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Reliability measure approach considering mixture uncertainties under insufficient input data

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Abstract: Reliability analysis and reliability-based optimization design require accurate measurement of failure probability under input uncertainties. A unified probabilistic reliability measure approach is proposed to calculate the probability of failure and sensitivity indices considering a mixture of uncertainties under insufficient input data. The input uncertainty variables are classified into statistical variables, sparse variables, and interval variables. The conservativeness level of the failure probability is calculated through uncertainty propagation analysis of distribution parameters of sparse variables and auxiliary parameters of interval variables. The design sensitivity of the conservativeness level of the failure probability at design points is derived using a semi-analysis and sampling-based method. The proposed unified reliability measure method is extended to consider *p*-box variables, multi-domain variables, and evidence theory variables. Numerical and engineering examples demonstrate the effectiveness of the proposed method, which can obtain an accurate confidence level of reliability index and sensitivity indices with lower function evaluation number.

Key words: Insufficient data; Reliability index; Sensitivity analysis; Sparse variable; Uncertainty propagation

1 Introduction

Uncertainties are ubiquitous in engineering products due to manufacturing error (Liu et al., 2022), lack of information, intrinsic random properties, etc. These uncertainties are quantified and propagated to uncertainties of product performance, which may lead to unexpected failure or performance fluctuation. Reliability analysis and reliability-based design optimization (RBDO) methodologies have been developed to obtain a reliable optimum design considering input uncertainties and have been applied in many engineering fields (Tostado-Véliz et al., 2021, 2022; Solazzi, 2022; Wakjira et al., 2022).

In traditional RBDO methodologies, the uncertainty variables are assumed to be determinate probabilistic variables (Sankararaman and Mahadevan, 2015). However, in many actual engineering applications, it is difficult to acquire the complete uncertainty information for calculating the accurate probability density functions of uncertainty variables under insufficient input data (Wang et al., 2016). According to the available amount of input sampling data, the uncertainty variables can be classified into statistical variables with sufficient input data (Type I), sparse variables with insufficient input data (Type II), and interval variables with little input data (Type III) (Oberkampf et al., 2004). With an increase of input sampling data, the interval variables can be converted to sparse variables or even to statistical variables; the sparse variables can be also converted to statistical variables when there are enough input sampling data.

The statistical variables (Type I) can be represented using determinate distribution type and accurate distribution parameters, such as normal distribution, gamma distribution, *F* distribution, and Weibull distribution (Chen et al., 2003). A series of probabilistic uncertainty representation, propagation, and

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optimization design methodologies have been proposed to deal with statistical variables (Gan et al., 2018; El Haj and Soubra, 2021). The interval variables (Type III) can be represented using non-probabilistic methodologies, such as the convex model, evidence theory, fuzzy number, and p-box (Ni et al., 2018). Many hybrid uncertainty analysis methodologies have also been proposed to deal with statistical variables and interval variables simultaneously (Hong et al., 2021).

The distribution parameters of sparse variables (Type II) cannot be fitted accurately due to the insufficiency of input sampling data, and the probabilistic uncertainty analysis methodologies for statistical variables cannot be used directly for the representation of sparse variables. If the sparse variable is represented using non-probabilistic methodologies for interval variables, much uncertainty information in the insufficient input data is missing. Therefore, how to represent the uncertainties of sparse variables accurately is one issue in reliability-based design optimization.

Initially, the sparse variables (Type II) are quantified using possibility-based approaches (Lee et al., 2013). For further parameterization of the sparse variables, likelihood-based approaches and Bayesian approaches are proposed to quantify their distribution types and distribution parameters. Uncertainty of distribution types can be estimated using many methodologies, such as the model identification method, Johnson distribution, and Kernel density estimation (Peng et al., 2017). Although the reliability index under sparse variables can be calculated, the algorithms are too computationally demanding due to the nesting estimation of uncertainty distribution types, distribution parameters, and uncertainty variables.

The second issue is how to accurately quantify the reliability index considering the three types of uncertainty variables simultaneously. The uncertainty propagation methodologies for statistical variables (Type I) have been widely studied, such as probability density evaluation (McFarland and DeCarlo, 2020), surrogation model (Yun et al., 2020), and importance sampling method (Liu and Elishakoff, 2020). Many uncertainty quantification methodologies of reliability index considering interval variables (Type III) have also been proposed, such as interval arithmetic techniques, global optimization approach, and perturbation methods, and are summarized by Faes and Moens (2020). The uncertainty propagation analysis for sparse variables (Type II) is a multiple-loop process, the distribution types and distribution parameters are estimated in the outer loops, and the reliability index is estimated in the inner loops using similar methods to those for statistical variables (Type I). To reduce computational complexity and increase the accuracy of the reliability index, many non-probabilistic reliability analysis methodologies for hybrid uncertainties have been proposed (Zhao et al., 2018; Wei et al., 2019). Although there are many reliability measure approaches for mixture uncertainties, there are multiple loops for the uncertainty quantification and propagation analysis of sparse variables, and the non-probabilistic reliability index is difficult to integrate with many probabilistic RBDO algorithms. Therefore, a probabilistic reliability measure approach is proposed and the reliability index and sensitivity indices are calculated considering the three types of uncertainties simultaneously.

The rest of this paper is organized as follows. The reliability measure and sensitivity analysis problem considering mixture uncertainties is described in Section 2. In Section 3, a unified calculation algorithm of reliability index is proposed with less sampling loops and less sampling points. The sensitivity indices are calculated through a semi-analytical method based on auxiliary variables in Section 4. The proposed algorithm is extended for considering p-box variables, multi-domain distribution variables, and evidence theory variables in Section 5. Three numerical and two engineering examples are demonstrated to verify the effectiveness of proposed methodology in Section 6. Conclusions are summarized in Section 7.

