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Pseudo-evolute curves and caustic surfaces*

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Abstract: In this study, osculating caustic developable surfaces and rectifying caustic developable surfaces were obtained by considering space curves and curves on surfaces as base curves and changing the direction of the light source reflected by the mirror surface. It was proved that pseudo-evolute curves represent the striction curves (regression edges) of these surfaces. For developable surfaces based on curves on surfaces, it was observed that osculating caustic developable surfaces are equivalent to rectifying caustic developable surfaces if the curve is geodesic. Additionally, when the base curve was taken over any surface, the caustic surfaces were characterized as flat or normal approximation surfaces, depending on the direction of the light source.

Key words: Pseudo-evolute curves; Caustic surfaces; Developable surfaces; Rectifying caustic developable surfaces; Osculating caustic developable surfaces; Normal caustic developable surfaces

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1 Introduction

Traditionally, conical, cylindrical, and tangent surfaces of space curves are considered to be the three principal categories of developable surfaces. In this study, we examine the tangent surfaces of space curves, which are the most common and well-studied type among these surfaces. In Hoffmann et al. (2022), the caustic surfaces of all three types of developable surfaces were successfully generated, and all required parameters were defined and characterized. Assuming that the developable surfaces obtained from the tangent surfaces of space curves act as mirror surfaces, the equation of the caustic surfaces in Hoffmann et al. (2022) is presented as follows:

$$\boldsymbol{\varepsilon}(s,u)=\boldsymbol{\gamma}(s)+u\boldsymbol{f}(s),\quad s\in[a,b],\ u\in\mathbb{R},\tag{1}$$

where s and u are the parameters of the caustic surface, $[a,b]$ is an arbitrary interval ($a,b\in\mathbb{R}$), $\boldsymbol{\gamma}(s)$ is an arbitrary space curve, and $\boldsymbol{f}(s)$ is the direction of the generators of the caustic envelope surface as

$$\begin{aligned}\boldsymbol{f}(s)=&\langle\boldsymbol{\gamma}'(s)\times\boldsymbol{d}(s),\boldsymbol{d}'(s)\rangle\boldsymbol{\gamma}'(s)\\&-\langle\boldsymbol{\gamma}'(s)\times\boldsymbol{d}(s),\boldsymbol{\gamma}''(s)\rangle\boldsymbol{d}(s),\end{aligned}\tag{2}$$

where $\boldsymbol{d}$ is the reflected vector, $\langle,\rangle$ denotes the inner product, and $\times$ is the cross product.

What is the precise meaning of the term "reflected vector" in this context? The reflected vector represents the direction of reflected light rays transmitted from a light source to a mirror surface, which is a developable surface. In this study, we follow the methodology proposed by Hoffmann et al. (2022) and fix the directions of the light sources and the reflected vectors at different angles.

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The question is how the reflected vector can be obtained. If the mirror surface Φ is derived from the surface $\boldsymbol{M}(s,u)$ based on the curve $\boldsymbol{\gamma}(s)$, and the normal vector field of the mirror surface is $\boldsymbol{S} = (s_1, s_2, s_3)$, we can find the reflected vector $\boldsymbol{d}$ of the transmitted light ray in the X direction as follows:

$$\boldsymbol{d} = -(\boldsymbol{I}_3 + 2\boldsymbol{S}^2)X, \tag{3}$$

$$(s_1, s_2, s_3) \cong \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}, \tag{4}$$

where $\boldsymbol{I}$ is a unit matrix, and $s_1, s_2, s_3 \in \mathbb{R}$ are the vector components of the normal vector.

The curve formed by the motion of the rectifying planes of a space curve $\boldsymbol{\gamma}(s)$ is defined as a pseudo-evolute curve. If the base curve $\boldsymbol{\gamma}(s)$ is situated on a surface designated as $\boldsymbol{M}(s,u)$, the pseudo-evolute curve is defined as the motion of the tangent planes of the given surface $\boldsymbol{M}(s,u)$ along $\boldsymbol{\gamma}(s)$. For detailed information about the pseudo-evolute curves of an arbitrary space curve, see Fuchs (2013) and Fuchs et al. (2024).

In analyzing caustic surfaces, we are interested in observing the generation of three types of caustic developable surfaces: the rectifying, osculating, and normal caustic developable surfaces.

In our investigation of rectifying caustic surfaces, two key questions emerge. The first concerns determining the optimal direction for the light source to generate these caustic surfaces. The second involves exploring potential tangent surfaces of curves that could be used to generate these surfaces. Through our research, we find that positioning the light source along the unit binormal of the base curve and using tangent surfaces of the base curve's pseudo-evolute curve yield these surfaces (Section 3).

The Darboux frame of a surface $\boldsymbol{M}(s,u)$ along a curve $\boldsymbol{\gamma}(s)$ is given by the set $\{\boldsymbol{e_1}(s), \boldsymbol{y}(s), \boldsymbol{u}(s)\}$, where $\boldsymbol{u}(s)$ is the unit normal vector of $\boldsymbol{M}(s,u)$ along $\boldsymbol{\gamma}(s)$, $\boldsymbol{e}_1(s)$ is the unit tangent vector, and $\boldsymbol{y}(s) = \boldsymbol{u}(s) \times \boldsymbol{e}_1(s)$. Using a similar approach as in Section 3, we investigate the case where the curve $\boldsymbol{\gamma}(s)$ lies on the surface $\boldsymbol{M}(s,u)$. The reflected surfaces are obtained as osculating caustic developable surfaces when the light source is positioned along the $\boldsymbol{y}$ direction, and as normal caustic developable surfaces when the light source is positioned along the $\boldsymbol{u}$ direction. Furthermore, the tangent surfaces of the pseudo-evolute curves on the surface apply to both types of caustics.

2 Basic concepts

Let $\boldsymbol{\gamma}\colon \boldsymbol{I} \to \mathbb{R}^3$ be a unit speed space curve. The Frenet frame of the curve $\boldsymbol{\gamma}(s)$ is denoted by the set $\{\boldsymbol{e}_1(s), \boldsymbol{e}_2(s), \boldsymbol{e}_3(s)\}$, where $\boldsymbol{e}_1 = \boldsymbol{\gamma}'(s)$ is the unit tangent vector, $\boldsymbol{e}_2(s) = \frac{\boldsymbol{\gamma}''(s)}{||\boldsymbol{\gamma}''(s)||}$ is the unit normal vector, and $\boldsymbol{e}_3(s) = \boldsymbol{e}_1(s) \times \boldsymbol{e}_2(s)$ is the unit binormal vector of $\boldsymbol{\gamma}(s)$. The Frenet–Serret formulae are as follows:

$$\boldsymbol{e}_1'(s) = \kappa(s)\boldsymbol{e}_2(s), \tag{5}$$

$$\boldsymbol{e}_2'(s) = -\kappa(s)\boldsymbol{e}_1(s) + \tau(s)\boldsymbol{e}_3(s), \tag{6}$$

$$\boldsymbol{e}_3'(s) = -\tau(s)\boldsymbol{e}_2(s), \tag{7}$$

where $\kappa(s)$ is the curvature and $\tau(s)$ is the torsion function of the curve $\boldsymbol{\gamma}(s)$. The vector field $\boldsymbol{D}(s)$ for any unit speed space curve $\boldsymbol{\gamma}(s)$ is defined as

$$\boldsymbol{D}(s) = (\tau\boldsymbol{e}_1 + \kappa\boldsymbol{e}_3)(s), \tag{8}$$

where $\boldsymbol{D}(s)$ is called the modified Darboux vector. The unit Darboux vector can be expressed as

$$\overline{\boldsymbol{D}}(s) = \frac{(\tau\boldsymbol{e}_1 + \kappa\boldsymbol{e}_3)(s)}{\sqrt{\tau^2(s) + \kappa^2(s)}}. \tag{9}$$

