}\ \boldsymbol{L}_{k+1}^{\mathrm{T}}\ \cdots\ \boldsymbol{L}_M^{\mathrm{T}}]^{\mathrm{T}}$, $\boldsymbol{b}_1=[\rho_k-\rho_1\ \rho_k-\rho_2\ \ldots\ \rho_k-\rho_{k-1}\ \rho_k-\rho_{k+1}\ \ldots\ \rho_k-\rho_M]^{\mathrm{T}}$, and $\boldsymbol{A}_2\Delta\boldsymbol{y}\geq\boldsymbol{b}_2$ represents the constraint such that the new contact position $\boldsymbol{r}_i'\in C_i$ after the movement of $\Delta\boldsymbol{y}$. Let $\Delta\boldsymbol{y}^*$ denote the solution to Eq. (32) and suppose $\|\Delta\boldsymbol{y}^*\|\neq 0$. We have a moving of $\chi\Delta\boldsymbol{y}^*$ which will certainly increase $h_A(\boldsymbol{n}_k)$ if χ is a sufficiently small positive number. Let σ be the termination tolerance and $n_{\max}$ be the maximum iteration number. By assigning $\chi\in(0, 1)$, we have the heuristic regrasp planning algorithm as follows:

Algorithm 2 Heuristic regrasp planning

Input: The initial force-closure unit grasp contact positions $\boldsymbol{r}_i$ and the constraint face C_i for m contact points, and the disturbance wrench set W_{d}.

Output: The final grasp quality ρ^* and all contact points $\boldsymbol{r}_i$.

```
Compute ρ_j and n_j for all the disturbance wrenches in W_d.
  Let ρ* ← min ρ_j. Select k such that ρ_k = ρ*
Formulate Eq. (32) and compute Δy*, n ← 0
repeat
    r_i^0 ← r_i for i=1, 2, ⋯, m, update all r_i by a moving of
      Δy*, and call Algorithm 1
    if ρ_m ≤ ρ* or it is not a force-closure unit grasp then
      repeat
        r_i ← r_i^0, Δy* ← χΔy*, update all r_i by a moving
          of Δy*, and call Algorithm 1
      until ‖Δy*‖ < σ or ρ_m > ρ*
      if ‖Δy*‖ < σ then
        break
      end if
    end if
    ρ* ← ρ_m, n ← n+1, formulate Eq. (32), and compute Δy*
until ‖Δy*‖ < σ or n ≥ n_max
return ρ* and r_i for i=1, 2, ⋯, m
```

6 Numerical examples

We verify the performance of the proposed algorithms in this section using three numerical examples. All the examples were implemented using MATLAB 7.5 on a 3.16 GHz Intel Core 2 Duo desktop computer. The coefficients of the friction for contact points and disturbance forces were set to 0.3 and 1.5, respectively. The unit of length for the objects was cm while the unit of force was set to N in the numerical examples. The termination tolerance ε in the ray-shooting algorithm was set to 10^{-5}. We adopted the same friction cone for a vertex as that defined by Strandberg and Wahlberg (2006). It was taken as the union of all the cones belonging to the faces forming the vertex. An additional cone in the direction of the averaged vertex normal was also added.

Example 1 This example was designed to verify the efficiency of the grasp evaluation algorithm in 2D grasps. Five 2D grasp cases were considered (Fig. 7).

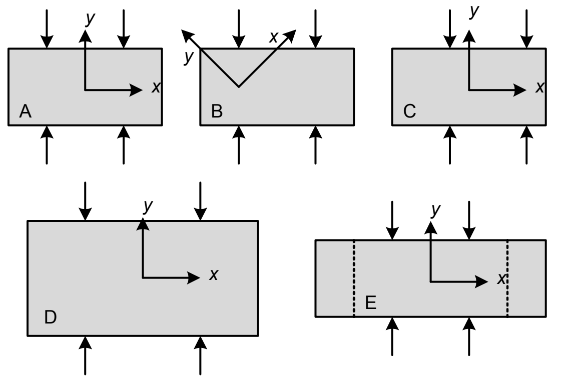

Fig. 7 Five 2D grasp examples

The original grasp object is a rectangle of 8×4 in case A. The reference frame is placed in the center of the rectangle, and the coordinates of the four contacts are $[\pm2\ \pm2]^T$. All the other cases are modifications of case A. In case B the origin is moved left by two units and the reference frame is rotated anticlockwise by 45°. In case C all the contact points are moved right by one unit. In case D the grasp and the object are scaled by a factor of 1.5. In case E two rectangles with dimensions 2×4 are added to the grasped object