2 Failure probability under insufficient input data

According to the available amount of input uncertainty data in RBDO, the input random variables can be divided into statistical variables $\boldsymbol{X}$, sparse variables $\boldsymbol{Y}$, and interval variables $\boldsymbol{Z}$ (Fig. 1). These random input uncertainty variables are assumed to be independent, and can each be decomposed into two parts as in Eqs. (1)–(3).

$$\boldsymbol{X}=\bar{\boldsymbol{x}}+\tilde{\boldsymbol{x}}, \tag{1}$$

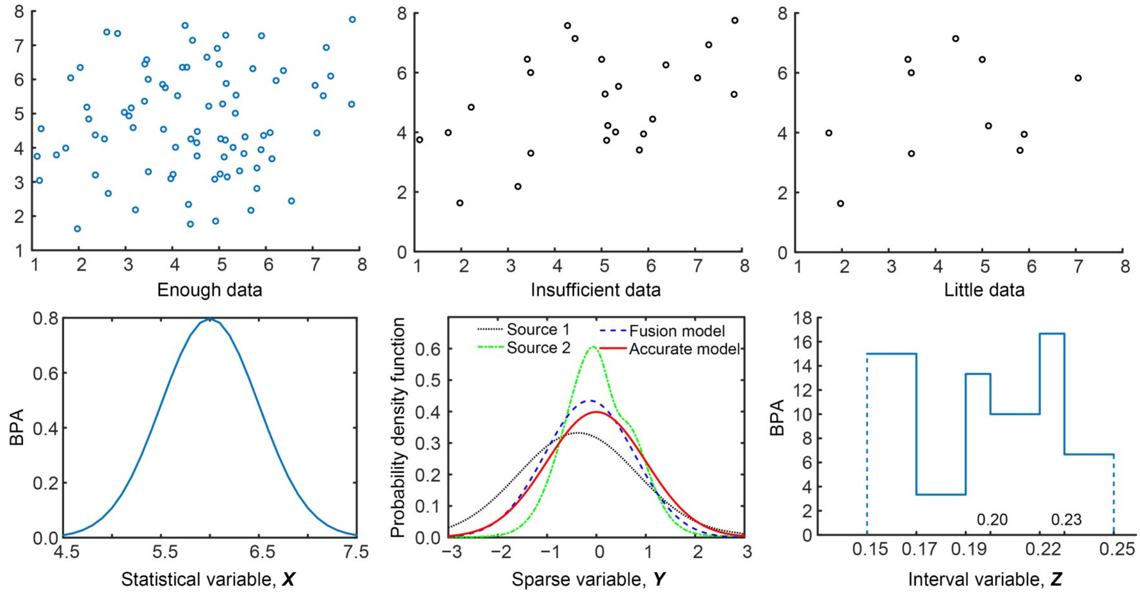

Fig. 1 Multiple uncertainty types due to insufficient input data. BPA represents basic probability assignment

$$\boldsymbol{Y}=\bar{\boldsymbol{y}}+\tilde{\boldsymbol{y}}, \tag{2}$$

$$\boldsymbol{Z}=\bar{\boldsymbol{z}}+\tilde{\boldsymbol{z}}, \tag{3}$$

where $\bar{\boldsymbol{x}}$, $\bar{\boldsymbol{y}}$, and $\bar{\boldsymbol{z}}$ are the mean values of $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$, respectively, which are also design variables and changeable in the RBDO process. $\tilde{\boldsymbol{x}}$, $\tilde{\boldsymbol{y}}$, and $\tilde{\boldsymbol{z}}$ are the dispersion parts of input uncertainty variables according to their available input data, which are maintained in the RBDO process. The failure probability p_G is defined using multi-dimensional integration in Eq. (4).

$$p_G=\int_{\mathbb{R}^N} I_{\Omega_G}(\boldsymbol{X},\ \boldsymbol{Y},\ \boldsymbol{Z})\, f_{\boldsymbol{X}}(\boldsymbol{X})\, f_{\boldsymbol{Y}}(\boldsymbol{Y})\, f_{\boldsymbol{Z}}(\boldsymbol{Z})\, \mathrm{d}\boldsymbol{X}\mathrm{d}\boldsymbol{Y}\mathrm{d}\boldsymbol{Z}, \tag{4}$$

where Ω_G is the failure domain such that the performance function $G(\boldsymbol{X},\ \boldsymbol{Y},\ \boldsymbol{Z})$ is larger than 0. $f_{\boldsymbol{X}}(\boldsymbol{X})$, n_x, $f_{\boldsymbol{Y}}(\boldsymbol{Y})$, n_y, and $f_{\boldsymbol{Z}}(\boldsymbol{Z})$, n_z are the probability density functions (PDFs) and total numbers of statistical variables $\boldsymbol{X}$, sparse variables $\boldsymbol{Y}$, and interval variables $\boldsymbol{Z}$, respectively. $N=n_x+n_y+n_z$ is the total number of uncertainty variables. The indicator function $I_{\Omega_G}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z})$ is defined in Eq. (5).

$$I_{\Omega_G}(\boldsymbol{X},\ \boldsymbol{Y},\ \boldsymbol{Z})=\begin{cases}1, & G(\boldsymbol{X},\ \boldsymbol{Y},\ \boldsymbol{Z})>0,\\ 0, & G(\boldsymbol{X},\ \boldsymbol{Y},\ \boldsymbol{Z})\leqslant 0.\end{cases} \tag{5}$$

The statistical variables $\boldsymbol{X}$ have complete uncertainty information, which can be represented with single distribution type ζ and determinate distribution parameters $\boldsymbol{\theta}$. The PDF $f_{\boldsymbol{X}}(\boldsymbol{X})$ can be determined directly according to the uncertainty representation function of $\boldsymbol{X}$. If there are only statistical variables $\boldsymbol{X}$, the failure probability p_G will be a fixed value.