It is known that a constant ratio $\frac{\tau(s)}{\kappa(s)}$, where $\kappa(s) \neq 0$, means that the curve is a generalized helix. Furthermore, if the value of $q(s) = \left(\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'\right)(s)$ is a constant, the curve is a slant helix, where $\kappa(s) \neq 0$.

Let $\boldsymbol{\gamma}\colon \boldsymbol{I} \to \mathbb{R}^3$ be the base curve, $\boldsymbol{v}\colon \boldsymbol{I} \to \mathbb{R}^3 \setminus \{0\}$ be the director curve, and $\boldsymbol{\Phi}(s,u)$ be defined as $\boldsymbol{\Phi}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{v}(s)$. Then the surfaces expressed as $\boldsymbol{\Phi}(s,u)$ are referred to as ruled surfaces. If these surfaces are represented in the form $\boldsymbol{\Phi}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{D}(s)$, they are known as rectifying developable surfaces. For details, please refer to Izumiya and Takeuchi (2004).

One of the fundamental formulae used in this study is the formula for the pseudo-evolute curve of a space curve, expressed as follows:

$$\begin{aligned}\boldsymbol{R}(s) &= \boldsymbol{\gamma}(s) - \frac{1}{\left(\frac{\tau}{\kappa}\right)'(s)}\left(\left(\frac{\tau}{\kappa}\right)\boldsymbol{e}_1 + \boldsymbol{e}_3\right)(s) \\ &= \boldsymbol{\gamma}(s) - \frac{1}{\left(\frac{\tau}{\kappa}\right)'(s)}\boldsymbol{D}(s).\end{aligned} \tag{10}$$

Let $\boldsymbol{M}(s,u)$ be a surface and $\boldsymbol{\gamma}\colon \boldsymbol{I}\to\boldsymbol{M}$ a unit speed curve on this surface. In this context, the Darboux frame of the surface $\boldsymbol{M}(s,u)$ along $\boldsymbol{\gamma}(s)$ is denoted by the set $\{\boldsymbol{e}_1(s),\boldsymbol{y}(s),\boldsymbol{u}(s)\}$. The normal curvature $\kappa_{\rm n}(s)$, the geodesic curvature $\kappa_{\rm g}(s)$, and the geodesic torsion $\tau_{\rm g}(s)$ of $\boldsymbol{\gamma}(s)$ are expressed as follows:

$$\kappa_{\rm n}(s)=\frac{\langle\boldsymbol{\gamma}''(s),\boldsymbol{u}(s)\rangle}{||\boldsymbol{\gamma}'(s)||^2}, \tag{11}$$

$$\kappa_{\rm g}(s)=\frac{\det(\boldsymbol{\gamma}'(s),\boldsymbol{\gamma}''(s),\boldsymbol{u}(s))}{||\boldsymbol{\gamma}'(s)||^3}, \tag{12}$$

$$\tau_{\rm g}(s)=\frac{\det(\boldsymbol{\gamma}'(s),\boldsymbol{u}(s),\boldsymbol{u}'(s))}{||\boldsymbol{\gamma}'(s)||^2}. \tag{13}$$

Then, the Darboux formulae are presented as follows:

$$\boldsymbol{e}_1'(s)=\kappa_{\rm g}(s)\boldsymbol{y}(s)+\kappa_{\rm n}(s)\boldsymbol{u}(s), \tag{14}$$

$$\boldsymbol{y}'(s)=-\kappa_{\rm g}(s)\boldsymbol{e}_1(s)+\tau_{\rm g}(s)\boldsymbol{u}(s), \tag{15}$$

$$\boldsymbol{u}'(s)=-\kappa_{\rm n}(s)\boldsymbol{e}_1(s)-\tau_{\rm g}(s)\boldsymbol{y}(s). \tag{16}$$

In Izumiya and Otani (2015), the authors defined two vector fields, osculating Darboux vector $\boldsymbol{D}_{\rm O}(s)$ along $\boldsymbol{\gamma}(s)$ and unit osculating Darboux vector $\overline{\boldsymbol{D}}_{\rm O}(s)$, which we will use in this work. These are defined as follows:

$$\boldsymbol{D}_{\rm O}(s)=(\tau_{\rm g}\boldsymbol{e}_1-\kappa_{\rm n}\boldsymbol{y})(s), \tag{17}$$

$$\overline{\boldsymbol{D}}_{\rm O}(s)=\left(\frac{\tau_{\rm g}\boldsymbol{e}_1-\kappa_{\rm n}\boldsymbol{y}}{\sqrt{\tau_{\rm g}^2+\kappa_{\rm n}^2}}\right)(s). \tag{18}$$

Izumiya and Otani (2015) defined the osculating developable surface of $\boldsymbol{M}(s,u)$ along $\boldsymbol{\gamma}(s)$ and provided the following equation for the surface:

$$\begin{aligned}{\rm OD}_{\boldsymbol{\gamma}}(s,u)&=\boldsymbol{\gamma}(s)+u\left(\frac{\tau_{\rm g}\boldsymbol{e}_1-\kappa_{\rm n}\boldsymbol{y}}{\sqrt{\tau_{\rm g}^2+\kappa_{\rm n}^2}}\right)(s)\\&=\boldsymbol{\gamma}(s)+u\overline{\boldsymbol{D}}_{\rm O}(s).\end{aligned} \tag{19}$$

In Hananoi and Izumiya (2017), similar methodologies were employed to derive the rectifying Darboux vector $\boldsymbol{D}_{\rm R}$ along $\boldsymbol{\gamma}(s)$, unit rectifying Darboux vector $\overline{\boldsymbol{D}}_{\rm R}$, and normal developable surface $\mathbf{ND}_{\boldsymbol{\gamma}}$, for which the formulae are provided as follows (for a more comprehensive understanding please refer to Hananoi and Izumiya (2017) and Köse and Yayli (2023)):

$$\boldsymbol{D}_{\rm R}(s)=(\tau_{\rm g}\boldsymbol{e}_1+\kappa_{\rm g}\boldsymbol{u})(s), \tag{20}$$

$$\overline{\boldsymbol{D}}_{\rm R}(s)=\left(\frac{\tau_{\rm g}\boldsymbol{e}_1+\kappa_{\rm g}\boldsymbol{u}}{\sqrt{\tau_{\rm g}^2+\boldsymbol{\kappa}_{\rm g}^2}}\right)(s), \tag{21}$$

$$\begin{aligned}\mathbf{ND}_{\boldsymbol{\gamma}}(s,u)&=\boldsymbol{\gamma}(s)+u\left(\frac{\tau_{\rm g}\boldsymbol{e}_1+\kappa_{\rm g}\boldsymbol{u}}{\sqrt{\tau_{\rm g}^2+\kappa_{\rm g}^2}}\right)(s)\\&=\boldsymbol{\gamma}(s)+u\overline{\boldsymbol{D}}_{\rm R}(s).\end{aligned} \tag{22}$$