We first set the resolution of the disturbance force directions to 5°, i.e., n=72, and calculated 164 disturbance wrenches in W_d for each case. Five grasp qualities were obtained, i.e., ρ_C=0.6477, ρ_E=0.5877, and ρ_A=ρ_B=ρ_D=0.7213 after calling Algorithm 1. Our grasp quality measure more closely resembled human intuition compared with other quality evaluation methods discussed by Borst *et al.* (2004). The results were independent of the scale and the choice of the reference frame. The computation costs measured by time are listed in Table 2. Next, we increased the resolution of the disturbance force directions to 0.1°, i.e., n=3600, and repeated the grasp evaluation procedure. To demonstrate the efficiency improvement yielded by our searching tree, we also implemented the grasp evaluation algorithm without the searching tree. The computation costs are also listed in Table 2. For n=72 and n=3600, the average CPU times of our searching tree method were only 4.64% and 2.02% respectively, of those calculated without the searching tree.

Example 2 This example was originally presented by Strandberg and Wahlberg (2006) for 3D grasp evaluation. A box of 2×2×5 was grasped with four point contacts (Fig. 8a). The object coordinate frame was attached to the center of the box. The contact positions were $[\pm1\ 0\ 0]^T$ and $[0\ \pm1\ 0]^T$. The

Table 2 Execution time for Example 1

Case	Execution time (ms)			
	With searching tree		Without searching tree	
	n=72	n=3600	n=72	n=3600
A	38.75	130.8	879.3	6426
B	42.71	134.8	929.6	6663
C	40.93	134.5	776.3	6641
D	33.95	127.8	763.4	6329
E	35.15	130.7	781.0	6463
Mean	38.30	131.7	825.9	6504

n is the number of discretized disturbance force directions

resolutions of the disturbance force directions were set to 10.59° and 10° for θ and φ, respectively. Thus, it resulted in a total of 578 force directions and 2664 disturbance wrenches in W_d. The grasp evaluation algorithm with the short-circuiting technique was implemented twice: in the first implementation gravity was ignored, while in the second one gravity of $\boldsymbol{g}=[0\ 0\ -0.3]^T$ was considered. Then, a contact point in the center of the bottom face was added (Fig. 8b) to increase the grasp quality. The algorithm proposed by Strandberg and Wahlberg (2006) adopts the UGWS as a convex combination of all linearized primitive wrenches for simplification. Thus, to compare computation time, we have to compute the Minkowski sum of primitive wrenches before implementing their method. We linearized the friction cone FC_i using a k-sided pyramid (k=3, 5, 7) and computed the elements of the Minkowski sum of primitive wrenches using the algorithm proposed by Zheng and Qian (2006). Then, we implemented Strandberg and Wahlberg's method using the convex hull of the computed Minkowski sum. The consumed CPU time and grasp quality ρ_m are listed in Table 3 in comparison with those obtained using our method.

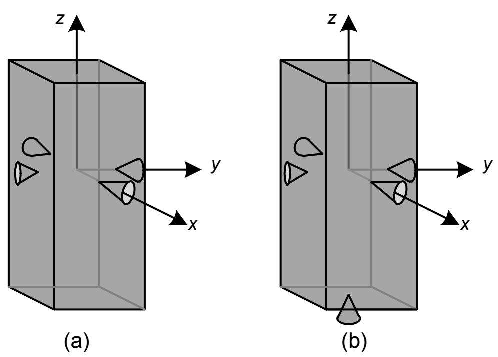

Fig. 8 Illustration of grasps evaluated in Example 2

(a) Four point contacts; (b) Five point contacts

Table 3 Comparison of consumed CPU time and grasp quality between our method with short-circuiting and Strandberg and Wahlberg (2006)'s method in Example 2