The ith sparse variable Y_i is the summation of design points $\bar{y}_i$ and its dispersion part $\tilde{y}_i$ in Eq. (2). Based on the α-dimensional available sparse sampling points $[a_{1,\tilde{y}_i},\ a_{2,\tilde{y}_i},\ \cdots,\ a_{\alpha,\tilde{y}_i}]$ for $\tilde{y}_i$, Y_i is represented using weight summation of multiple distribution types ζ with uncertain distribution parameters $\boldsymbol{\theta}$ in Eq. (6). The detailed representation method is shown in the electronic supplementary materials (ESM).

$$Y_i\sim \sum w_{k,i}\zeta_{k,i}\left(\theta_{k,i}^1,\ \theta_{k,i}^2\right), \tag{6}$$

where $w_{k,i}$ is the weight ratio for the kth distribution type, $\zeta_{k,i}$ is the kth distribution type, and $\theta_{k,i}^1$ and $\theta_{k,i}^2$ are the distribution parameters for the kth distribution type.

In the following, we assume that the value of $\boldsymbol{\theta}$ for every distribution type ζ is 2 (Sankararaman and Mahadevan, 2013; Kang et al., 2016). Due to the uncertainties of distribution parameters $\boldsymbol{\theta}$, the PDF $f_{\boldsymbol{Y}}(\boldsymbol{Y})$ of $\boldsymbol{Y}$ is a distribution family of those uncertain distribution parameters $\boldsymbol{\theta}$. Therefore, the failure probability p_G is also an uncertainty variable.

The uncertainty information of interval variables $\boldsymbol{Z}$ is missing, only lower bound $\underline{\boldsymbol{Z}}$ and upper bound $\bar{\boldsymbol{Z}}$

are available in Eq. (7). Therefore, the PDF $f_Z(\boldsymbol{Z})$ of $\boldsymbol{Z}$ cannot be determined directly.

$$\boldsymbol{Z} \in \left[\underline{\boldsymbol{Z}}, \bar{\boldsymbol{Z}}\right]. \tag{7}$$

Because the interval variables $\boldsymbol{Z}$ do not have determinate PDFs, the auxiliary variables $\boldsymbol{\psi}$ are introduced, and $\boldsymbol{Z}$ are transformed to different PDFs according to the values of $\boldsymbol{\psi}$. Eq. (4) is transformed into Eq. (8) relying on the insufficient input data $\boldsymbol{a}$ of sparse variables $\boldsymbol{Y}$ and auxiliary variables $\boldsymbol{\psi}$ of interval variables $\boldsymbol{Z}$.

$$p_G = \int_{\mathbb{R}^N} I_{\Omega_G}(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z})\, f_X(\boldsymbol{X})\, f_Y(\boldsymbol{Y}|\boldsymbol{a})\, f_Z\left(\boldsymbol{Z}\middle|\boldsymbol{\psi}\right) \mathrm{d}\boldsymbol{X}\mathrm{d}\boldsymbol{Y}\mathrm{d}\boldsymbol{Z}, \tag{8}$$

where $f_Y(\boldsymbol{Y}|\boldsymbol{a})$ is the PDF of sparse variables $\boldsymbol{Y}$ under available insufficient input data $\boldsymbol{a}$, which is also an uncertainty variable due to the lack of information of $\tilde{\boldsymbol{y}}$; $f_Z\left(\boldsymbol{Z}\middle|\boldsymbol{\psi}\right)$ is the PDF of $\boldsymbol{Z}$ under auxiliary variables $\boldsymbol{\psi}$.

3 Unified calculation of probability of failure probability

3.1 Reliability measure based on auxiliary variable method

A two-level sampling-based reliability measure method based on auxiliary variables is proposed to calculate the PDF f_{p_G} considering the three types of uncertainty variables simultaneously, where the auxiliary distribution parameters $\boldsymbol{\theta}$ for sparse variables $\boldsymbol{Y}$ and auxiliary variables $\boldsymbol{\psi}$ for interval variables $\boldsymbol{Z}$ are employed.

The failure probability p_G is a determinate value when the distribution parameters $\boldsymbol{\theta}$ for sparse variables $\boldsymbol{Y}$ and auxiliary variables $\boldsymbol{\psi}$ for interval variables $\boldsymbol{Z}$ are determinate values, as shown in Eq. (9).

$$p_G\middle|\boldsymbol{\theta}, \boldsymbol{\psi}, \boldsymbol{a} = \int_{\mathbb{R}^N} I_{\Omega_G}(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z})\, f_X(\boldsymbol{X})\, f_Y\left(\boldsymbol{Y}\middle|\boldsymbol{\theta}, \boldsymbol{a}\right) f_Z\left(\boldsymbol{Z}\middle|\boldsymbol{\psi}\right) \mathrm{d}\boldsymbol{X}\mathrm{d}\boldsymbol{Y}\mathrm{d}\boldsymbol{Z}. \tag{9}$$

Due to the uncertainties of $\boldsymbol{\theta}$ and $\boldsymbol{\psi}$, the PDF f_{p_G} of failure probability p_G is obtained using Eq. (10). In Eq. (10), the joint PDF of p_G is a product of three conditional PDFs. The $f_{\boldsymbol{\theta}}(\boldsymbol{\theta}|\boldsymbol{a})$ is the PDF of distribution parameters $\boldsymbol{\theta}$ for sparse variables $\boldsymbol{Y}$, and the PDF $f_{\psi}(\boldsymbol{\psi})$ of auxiliary variables $\boldsymbol{\psi}$ is the random variable to represent the interval variables $\boldsymbol{Z}$.