Another term that is repeatedly referenced in our study is the striction curve, defined as the regression edge of the surface $\boldsymbol{M}(s,u)$, which is expressed as follows:

$$\boldsymbol{S}(s)=\boldsymbol{\gamma}(s)-\frac{\langle\boldsymbol{\gamma}'(s),\boldsymbol{v}'(s)\rangle}{\langle\boldsymbol{v}'(s),\boldsymbol{v}'(s)\rangle}\boldsymbol{v}(s),$$

where $\boldsymbol{\gamma}(s)$ is the base curve and $\boldsymbol{v}\colon \boldsymbol{I}\to\mathbb{R}^3\setminus\{0\}$ is the director curve.

In conclusion, we present a fundamental overview of approximation surfaces. If the normal vector field of a ruled surface takes $\boldsymbol{\gamma}(s)$ as its base curve and coincides with the normal vector field of the surface $\boldsymbol{M}(s,u)$ along $\boldsymbol{\gamma}(s)$, this ruled surface is called the flat approximation surface of $\boldsymbol{M}(s,u)$ and is described as

$$\begin{aligned}\boldsymbol{F}_{\rm b}(s,u)&=\boldsymbol{b}(s)+u\left(\frac{\tau_{\rm g}\boldsymbol{e}_1-\kappa_{\rm g}\boldsymbol{y}}{\sqrt{\tau_{\rm g}^2+\kappa_{\rm n}^2}}\right)(s)\\&=\boldsymbol{b}(s)+u\overline{\boldsymbol{D}}_{\rm O}(s),\end{aligned} \tag{23}$$

$$\boldsymbol{b}(s)=\boldsymbol{\gamma}(s)+r\boldsymbol{u}(s),\quad r\in\mathbb{R}, \tag{24}$$

where r is the offset distance from the base surface $\boldsymbol{M}(s,u)$ to the parallel surface. When $r=0$, $\boldsymbol{F}_{\rm b}(s,u)$ is a flat approximation surface of $\boldsymbol{M}(s,u)$ along $\boldsymbol{\gamma}(s)$. When $r\neq 0$, $\boldsymbol{F}_{\rm b}(s,u)$ is a flat approximation surface parallel to $\boldsymbol{M}(s,u)$ along $\boldsymbol{b}(s)$.

Similarly, if the normal vector field of the surface $\boldsymbol{M}(s,u)$ and the normal vector field of the ruled surface—taking $\boldsymbol{\gamma}(s)$ as the base curve—are orthogonal to each other along the curve $\boldsymbol{\gamma}(s)$, then this ruled surface is called the normal approximation surface of $\boldsymbol{M}(s,u)$ and can be expressed as follows:

$$\begin{aligned}\boldsymbol{N}_{\rm b}(s,u)&=\boldsymbol{b}(s)+u\left(\frac{\tau_{\rm g}\boldsymbol{e}_1+\kappa_{\rm g}\boldsymbol{u}}{\sqrt{\tau_{\rm g}^2+\kappa_{\rm g}^2}}\right)(s)\\&=\boldsymbol{b}(s)+u\overline{\boldsymbol{D}}_{\rm R},\end{aligned} \tag{25}$$

$$\boldsymbol{b}(s)=\boldsymbol{\gamma}(s)+r\boldsymbol{y}(s),\quad r\in\mathbb{R},\tag{26}$$

where r is the offset distance from the base surface $\boldsymbol{M}(s,u)$ to the parallel surface. When $r=0$, $\boldsymbol{N}_{\rm b}(s,u)$ is a normal approximation surface of $\boldsymbol{M}(s,u)$ along $\boldsymbol{\gamma}(s)$. When $r\neq 0$, $\boldsymbol{N}_{\rm b}(s,u)$ is a normal approximation surface parallel to $\boldsymbol{M}(s,u)$ along $\boldsymbol{b}(s)$.

The pseudo-evolute curve $\overline{\boldsymbol{R}}(s)$ of $\boldsymbol{\gamma}(s)$ on a surface $\boldsymbol{M}(s,u)$ is obtained by

$$\begin{aligned}\overline{\boldsymbol{R}}(s)=&\boldsymbol{\gamma}(s)-\overline{\boldsymbol{D}}_{\rm O}(s)\\&\cdot\left(\frac{\kappa_{\rm g}}{\left(\kappa_{\rm g}+\frac{\kappa_{\rm g}\kappa_{\rm g}'-\kappa_{\rm n}'\kappa_{\rm g}}{\kappa_{\rm n}^2+\kappa_{\rm g}^2}\right)\sqrt{\kappa_{\rm n}^2+\kappa_{\rm g}^2}}\right)(s)\\=&\boldsymbol{\gamma}(s)-\frac{\kappa_{\rm n}(s)\left(\kappa_{\rm g}\boldsymbol{e}_1-\kappa_{\rm n}\boldsymbol{y}\right)(s)}{\left(\kappa_{\rm g}^3+\kappa_{\rm g}\kappa_{\rm n}^2+\kappa_{\rm n}\kappa_{\rm g}'-\kappa_{\rm g}'\kappa_{\rm g}\right)(s)}.\end{aligned}\tag{27}$$

3 Rectifying caustic developable surfaces

In this section, we consider a unit speed space curve $\boldsymbol{\gamma}$: $\boldsymbol{I}\rightarrow\mathbb{E}^3$ ($\mathbb{E}^3$ is a three-dimensional Euclidean space), and analyze in which conditions a rectifying developable surface becomes a caustic surface and in which direction of the light source the reflected surface becomes a rectifying caustic developable surface. Furthermore, it is demonstrated that the surface derived from the tangents of the pseudo-evolute curve of $\boldsymbol{\gamma}(s)$ is a rectifying caustic developable surface, i.e., the base curve of this surface is the pseudo-evolute curve of $\boldsymbol{\gamma}(s)$.