Grasp configuration	CPU time (s)				ρ_m			
	Ours	Strandberg and Wahlberg (2006)			Ours	Strandberg and Wahlberg (2006)		
		k=3	k=5	k=7		k=3	k=5	k=7
Fig. 8a, $\boldsymbol{g}$=[0 0 0]^T	2.69	10.1	37.0	68.7	0.2236	0.1591	0.2039	0.2169
Fig. 8a, $\boldsymbol{g}$=[0 0 −0.3]^T	2.65	10.5	38.2	75.9	0.1949	0.1208	0.1710	0.1844
Fig. 8b, $\boldsymbol{g}$=[0 0 −0.3]^T	2.73	38.6	139.4	334.2	0.3317	0.1951	0.2790	0.3037

k is the number of force vectors to linearize the friction cone

We can see that the consumed CPU time of our method was less than that of Strandberg and Wahlberg's method even when k=3 (the minimum number to linearize the friction cone for 3D grasps) and our method obtained much higher accuracy on grasp quality (Table 3). The CPU time consumed using their method increased dramatically with the growth of the number to linearize the friction cone and the number of contacts because there were $(k+1)^m-1$ elements in the Minkowski sum of primitive wrenches. For instance, we obtained 32 767 elements in computing the Minkowski sum and found 977 044 hyperplanes when implementing their method for k=7 and m=5. Therefore, their method is not suitable for grasp evaluation based on the UGWS described in this paper when the number of contacts is relatively large.

Example 3 The contact points planning was tested in this example for a triangular prism with height 2 (Fig. 9), given by Watanabe and Yoshikawa (2007). The base of this triangular prism was a right isosceles triangle with a size of $4\times4\times4\sqrt{2}$. We placed the object coordinate frame at the isosceles vertex of the bottom triangular; thus, the object's centroid translated to [4/3 4/3 1]^T. The gravitational force applied to the object was $\boldsymbol{g}$=[0 0 −0.16]^T, and the contact points were restricted to three lateral faces. Five thousand grasp configurations were sampled on the object, and 279 force-closure unit grasps were found by running the grasp test part, i.e., lines 1 to 14, of Algorithm 1. The average CPU time for a grasp test was 10.38 ms. Next, we planned the optimal grasp configuration using the heuristic regrasp planning algorithm proposed in Section 5. The resolutions of the disturbance force directions were set to 15° for both θ and φ, resulting in a total of 266 force directions and 932 disturbance wrenches in W_d. The termination tolerance σ and the maximum iteration number n_{max} were taken to be 10^{-4} and 20, respectively. We set χ=0.2 and took the first 50 force-closure unit grasps in the grasp test procedure as the initial grasps. Fig. 10 shows the correlation between the initial and the final grasp quality. The average grasp qualities for the initial grasp and the final grasp were 0.0206 and 0.1069, respectively. The average CPU time for a single heuristic searching procedure was 180.7 s.

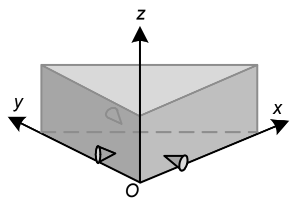

Fig. 9 Contact points planning in Example 3

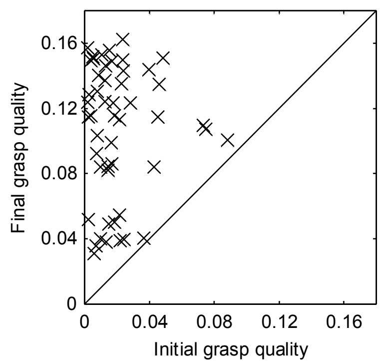

Fig. 10 Initial vs. final grasp quality for contact points regrasp planning in Example 3

7 Conclusions

Like the evaluation procedure based on the ability of the grasp to reject disturbance forces proposed by Strandberg and Wahlberg (2006), our approach incorporates the object geometry, can be visualized easily for 3D grasps, and is independent of the scale and choice of the reference frame. However, our method is based on an enhanced ray-shooting algorithm in which the geometry of the grasp wrench space is read by the support mapping. Therefore, a common and more natural grasp wrench space which limits the normal component of each individual contact force to one, can be implemented in our grasp evaluation without linearization of friction cones. Contact points regrasp planning is also discussed in this paper. Starting from the initial force-closure unit grasp, a heuristic searching algorithm to iteratively improve the grasp quality is proposed based on a necessary improvable condition. The efficiency and effectiveness of the proposed algorithms are illustrated by three numerical examples.

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