Furthermore, the cumulative density function (CDF) of p_G is calculated by integrating Eq. (10) and the result is shown in Eq. (11).

$$f_{p_G}(p_G, \boldsymbol{\theta}, \boldsymbol{\psi}|\boldsymbol{a}) = \int_{\Omega_\theta}\int_{\Omega_\psi} p_G\middle|\boldsymbol{\theta}, \boldsymbol{\psi}, \boldsymbol{a} \times f_{\theta}(\boldsymbol{\theta}|\boldsymbol{a}) \times f_{\psi}(\boldsymbol{\psi})\, \mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{\psi}, \tag{10}$$

$$F_{p_G}(\hat{p}_G|\boldsymbol{a}) = \int_0^{\hat{p}_G}\int_{\Omega_\theta}\int_{\Omega_\psi} f_{p_G}(p_G, \boldsymbol{\theta}, \boldsymbol{\psi}|\boldsymbol{a})\, \mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{\psi}\, \mathrm{d}\phi, \tag{11}$$

where ϕ is the variable which corresponds to the failure probability p_G. The CDF of p_G represents the probability that p_G is less than the specific value $\hat{p}_G$. If there is complete sampling information for sparse variables $\boldsymbol{Y}$ and interval variables $\boldsymbol{Z}$, $f_{\theta}\left(\boldsymbol{\theta}\middle|\tilde{\boldsymbol{y}}\right)$ and $f_{\psi}(\boldsymbol{\psi})$ are determinate values, and the f_{p_G} will be a determinate value, which is the $p_G|\boldsymbol{\theta}, \boldsymbol{\psi}, \boldsymbol{a}$ in Eq. (9).

3.2 Calculation procedure of probability of failure probability

The step-to-step procedure is listed as follows, and the calculation flowchart is shown in Fig. 2.

Step 1: The uncertainties of sparse variables $\boldsymbol{Y}$ are represented. The distribution types $\boldsymbol{\zeta}$, weight ratios $\boldsymbol{w}$, and uncertainties of distribution parameters $\boldsymbol{\theta}$ are calculated based on design points $\bar{\boldsymbol{y}}$ and the available insufficient input data $\boldsymbol{a}$ for dispersion part $\tilde{\boldsymbol{y}}$ of $\boldsymbol{Y}$.

Step 2: The PDF $f_{\boldsymbol{\theta}, \tilde{y}_i}(\boldsymbol{\theta})$ of distribution parameters $\boldsymbol{\theta}$ for the ith dispersion part $\tilde{y}_i$ is calculated. The auxiliary variables $\boldsymbol{\psi}$ for interval variables $\boldsymbol{Z}$ are assumed to be uniform distribution variables in [0, 1]. Therefore, N_1 sampling points of distribution parameters $\boldsymbol{\theta}$ and auxiliary variables $\boldsymbol{\psi}$ are randomly selected using Latin hypercube sampling method according to their PDFs.

Step 3: For the mth sampling points of $\boldsymbol{\theta}$ and $\boldsymbol{\psi}$, the PDFs of $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are calculated. The PDF $f_Y\left(\boldsymbol{Y}\middle|\boldsymbol{\theta}, \boldsymbol{a}\right)$ in Eq. (9) is a certain value under determinate distribution parameters $\boldsymbol{\theta}^{(m)}$ and available insufficient input data $\boldsymbol{a}$ for $\tilde{\boldsymbol{y}}$. The interval variables $\boldsymbol{Z}$ are transformed to probabilistic values using auxiliary

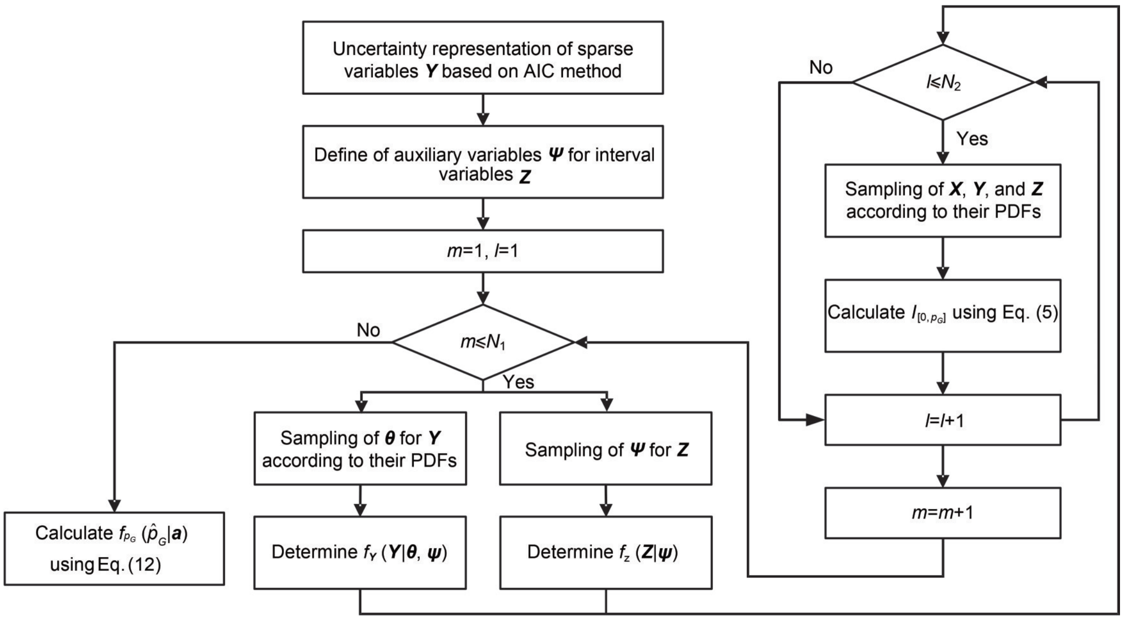

Fig. 2 Calculation flow chart for failure probability. AIC is the Akaike information criterion

variables $\boldsymbol{\psi}^{(m)}$. A series of sampling points for $\boldsymbol{Z}$ are obtained using random sampling method with random factor $\boldsymbol{\psi}^{(m)}$ in their uncertainty spaces $\left[\underline{\boldsymbol{Z}}, \bar{\boldsymbol{Z}}\right]$, and their PDF $f_{\boldsymbol{Z}}\left(\boldsymbol{Z}\middle|\boldsymbol{\psi}\right)$ is available through reverse fitting based on sampling information.

Step 4: N_2 random sampling points of $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$ are obtained using Latin hypercube sampling method according to their PDFs $f_{\boldsymbol{X}}(\boldsymbol{X})$, $f_{\boldsymbol{Y}}(\boldsymbol{Y}|\boldsymbol{\theta}, \boldsymbol{a})$, and $f_{\boldsymbol{Z}}(\boldsymbol{Z}|\boldsymbol{\psi})$, respectively.