Theorem 1 Let γ: $\boldsymbol{I}\rightarrow\mathbb{E}^3$ be a unit speed space curve, the light source be in the negative direction of the unit binormal vector of the base curve $\boldsymbol{\gamma}(s)$ along the tangent plane of the mirror surface $\boldsymbol{\Phi}(s,u)$ ($\boldsymbol{\Phi}(s,u)=\boldsymbol{\gamma}(s)+u\boldsymbol{\gamma}'(s)$, $u\in\mathbb{R}$), and the reflected vector be the unit binormal vector $\boldsymbol{e}_3(s)$ of $\boldsymbol{\gamma}(s)$ along the tangent plane. In this case, the caustic of the mirror surface (a rectifying developable surfac) is

$$\begin{aligned}\boldsymbol{\varepsilon}_{\rm R}(s,u)&=\boldsymbol{\gamma}(s)+u\left(\tau\boldsymbol{e}_1+\kappa\boldsymbol{e}_3\right)(s)\\&=\boldsymbol{\gamma}(s)+u\boldsymbol{D}(s).\end{aligned}\tag{28}$$

Furthermore, the caustic surface can be presented as

$$\boldsymbol{\varepsilon}_{\rm R}(s,u)=\boldsymbol{R}(s)+u\boldsymbol{R}'(s),\tag{29}$$

where $\boldsymbol{R}(s)$ is the pseudo-evolute curve of $\boldsymbol{\gamma}(s)$.

Proof From Eqs. (1) and (2), we know that the equation of a caustic developable surface is

$$\boldsymbol{\varepsilon}_{\rm R}(s,u)=\boldsymbol{\gamma}(s)+u\boldsymbol{f}(s),\quad u\in\mathbb{R},\tag{30}$$

where

$$\begin{aligned}\boldsymbol{f}(s)=&\langle\boldsymbol{\gamma}'(s)\times\boldsymbol{e}_3(s),\boldsymbol{e}_3'(s)\rangle\boldsymbol{\gamma}'(s)\\&-\langle\boldsymbol{\gamma}'(s)\times\boldsymbol{e}_3(s),\boldsymbol{\gamma}''(s)\rangle\boldsymbol{e}_3)(s)\\=&(\tau\boldsymbol{e}_1+\kappa\boldsymbol{e}_3)(s).\end{aligned}\tag{31}$$

Thus, the caustic equation of a developable surface can take the form

$$\boldsymbol{\varepsilon}_{\rm R}(s,u)=\boldsymbol{\gamma}(s)+u\left(\tau\boldsymbol{e}_1+\kappa\boldsymbol{e}_3\right)(s).$$

The striction curve of the caustic developable surface $\boldsymbol{c}_{\rm R}(s)$ is defined as (Hoffmann et al., 2022)

$$\boldsymbol{c}_{\rm R}(s)=\boldsymbol{\gamma}(s)+\lambda_{\rm R}(s)\boldsymbol{f}(s).\tag{32}$$

Furthermore, according to our previous assumptions, we can calculate $\lambda_{\rm R}(s)$ as (Hoffmann et al., 2022)

$$\begin{aligned}\lambda_{\rm R}(s)&=\frac{-\langle\boldsymbol{e}_1(s),\boldsymbol{e}_1'(s)\times\boldsymbol{e}_3(s)\rangle}{\langle\boldsymbol{f}'(s),\boldsymbol{e}_1'(s)\times\boldsymbol{e}_3(s)+\boldsymbol{e}_1(s)\times\boldsymbol{e}_3'(s)\rangle}\\&=\left(\frac{\kappa}{\kappa'\tau-\kappa\tau'}\right)(s).\end{aligned}\tag{33}$$

Then, from Eqs. (10) and (32), we have

$$\begin{aligned}\boldsymbol{c}_{\rm R}(s)&=\boldsymbol{\gamma}(s)+\left(\frac{\kappa}{\kappa'\tau-\kappa\tau'}\right)(s)\left(\tau\boldsymbol{e}_1+\kappa\boldsymbol{e}_3\right)(s)\\&=\boldsymbol{R}(s).\end{aligned}\tag{34}$$

Thus, we have

$$\begin{aligned}\boldsymbol{\varepsilon}_{\rm R}(s,u)&=\boldsymbol{c}_{\rm R}(s)+u\boldsymbol{c}_{\rm R}'(s)\\&=\boldsymbol{R}(s)+u\boldsymbol{R}'(s),\ u\in\mathbb{R}.\end{aligned}\tag{35}$$

In this section, the curve $\boldsymbol{\gamma}(s)$ is taken to be a space curve. We now modify our question by considering $\boldsymbol{\gamma}(s)$ on an arbitrary surface $\boldsymbol{M}(s,u)$ as follows: in which direction of the light source does the reflected surface indicate an osculating caustic developable surface or a normal caustic developable surface? Additionally, for osculating caustic developable surfaces and normal caustic developable surfaces, even if the direction of the light source is changed, the basis curves of these surfaces remain the pseudo-evolute curve of $\boldsymbol{\gamma}(s)$.

4 Osculating caustic developable surfaces

Theorem 2 Let $\boldsymbol{\gamma}: \boldsymbol{I} \to \boldsymbol{M}$ be a unit speed curve on a surface $\boldsymbol{M}(s,u)$, the light source be in the direction of $-\boldsymbol{y}(s)$ of the base curve $\boldsymbol{\gamma}(s)$ along the tangent plane of the mirror surface $\boldsymbol{\Phi}(s,u)$ ($\boldsymbol{\Phi}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{\gamma}'(s)$, $u \in \mathbb{R}$), and the reflected vector be in the direction of $\boldsymbol{y}(s)$ of $\boldsymbol{\gamma}(s)$ along the tangent plane. In this case, the caustic of the mirror surface is

$$\begin{aligned}\boldsymbol{\varepsilon}_{\mathrm{O}}(s,u) &= \boldsymbol{\gamma}(s) + u(\tau_{\mathrm{g}}\boldsymbol{e}_1 - \kappa_{\mathrm{n}}\boldsymbol{y})(s)\\ &= \boldsymbol{\gamma}(s) + u\boldsymbol{D}_{\mathrm{O}}(s),\end{aligned} \tag{36}$$

which is an osculating developable surface. Furthermore, the caustic surface can be presented as

$$\boldsymbol{\varepsilon}_{\mathrm{O}}(s,u) = \boldsymbol{R}(s) + u\boldsymbol{R}'(s), \tag{37}$$

where $\boldsymbol{R}(s)$ is the pseudo-evolute curve of $\boldsymbol{\gamma}(s)$ on the surface $\boldsymbol{M}(s,u)$.

Proof From Eq. (1), the caustic of $\boldsymbol{\Phi}(s,u)$ is

$$\boldsymbol{\varepsilon}_{\mathrm{O}}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{f}(s). \tag{38}$$

According to our assumptions, it follows from Eq. (2) that the function $\boldsymbol{f}$ here is

$$\begin{aligned}\boldsymbol{f}(s) =& \langle \boldsymbol{e}_1(s) \times \boldsymbol{y}(s), \boldsymbol{y}'(s)\rangle \boldsymbol{e}_1(s)\\ &- \langle \boldsymbol{e}_1(s) \times \boldsymbol{y}(s), \boldsymbol{e}_1'(s)\rangle \boldsymbol{y}(s)\\ =& (\tau_{\mathrm{g}}\boldsymbol{e}_1 - \kappa_{\mathrm{n}}\boldsymbol{y})(s).\end{aligned} \tag{39}$$

Then we have

$$\boldsymbol{\varepsilon}_{\mathrm{O}}(s,u) = \boldsymbol{\gamma}(s) + u(\tau_{\mathrm{g}}\boldsymbol{e}_1 - \kappa_{\mathrm{n}}\boldsymbol{y})(s). \tag{40}$$

From Eq. (23), $\boldsymbol{\varepsilon}(s,u)$ can be identified as the flat approximation surface of $\boldsymbol{M}(s,u)$. The striction curve of the caustic surface is

$$\begin{aligned}\boldsymbol{c}_{\mathrm{O}}(s) &= \boldsymbol{\gamma}(s) + \lambda_{\mathrm{O}}(s)\boldsymbol{f}(s)\\ &= \boldsymbol{\gamma}(s) + \lambda_{\mathrm{O}}(s)(\tau_{\mathrm{g}}\boldsymbol{e}_1 - \kappa_{\mathrm{n}}\boldsymbol{y})(s).\end{aligned} \tag{41}$$

Then we have (Hoffmann et al., 2022)

$$\begin{aligned}\lambda_{\mathrm{O}}(s) &= \frac{-\langle \boldsymbol{e}_1(s), \boldsymbol{e}_1'(s) \times \boldsymbol{y}(s)\rangle}{\langle \boldsymbol{f}'(s), \boldsymbol{e}_1'(s) \times \boldsymbol{y}(s) + \boldsymbol{e}_1(s) \times \boldsymbol{y}'(s)\rangle}\\ &= \left(\frac{\kappa_{\mathrm{n}}}{-\kappa_{\mathrm{g}}\left(\kappa_{\mathrm{n}}^2 + \tau_{\mathrm{g}}^2\right) - \kappa_{\mathrm{n}}\tau_{\mathrm{g}}' + \kappa_{\mathrm{n}}'\tau_{\mathrm{g}}}\right)(s).\end{aligned} \tag{42}$$

Therefore, the equation of the striction curve of the caustic surface takes the form

$$\begin{aligned}\boldsymbol{c}_{\mathrm{O}}(s) =& \boldsymbol{\gamma}(s) - \left(\frac{\kappa_{\mathrm{n}}}{-\kappa_{\mathrm{g}}\left(\kappa_{\mathrm{n}}^2 + \tau_{\mathrm{g}}^2\right) - \kappa_{\mathrm{n}}\tau_{\mathrm{g}}' + \kappa_{\mathrm{n}}'\tau_{\mathrm{g}}}\right)(s)\\ &\cdot (\tau_{\mathrm{g}}\boldsymbol{e}_1 - \kappa_{\mathrm{n}}\boldsymbol{y})(s)\\ =& \boldsymbol{\gamma}(s) - \overline{\boldsymbol{D}}_{\mathrm{O}}(s)\\ &\cdot \left(\frac{\kappa_{\mathrm{n}}}{\left(\kappa_{\mathrm{g}} + \frac{\kappa_{\mathrm{n}}\kappa_{\mathrm{g}}' - \kappa_{\mathrm{n}}'\kappa_{\mathrm{g}}}{\kappa_{\mathrm{n}}^2 + \kappa_{\mathrm{g}}^2}\right)\sqrt{\kappa_{\mathrm{n}}^2 + \kappa_{\mathrm{g}}^2}}\right)(s),\end{aligned} \tag{43}$$

which is the pseudo-evolute curve of the developable surface from Eq. (27).

The caustic surface formed by reflected rays in the direction of $\boldsymbol{y}(s)$ on the mirror surface $\boldsymbol{\Phi}(s,u)$ is

$$\boldsymbol{\varepsilon}_{\mathrm{O}}(s,u) = \overline{\boldsymbol{R}}(s) + u\overline{\boldsymbol{R}}'(s). \tag{44}$$

For instance, in the event where $\boldsymbol{M}(s,u)$ is chosen as

$$\begin{aligned}\boldsymbol{M}(s,u) =& \left\{\frac{2}{3}\sqrt{1 + \frac{3u^2}{4}}\cos s,\right.\\ &\left.\frac{2}{3}\sqrt{1 + \frac{3u^2}{4}}\sin s, u\right\},\end{aligned} \tag{45}$$

The osculating developable surface $\boldsymbol{\varepsilon}_{\mathrm{O}}(s,u)$ of the surface $\boldsymbol{M}(s,u)$ (represents as a circular hyperboloid of one sheet) and the mirror surface $\boldsymbol{\Phi}(s,u)$ and the base curve $\boldsymbol{\gamma}(s)$ mentioned in Theorem 2, are illustrated in Figs. 1–3 and Fig. 4.

To provide a clearer visualization, Figs. 3–5 are zoomed in, and the value ranges are restricted to the region where the curve lies on the surface.

Corollary 1 If we set $\kappa_{\mathrm{g}}(s) = 0$ in Eq. (43), we obtain the striction curve of the curve $\boldsymbol{\gamma}(s)$, which is a space curve. In this particular instance, it can be observed that the Frenet frame and the Darboux frame coincide.

Proposition 1 Let $\boldsymbol{\gamma}: \boldsymbol{I} \to \boldsymbol{M}$ be a unit speed curve on a surface $\boldsymbol{M}(s,u)$. Then the following statements are satisfied for curve $\boldsymbol{\gamma}(s)$:

1. The developable surface $\boldsymbol{\Phi}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{\gamma}'(s)$ is the mirror surface.
2. $\boldsymbol{\varepsilon}_{\mathrm{O}}(s,u) = \boldsymbol{\gamma}(s) + u\left(\tau_{\mathrm{g}}\boldsymbol{e}_1 - \kappa_{\mathrm{n}}\boldsymbol{y}\right)(s)$ is the flat approximation surface of $\boldsymbol{M}(s,u)$ along $\boldsymbol{\gamma}(s)$.
3. $\boldsymbol{\varepsilon}_{\mathrm{O}}(s,u)$ is the osculating caustic surface of the mirror surface $\boldsymbol{\Phi}(s,u)$.
4. $\boldsymbol{c}_{\mathrm{O}}(s)$ is the pseudo-evolute curve of the curve $\boldsymbol{\gamma}$ on the surface $\boldsymbol{M}(s,u)$.