Step 5: The indicator functions $I_{[0,p_G]}[p_G(\boldsymbol{\theta}^{(m)}, \boldsymbol{\psi}^{(m)}, \boldsymbol{X}^{(l)}, \boldsymbol{Y}^{(l)}, \boldsymbol{Z}^{(l)})]$ under the mth sampling points of $\boldsymbol{\theta}$ and $\boldsymbol{\psi}$ and the lth sampling points of $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$ are calculated using Eq. (5), where the value of $I_{[0,p_G]}[\Theta]$ is 1 when $\Theta \in [0, p_G]$, and 0 otherwise.

Step 6: The CDF of p_G is calculated using the two-level Monte Carlo sampling (MCS) method in Eq. (12).

$$F_{p_G}(\hat{p}_G|\boldsymbol{a}) = \frac{1}{N_1 N_2}\sum_{m=1}^{N_1}\sum_{l=1}^{N_2} I_{[0,p_G]}\left[p_G(\boldsymbol{\theta}^{(m)}, \boldsymbol{\psi}^{(m)}, \boldsymbol{X}^{(l)}, \boldsymbol{Y}^{(l)}, \boldsymbol{Z}^{(l)})\right]. \quad (12)$$

4 Sensitivity analysis of reliability index

The failure probability p_G is an uncertainty variable due to insufficient input data. However, it is also a critical constraint in many reliability-based design optimization problems (Liu et al., 2016; Keshtegar and Hao, 2018; Chen et al., 2020). The design sensitivity of failure probability can be obtained using the first-order score function and chain rules in Eq. (13) (Cho et al., 2016b).

$$\frac{\partial}{\partial \bar{\boldsymbol{d}}} F_{p_G}(\hat{p}_G|\boldsymbol{a}) = \int_0^{\hat{p}_G}\int_{\Omega_\theta}\int_{\Omega_\psi} f(p_G, \boldsymbol{\theta}, \boldsymbol{\psi}|\boldsymbol{a})\frac{\partial}{\partial \bar{\boldsymbol{d}}}\ln f_{\boldsymbol{d}}(\boldsymbol{d}|\boldsymbol{a})\,\mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{\psi}\mathrm{d}\phi, \quad (13)$$

where the input variables $\boldsymbol{d}$ contain $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$, and the design points $\bar{\boldsymbol{d}}$ are their corresponding mean values $\bar{\boldsymbol{x}}$, $\bar{\boldsymbol{y}}$, and $\bar{\boldsymbol{z}}$, respectively.

The design sensitivity of sparse variables $\boldsymbol{Y}$ can be calculated using the two-level MCS sampling method of probability of failure probability. In the sampling loop of $\boldsymbol{\theta}$ and $\boldsymbol{\psi}$, the distribution parameters $\boldsymbol{\theta}$ for $\boldsymbol{Y}$ are determinate, and the coefficient for sensitivity analysis $\mathrm{SF}\left(\bar{y}_i, \theta_{k,i}^1, \theta_{k,i}^2\middle|\boldsymbol{a}\right)$ can be calculated according to their weight ratios, distribution types, and distribution parameters. The sensitivity index of sparse variables $\boldsymbol{Y}$ is calculated using Eq. (14).

$$\frac{\partial}{\partial \bar{y}_i} F_{p_G}(\hat{p}_G|\boldsymbol{a}) = \frac{1}{N_1 N_2}\sum_{m=1}^{N_1}\sum_{l=1}^{N_2} I_{[0,p_G]}\left[p_G(\boldsymbol{\theta}^{(m)}, \boldsymbol{\psi}^{(m)}, \boldsymbol{X}^{(l)}, \boldsymbol{Y}^{(l)}, \boldsymbol{Z}^{(l)})\right]\mathrm{SF}(\bar{y}_i, \boldsymbol{\theta}_i^{(m)}|\boldsymbol{a}), \quad (14)$$

$$\mathrm{SF}\left(\bar{y}_i,\ \theta^1_{k,i},\ \theta^2_{k,i}\middle|\,\boldsymbol{a}\right)=\frac{\sum w_{k,i}\dfrac{\partial}{\partial \bar{y}_i}\left[\zeta_{k,i}\left(\theta^1_{k,i},\ \theta^2_{k,i}\middle|\,\boldsymbol{a}\right)\right]}{\sum w_{k,i}\zeta_{k,i}\left(\theta^1_{k,i},\ \theta^2_{k,i}\middle|\,\boldsymbol{a}\right)}. \quad (15)$$

For the statistical variables $\boldsymbol{X}$, the distribution type and distribution parameters are determinate, the PDF $f_X\left(X,\ \theta^1_{x,i},\ \theta^2_{x,i}\right)$ is also a determinate value, $\mathrm{SF}\left(\bar{x}_i,\ \theta^1_{x,i},\ \theta^2_{x,i}\right)$ are changeable in the outer loop of the two-level sampling, and the design sensitivity of statistical variables is calculated using Eq. (16).

$$\frac{\partial}{\partial \bar{x}_i}F_{p_G}(\hat{p}_G|\boldsymbol{a})=\frac{1}{N_1N_2}\sum_{l=1}^{N_2}\mathrm{SF}\left(\bar{x}_i,\ \theta^1_{x,i},\ \theta^2_{x,i}\right)\times \sum_{m=1}^{N_1}I_{[0,p_G]}\left[p_G(\boldsymbol{\theta}^{(m)},\ \boldsymbol{\psi}^{(m)},\ \boldsymbol{X}^{(l)},\ \boldsymbol{Y}^{(l)},\ \boldsymbol{Z}^{(l)})\right], \quad (16)$$

$$\mathrm{SF}\left(\bar{x}_i,\ \theta^1_{x,i},\ \theta^2_{x,i}\right)=\frac{\dfrac{\partial}{\partial \bar{x}_i}f_X\left(X,\ \theta^1_{x,i},\ \theta^2_{x,i}\right)}{f_X\left(X,\ \theta^1_{x,i},\ \theta^2_{x,i}\right)}. \quad (17)$$