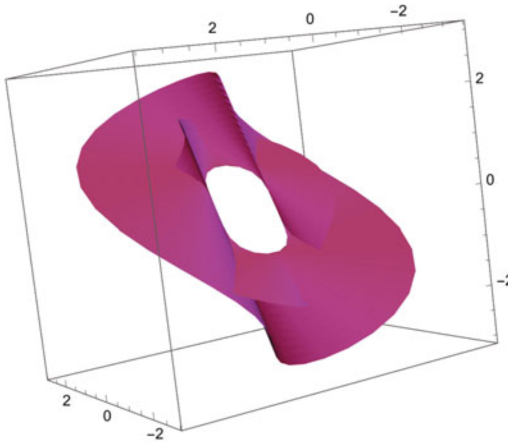

Fig. 1 Mirror surface $\boldsymbol{\Phi}(s, u)$

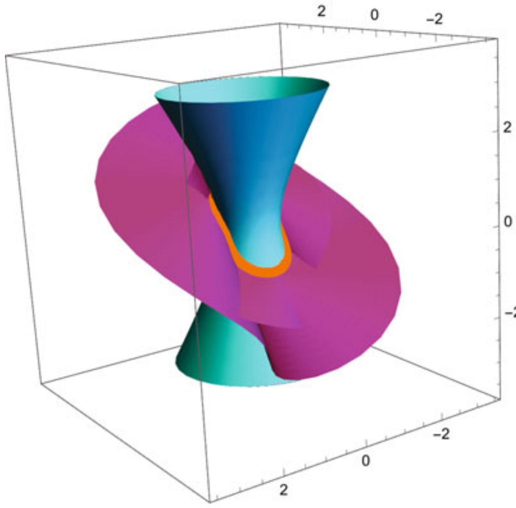

Fig. 2 Base curve $\gamma(s)$ (orange), mirror surface $\boldsymbol{\Phi}(s, u)$ (purple), and surface $\boldsymbol{M}(s, u)$ (cyan). References to color refer to the online version of this figure

5 Normal caustic developable surfaces

Theorem 3 Let $\boldsymbol{\gamma}: \boldsymbol{I} \to \boldsymbol{M}$ be a unit speed curve on a surface $\boldsymbol{M}(s, u)$, the reflected vector be in the direction $\boldsymbol{u}(s)$ of the base curve $\boldsymbol{\gamma}(\mathrm{s})$ along the tangent plane of the mirror surface $\boldsymbol{\Phi}(s, u)$ ($\boldsymbol{\Phi}(s, u) = \boldsymbol{\gamma}(s) + u\boldsymbol{\gamma}'(s)$, $u \in \mathbb{R}$), and the reflected vector be in the direction $\boldsymbol{u}(s)$ of $\boldsymbol{\gamma}(s)$ along the tangent plane. In this case, the caustic surface of the mirror is obtained by

$$\begin{aligned} \varepsilon_{\mathrm{N}}(s, u) &= \boldsymbol{\gamma}(s) + u(\tau_{\mathrm{g}}\boldsymbol{e}_1 + \kappa_{\mathrm{g}}\boldsymbol{u})(s) \\ &= \boldsymbol{\gamma}(s) + u\boldsymbol{DR}(s), \end{aligned} \tag{46}$$

which is a normal developable surface.

Proposition 2 Let $\boldsymbol{\gamma}: \boldsymbol{I} \to \boldsymbol{M}$ be a unit speed curve on a surface $\boldsymbol{M}(s, u)$ and the reflected vector be in the direction $\boldsymbol{u}(s)$ of the base curve $\boldsymbol{\gamma}(s)$ along

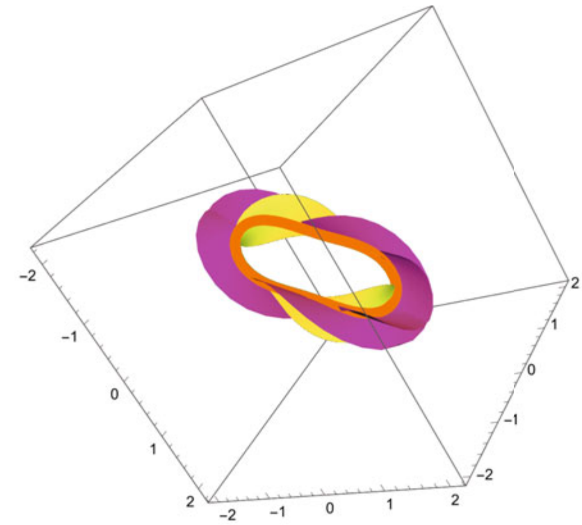

Fig. 3 Osculating caustic developable $\varepsilon_{\mathrm{O}}(s, u)$ (yellow), mirror surface $\boldsymbol{\Phi}(s, u)$ (purple), and base curve $\gamma(s)$ (orange). References to color refer to the online version of this figure

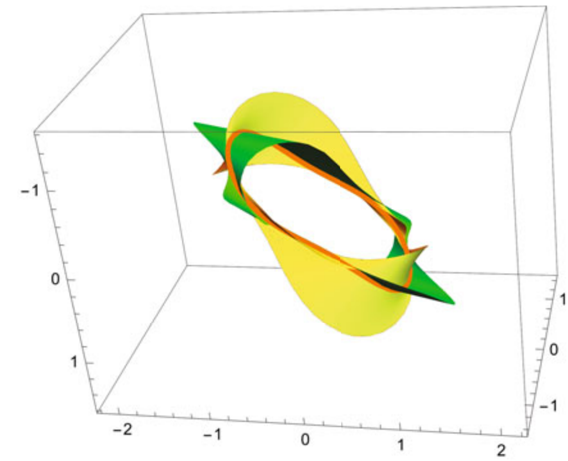

Fig. 4 Osculating caustic developable surface $\varepsilon_{\mathrm{O}}(s, u)$ (yellow), normal caustic developable surface $\varepsilon_{\mathrm{N}}(s, u)$ (green), and curve $\gamma(s)$ (orange). References to color refer to the online version of this figure

the tangent plane of the developable surface $\boldsymbol{\Phi}(s, u)$. Then, the caustic surface $\varepsilon_{\mathrm{N}}(s, u)$ is the normal approximation surface of $\boldsymbol{M}(s, u)$.

The surface $\boldsymbol{M}(s, u)$ in Eq. (45) and the normal caustic developable surface $\varepsilon_{\mathrm{N}}(s, u)$ of the surface $\boldsymbol{M}(s, u)$ mentioned in Theorem 3 and Proposition 2 are illustrated in Figs. 4 and 5.

Theorem 4 Let $\boldsymbol{\gamma}: \boldsymbol{I} \to \boldsymbol{M}$ be a unit speed curve on a surface $\boldsymbol{M}(s, u)$ and the reflected vector be in the direction $\boldsymbol{u}(s)$ of the base curve $\gamma(s)$ along the tangent plane of the developable surface $\boldsymbol{\Phi}(s, u)$. The

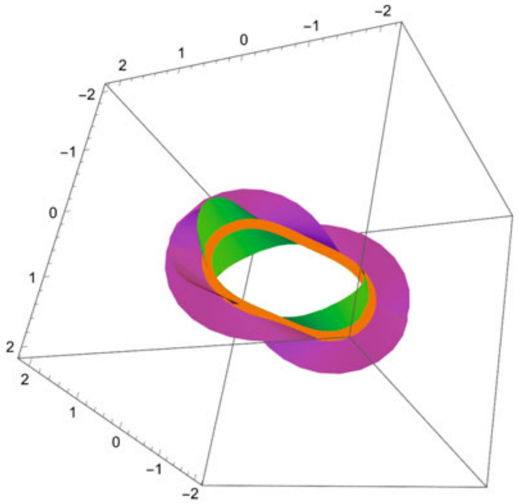

Fig. 5 Normal caustic developable $\varepsilon_{\mathrm{O}}(s,u)$ (green), mirror surface $\Phi(s,u)$ (purple), and base curve $\gamma(s)$ (orange). References to color refer to the online version of this figure

striction curve of the caustic surface $\varepsilon_{\mathrm{N}}(s,u)$ is

$$\begin{aligned}\overline{\boldsymbol{R}}(s) =&\boldsymbol{\gamma}(s)\\ &+\left(\frac{-\kappa_g}{\kappa_{\mathrm{g}}\left(\tau_{\mathrm{g}}'-\kappa_{\mathrm{n}}\kappa_{\mathrm{g}}\right)-\tau_{\mathrm{g}}\left(\kappa_{\mathrm{g}}'+\tau_{\mathrm{g}}\kappa_{\mathrm{n}}\right)}\right)(s)\\ &\cdot\left(\tau_{\mathrm{g}}\boldsymbol{e}_1+\kappa_{\mathrm{g}}\boldsymbol{u}\right)(s). \end{aligned} \tag{47}$$

Proof The striction curve of $\varepsilon_{\mathrm{N}}(s,u)$ is

$$\boldsymbol{c}_{\mathrm{N}}(s)=\boldsymbol{\gamma}(s)+\lambda_{\mathrm{N}}(s)\boldsymbol{f}(s). \tag{48}$$

From Eqs. (2) and (33), we have

$$\lambda_{\mathrm{N}}(s)=\frac{-\kappa_{\mathrm{g}}}{\kappa_{\mathrm{g}}(\tau_{\mathrm{g}}'-\kappa_{\mathrm{n}}\kappa_{\mathrm{g}})-\tau_{\mathrm{g}}(\kappa_{\mathrm{g}}'+\tau_{\mathrm{g}}\kappa_{\mathrm{n}})}(s), \tag{49}$$

$$\boldsymbol{f}(s)=(\tau_{\mathrm{g}}\boldsymbol{e}_1+\kappa_{\mathrm{g}}\boldsymbol{u})(s). \tag{50}$$

Corollary 2 If we take $\kappa_{\mathrm{n}}(s)=0$ in Eq. (47), we have the striction curve of the curve $\boldsymbol{\gamma}(s)$, which is a space curve. In this particular instance, it can be observed that the Frenet frame and the Darboux frame coincide.

Proposition 3 Let $\boldsymbol{\gamma}\colon \boldsymbol{I}\to\boldsymbol{M}$ be a unit speed curve on a surface $\boldsymbol{M}(s,u)$ and the reflected vector be in the direction $\boldsymbol{u}(s)$ of the base curve $\boldsymbol{\gamma}(s)$ along the tangent plane of the developable surface $\boldsymbol{\Phi}(s,u)$. The striction curve of the caustic surface $\boldsymbol{c}_{\mathrm{N}}(s)$ is simultaneously considered as the pseudo-evolute curve of $\varepsilon_{\mathrm{N}}(s,u)$, denoted by $\overline{\boldsymbol{R}}(s)$.

Example 1 Let $\boldsymbol{\gamma}\colon \boldsymbol{I}\to\boldsymbol{M}$ be a space curve as

$$\gamma(s)=\big(s\cos s, s\sin s, s\big). \tag{51}$$

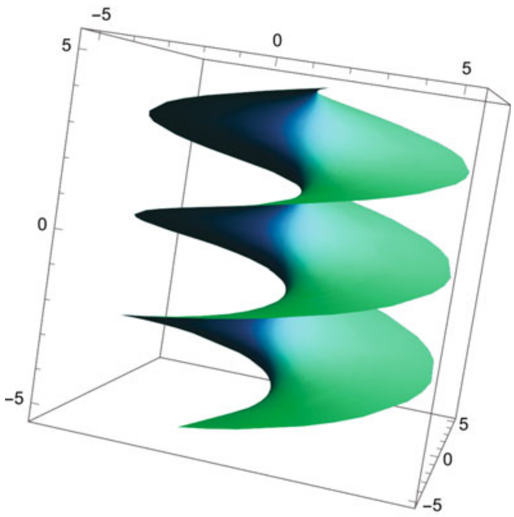

Fig. 6 Surface $M(s,u)$, which is given with Eq. (52)

The curve $\boldsymbol{\gamma}(s)$ is located on the surface $\boldsymbol{M}(s,u)$ (Fig. 6), which is a helicoid:

$$\boldsymbol{M}(s,u)=\big\{s\cos u, s\sin u, u\big\}. \tag{52}$$

As can be observed in Fig. 7, the curve $\boldsymbol{\gamma}(s)$ lies above the mirror surface and the surface $\boldsymbol{M}(s,u)$. The Darboux frame of the surface $\boldsymbol{M}(s,u)$ along $\boldsymbol{\gamma}(s)$ is

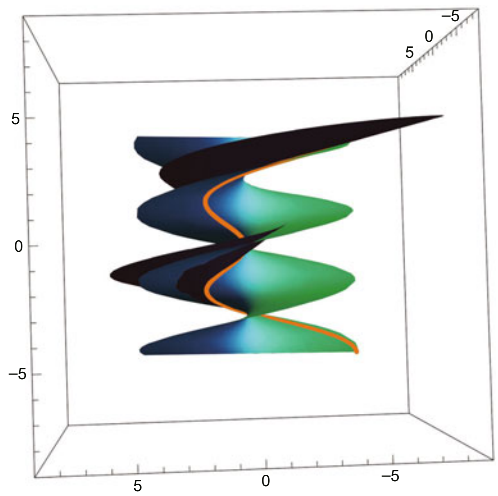

Fig. 7 Base curve $\gamma(s)$ (Eq. (51), orange) on the mirror surface $\Phi(s,u)$ (purple) and the surface $M(s,u)$ (Eq. (52), cyan). References to color refer to the online version of this figure

$$\boldsymbol{e}_1(s)=\frac{1}{\sqrt{1+s^2}}(\cos s - s\sin s, s\cos s + \sin s, 1), \tag{53}$$

$$\boldsymbol{y}(s)=\left(-\cos s-\frac{s\sin s}{1+s^2}, -\sin s+\frac{s\cos s}{1+s^2}, \frac{1}{1+s^2}\right), \tag{54}$$

$$\boldsymbol{u}(s)=\frac{1}{\sqrt{1+s^2}}(\sin s, -\cos s, s), \tag{55}$$

where

$$\kappa_{\mathrm{n}}(s)=-\frac{2}{(1+s^2)^{\frac{3}{2}}}, \tag{56}$$

$$\kappa_{\mathrm{g}}(s)=\frac{s(3+s^2)}{(1+s^2)^2}, \tag{57}$$

$$\tau_{\mathrm{g}}(s)=\frac{s}{(1+s^2)^2}. \tag{58}$$

The osculating caustic developable surface of $\boldsymbol{M}(s,u)$ along $\boldsymbol{\gamma}(s)$ is

$$\boldsymbol{\varepsilon}_{\mathrm{O}}(s,u)=\frac{1}{(1+s^2)^{\frac{5}{2}}}\{s\cos s + u((-2+s^2)(1+s^2)\cos s - s(2+s^2+s^4)\sin s), s\sin s + u(s(2+s^2+s^4)\cos s + (-2+s^2)(1+s^2)\sin s), s+u(2+s^2+s^4)\}. \tag{59}$$