For the interval variables $\boldsymbol{Z}$, their PDFs are assumed to be the weight summation of two distribution functions according to the auxiliary variables $\boldsymbol{\psi}$ in Eq. (18). The two distribution types are chosen as two common distribution types: normal distribution and Weibull distribution. To represent the randomness of interval variables $\boldsymbol{Z}$, the auxiliary variables $\boldsymbol{\psi}$ are random variables, which is the same as the calculation of failure probability p_G. Therefore, the sensitivity index of $\boldsymbol{Z}$ can be calculated in Eq. (19) using the similar method for sparse variables $\boldsymbol{Y}$.

$$Z_i=\psi_z\zeta_{1,i}\left(\boldsymbol{\theta}_{1,i}\right)+(1-\psi_z)\zeta_{2,i}\left(\boldsymbol{\theta}_{2,i}\right), \quad (18)$$

$$\frac{\partial}{\partial \bar{z}_i}F_{p_G}(\hat{p}_G|\boldsymbol{a})=\int_0^{\hat{p}_G}\int_{\Omega_\theta}\int_{\Omega_\psi}f(p_G,\ \boldsymbol{\theta},\ \boldsymbol{\psi}|\boldsymbol{a})\,\mathrm{SF}(\bar{z}_i,\ \psi_z)\,\mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{\psi}\mathrm{d}\phi, \quad (19)$$

$$\mathrm{SF}(\bar{z}_i,\ \psi_z)=\frac{\psi_z\dfrac{\partial}{\partial z_i}\left[\zeta_{1,i}\left(\theta^1_{1,i},\ \theta^2_{1,i}\right)\right]+(1-\psi_z)\dfrac{\partial}{\partial z_i}\left[\zeta_{2,i}\left(\theta^1_{2,i},\ \theta^2_{2,i}\right)\right]}{\psi_z\zeta_{1,i}\left(\theta^1_{1,i},\ \theta^2_{1,i}\right)+(1-\psi_z)\zeta_{2,i}\left(\theta^1_{2,i},\ \theta^2_{2,i}\right)}. \quad (20)$$

A two-level sampling method can be applied to the calculation of the sensitivity index of interval variables $\boldsymbol{Z}$, as shown in Eq. (21).

$$\frac{\partial}{\partial \bar{z}_i}F_{p_G}(\hat{p}_G|\boldsymbol{a})=\frac{1}{N_1N_2}\sum_{m=1}^{N_1}\sum_{l=1}^{N_2}I_{[0,p_G]}\left[p_G(\boldsymbol{\theta}^{(m)},\ \boldsymbol{\psi}^{(m)},\ \boldsymbol{X}^{(l)},\ \boldsymbol{Y}^{(l)},\ \boldsymbol{Z}^{(l)})\right]\mathrm{SF}\left(\bar{z}_i,\ \psi^{(m)}_{z,i}\right). \quad (21)$$

5 Extension of the proposed method to more uncertainty presentation types

The proposed methodology can be extended to the reliability measure of multiple types of epistemic uncertainties, such as p-box variables, multi-modal variables, and evidence theory variables.

5.1 p-box uncertainty variables

The p-box uncertainty variables $\boldsymbol{U}$ are represented with determinate single distribution type ζ and uncertain distribution parameters $\boldsymbol{\theta}$ in Eq. (22).

$$U_i\sim\zeta_i(\theta^1_i,\ \theta^2_i). \quad (22)$$

In the calculation of the probability of p_G, the uncertain distribution parameters of $\boldsymbol{U}$ and the auxiliary variables $\boldsymbol{\psi}$ for $\boldsymbol{Z}$ are sampled in the inner loop simultaneously, and the CDF of p_G considering $\boldsymbol{X}$, $\boldsymbol{U}$, and $\boldsymbol{Z}$ is calculated using Eq. (12).

Compared with the sensitivity analysis for $\boldsymbol{Y}$, the $\mathrm{SF}(\bar{u}_i,\ \theta^1_i,\ \theta^2_i)$ for $\boldsymbol{U}$ is calculated using Eq. (23), which is similar to that for the statistical variables $\boldsymbol{X}$, but the distribution parameters for $\boldsymbol{U}$ are uncertain. The sensitivity index of $\boldsymbol{U}$ can be calculated using Eq. (24).

$$\mathrm{SF}(\bar{u}_i,\ \theta^1_i,\ \theta^2_i)=\frac{\dfrac{\partial}{\partial \bar{u}_i}\left[\zeta_i(\theta^1_i,\ \theta^2_i)\right]}{\zeta_i(\theta^1_i,\ \theta^2_i)}, \quad (23)$$

$$\frac{\partial}{\partial \bar{u}_i}F_{p_G}(\hat{p}_G|\boldsymbol{a})=\frac{1}{N_1N_2}\sum_{m=1}^{N_1}\sum_{l=1}^{N_2}I_{[0,p_G]}\left[p_G(\boldsymbol{\theta}^{(m)},\ \boldsymbol{\psi}^{(m)},\ \boldsymbol{X}^{(l)},\ \boldsymbol{U}^{(l)},\ \boldsymbol{Z}^{(l)})\right]\mathrm{SF}(\bar{u}_i,\ \boldsymbol{\theta}^{(m)}_i). \quad (24)$$

5.2 Multi-modal distribution variables

The multi-modal distribution variables $\boldsymbol{Q}$ can be represented with weight summation of multiple types of distribution functions in Eq. (25) (Zhang et al., 2019). However, the information of distribution parameters and weight ratios for $\boldsymbol{Q}$ are known in advance, which is different from $\boldsymbol{Y}$. Therefore, the calculation method for sparse variables $\boldsymbol{Y}$ is applied in the reliability measure and sensitivity calculation considering $\boldsymbol{Q}$.

$$Q_i \sim \sum w_{k,i}\zeta_{k,i}\left(\theta_{k,i}^1,\ \theta_{k,i}^2\right). \tag{25}$$

In the calculation of sensitivity index, $\mathrm{SF}\left(\bar{q}_i, \theta_{k,i}^1, \theta_{k,i}^2\right)$ in Eq. (26) is determinate in the inner sampling loop for $\boldsymbol{\theta}$ and $\boldsymbol{\psi}$. Therefore, the sensitivity index for $\boldsymbol{Q}$ is calculated using Eq. (27).

$$\mathrm{SF}\left(\bar{q}_i,\ \theta_{k,i}^1,\ \theta_{k,i}^2\right) = \frac{\sum w_{k,i}\frac{\partial}{\partial \bar{q}_i}\left[\zeta_{k,i}\left(\theta_{k,i}^1,\ \theta_{k,i}^2\right)\right]}{\sum w_{k,i}\zeta_{k,i}\left(\theta_{k,i}^1,\ \theta_{k,i}^2\right)}, \tag{26}$$

$$\frac{\partial}{\partial \bar{q}_i}F_{p_G}(\hat{p}_G|\boldsymbol{a}) = \frac{1}{N_1 N_2}\sum_{l=1}^{N_2}\mathrm{SF}\left(\bar{q}_i,\ \boldsymbol{\theta}_{q,i}\right)\times \sum_{m=1}^{N_1} I_{[0,p_G]}\left[p_G(\boldsymbol{\theta}^{(m)},\ \boldsymbol{\psi}^{(m)},\ \boldsymbol{Q}^{(l)},\ \boldsymbol{Y}^{(l)},\ \boldsymbol{Z}^{(l)})\right]. \tag{27}$$

5.3 Evidence theory variables

The evidence theory variables $\boldsymbol{R}$ are represented using some subintervals with basic probability assignments (BPAs). Compared with interval variables $\boldsymbol{Z}$, $\boldsymbol{R}$ is divided into many subintervals, and every subinterval is an interval variable. Therefore, the evidence theory variables $\boldsymbol{R}$ can be handled using a method similar to that for interval variables $\boldsymbol{Z}$.

In the sensitivity analysis, the PDFs of evidence theory variables $\boldsymbol{R}$ are assumed to be the weight summation of two distribution functions in every subinterval according to the auxiliary variables $\boldsymbol{\psi}$ in Eq. (28), and the sensitivity index of $\boldsymbol{R}$ is calculated in Eq. (29) using a similar method as for interval variables $\boldsymbol{Z}$.

A two-level sampling method considering $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{R}$ is applied to the calculation of the sensitivity index of $\boldsymbol{R}$, as shown in Eq. (31).

$$\boldsymbol{R} = \sum m_k\left(\psi_{k,1}\zeta_{k,1}(\boldsymbol{\theta}^i) + \left(1-\psi_{k,1}\right)\zeta_{k,2}(\boldsymbol{\theta}^i)\right), \tag{28}$$

$$\frac{\partial}{\partial \bar{r}_i}F_{p_G}(\hat{p}_G|\boldsymbol{a}) = \int_0^{\hat{p}_G}\int_{\Omega_\theta}\int_{\Omega_\psi} f(p_G,\ \boldsymbol{\theta},\ \boldsymbol{\psi}|\boldsymbol{a})\,\mathrm{SF}(\bar{r}_i,\ \psi_r)\,\mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{\psi}\mathrm{d}\phi, \tag{29}$$

$$\mathrm{SF}(\bar{r}_i,\ \psi_r) = \sum m_k\left\{\psi_{k,1}\frac{\partial}{\partial r_i}\left[\zeta_{i,k,1}\left(\theta_{1,i}^1,\ \theta_{1,i}^2\right)\right] + \left(1-\psi_{k,1}\right)\frac{\partial}{\partial r_i}\left[\zeta_{i,k,2}\left(\theta_{2,i}^1,\ \theta_{2,i}^2\right)\right]\right\}\Big/ \sum m_k\left[\psi_{k,1}\zeta_{i,k,1}\left(\theta_{1,i}^1, \theta_{1,i}^2\right) + \left(1-\psi_{k,1}\right)\zeta_{i,k,2}\left(\theta_{2,i}^1, \theta_{2,i}^2\right)\right], \tag{30}$$

$$\frac{\partial}{\partial \bar{r}_i}F_{p_G}(\hat{p}_G|\boldsymbol{a}) = \frac{1}{N_1 N_2}\sum_{m=1}^{N_1}\sum_{l=1}^{N_2} I_{[0,p_G]}\left[p_G(\boldsymbol{\theta}^{(m)},\ \boldsymbol{\psi}^{(m)},\ \boldsymbol{X}^{(l)},\ \boldsymbol{Y}^{(l)},\ \boldsymbol{R}^{(l)})\right]\mathrm{SF}\left(\bar{r}_i,\ \psi_{r,i}^{(m)}\right). \tag{31}$$

6 Application examples

6.1 Numerical example 1

To demonstrate the effectiveness of the proposed reliability measure approach under insufficient input data, the 2D mathematical performance functions in Eqs. (32)–(34) (Cho et al., 2016a) are introduced.

$$G_1(\boldsymbol{Y}) = 1 - \frac{Y_1^2 Y_2}{20}, \tag{32}$$

$$G_2(\boldsymbol{Y}) = -1 + (0.9063Y_1 + 0.4226Y_2 - 6)^2 + (0.9063Y_1 + 0.4226Y_2 - 6)^3 - 0.6(0.9063Y_1 + 0.4226Y_2 - 6)^4 - (-0.4226Y_1 + 0.9063Y_2), \tag{33}$$

$$G_3(\boldsymbol{Y}) = 1 - \frac{80}{Y_1^2 + 8Y_2 + 5}, \tag{34}$$

where the true distributions of independent input random variables are $Y_1 \sim N(\bar{y}_1, 0.3)$ and $Y_2 \sim N(\bar{y}_2, 0.3)$, and the design point $\boldsymbol{y}^0 = [\bar{y}_1, \bar{y}_2]^{\mathrm{T}} = [4.7, 1.6]^{\mathrm{T}}$ is considered.