The normal caustic developable surface of $\boldsymbol{M}(s,u)$ along $\boldsymbol{\gamma}(s)$ is

$$\boldsymbol{\varepsilon}_{\mathrm{N}}(s,u)=\left\{s\left(\cos s+\frac{u(s+s^3)\cos s-(-3+s^4)\sin s}{(1+s^2)^{\frac{5}{2}}}\right), s\left(\sin s+\frac{u(-3+s^4)\cos s+s(1+s^2)\sin s}{(1+s^2)^{\frac{5}{2}}}\right), s+\frac{2s^2(2+s^2)u}{(1+s^2)^{\frac{5}{2}}}\right\}. \tag{60}$$

In Figs. 8 and 9, the pseudo-evolute curve $\boldsymbol{c}_{\mathrm{O}}(s)$ is observed as the striction curve of the osculating caustic developable surface $\boldsymbol{\varepsilon}_{\mathrm{O}}(s,u)$.

Similarly, as can be observed in Figs. 10 and 11, the pseudo-evolute curve $\boldsymbol{c}_{\mathrm{N}}(s)$ coincides with the striction curve of the normal caustic developable surface $\boldsymbol{\varepsilon}_{\mathrm{N}}(s,u)$.

6 Conclusions

In this study, we investigate which base curves can be employed to generate these mirror surfaces and in which directions the light sources should be positioned so that the resulting surfaces are characterized as rectifying, osculating, and normal caustic developable surfaces. The methods used to obtain these surfaces are fully detailed in this paper. In our forthcoming research, we intend to extend the methods described here to the Lorentz–Minkowski space.

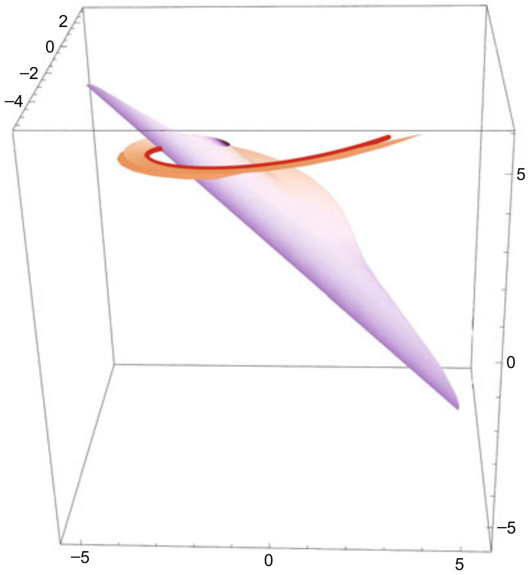

Fig. 8 Pseudo-evolute curve $c_{\mathrm{O}}(s)$ (red) of the osculating caustic developable surface $\varepsilon_{\mathrm{O}}(s,u)$ (Eq. (59)). References to color refer to the online version of this figure

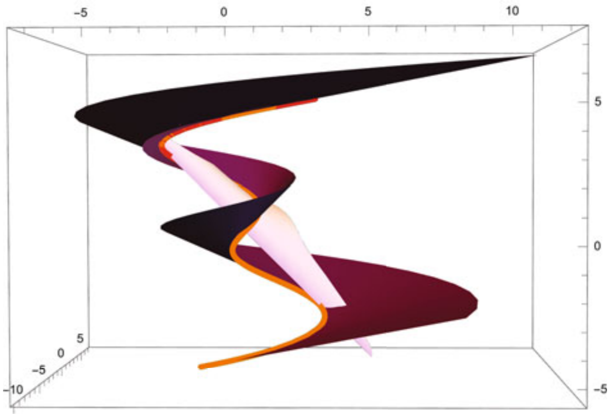

Fig. 9 Base curve $\gamma(s)$ (Eq. (51), orange), pseudo-evolute curve $c_{\mathrm{O}}(s)$ (red), osculating caustic developable surface $\varepsilon_{\mathrm{O}}(s,u)$ (Eq. (59)), and mirror surface $\Phi(s,u)$ (purple). References to color refer to the online version of this figure

Contributors

Hande Nur DALKILIÇ and Yusuf YAYLI designed the research and drafted the paper. Hande Nur DALKILIÇ

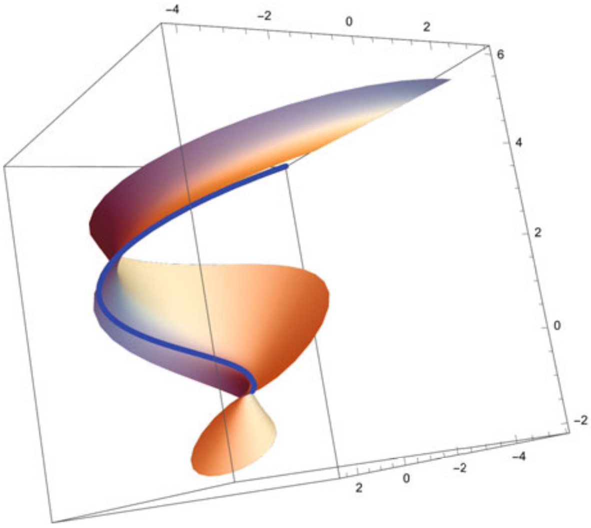

Fig. 10 Pseudo-evolute curve $c_{\mathrm{N}}(s)$ (blue) of the normal caustic developable surface $\varepsilon_{\mathrm{N}}(s,u)$ (Eq. (60)). References to color refer to the online version of this figure

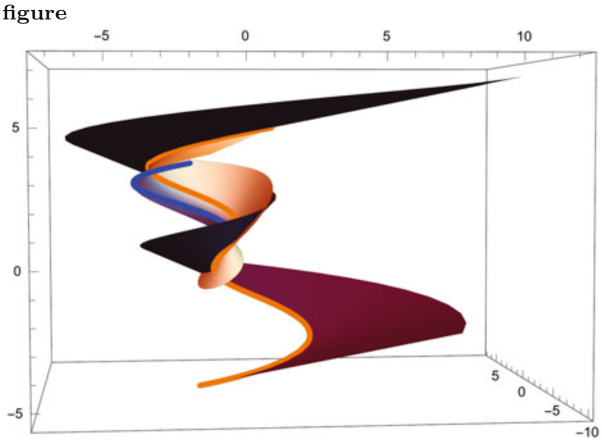

Fig. 11 Base curve $\gamma(s)$ (Eq. (51), orange), pseudo-evolute curve $c_{\mathrm{N}}(s)$ (blue), normal caustic developable surface $\varepsilon_{\mathrm{N}}(s,u)$ (Eq. (60)), and mirror surface $\Phi(s,u)$ (purple). References to color refer to the online version of this figure

revised and finalized the paper.

Conflict of interest

Both authors declare that they have no conflict of interest.

Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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