To verify the effectiveness of the proposed method for reliability measure under insufficient input data, 10 random sampling points for dispersion parts of input variables are available based on the true distribution functions of Y_1 and Y_2, which are the same as that in (Cho et al., 2016a). The selected distribution types for Y_1 are normal, lognormal, and Weibull types, and the corresponding weight ratios are 0.391, 0.492, and 0.117, respectively. The selected distribution types for Y_2 are normal, Weibull, and extreme value types, and the corresponding weight ratios are 0.193, 0.357, and 0.450, respectively. The probability of p_G is calculated and shown in Fig. 3a.

The true failure probabilities at $\boldsymbol{y}^0$ are 1.79%, 1.49%, and 0.00% for G_1, G_2, and G_3, respectively. When there are only 10 input data, the conservativeness levels at failure probability $p_G = 2.275\%$ are

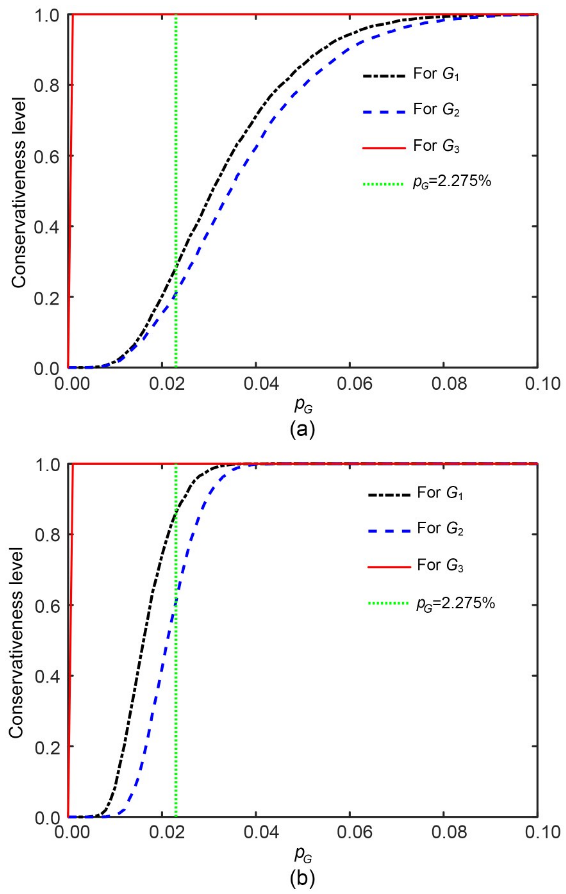

Fig. 3 Conservativeness level of failure probability considering sparse variables: (a) 10 input data; (b) 100 input data

28.01% for G_1, 20.03% for G_2, and 99.99% for G_3. However, the selected distribution types for Y_1 and Y_2 are determinate; only two-level sampling loops (distribution parameters and design variables) are implemented, which decreases the computational complexity compared to the three-level sampling loop method (distribution types, distribution parameters, and design variables) in (Cho et al., 2016a).

Due to the limited input data, the probability that the design meets the target failure probability is less than 50% for G_1 and G_2. To analyze the influence of sampling number, 100 random sampling points for Y_1 and Y_2 are selected according to their true distribution functions, and the calculated probability of p_G is shown in Fig. 3b. The conservativeness levels at $p_G=2.275\%$ are 85.02% for G_1, 57.97% for G_2, and 99.99% for G_3, which is better than the results (59.9% for G_1, 63.6% for G_2, and 99.9% for G_3) in (Cho et al., 2016a). The proposed reliability measure method can obtain more accurate results because the optimum distribution types are obtained due to more insufficient input data, which decreases the uncertainties of distribution types, and only uncertainties of distribution parameters are considered.

To verify the accuracy of the derived sensitivity analysis method in Section 4, the design sensitivity for $p_G \in [0,\ 0.1]$ at $\boldsymbol{y}^0=[4.7,\ 1.6]^{\mathrm{T}}$ under 10 input data is computed using the proposed method and the finite difference method (FDM), respectively (Lee et al., 2011). The details of the FDM method are shown in the ESM. The sensitivity indices S between Y_1, Y_2 and reliability indices of G_1, G_2 are summarized in Fig. 4. The relative errors between the proposed method and the FDM method are less than 4.3% for $S_{Y_1\_G_1}$, 4.8% for $S_{Y_1\_G_2}$, 3.7% for $S_{Y_2\_G_1}$, and 5.2% for $S_{Y_2\_G_2}$. In the calculation of the FDM method, the conservativeness levels of p_G at $[4.70,\ 1.60]^{\mathrm{T}}$, $[4.71,\ 1.60]^{\mathrm{T}}$, $[4.69,\ 1.60]^{\mathrm{T}}$, $[4.70,\ 1.59]^{\mathrm{T}}$, and $[4.70,\ 1.61]^{\mathrm{T}}$ are calculated, and the calculation burden is 5 times that of calculating the probability of p_G. However, the sensitivity indices can be calculated using the proposed method with a small computation increase compared with the calculation of the probability of p_G.

6.2 Numerical example 2

To demonstrate the effectiveness of the proposed reliability measure approach under hybrid uncertainties, the 2D mathematical functions in Section 6.1 are extended to 3D functions in Eqs. (35) and (36).

$$G_1(X,\ Y,\ Z)=1-\frac{XYZ}{20},\tag{35}$$

$$\begin{aligned}G_2(X,\ Y,\ Z)=&-1+(0.9063X+0.4226Y-6)^2+\\&(0.4226Y+0.9063Z-6)^3-0.6(0.9063X+\\&0.4226Y-6)^4-(0.9063Y-0.4226Z),\end{aligned}\tag{36}$$

where the statistical variable X is a normal distribution function with mean $\bar{x}$ and standard deviation 0.3. The available 10 random sampling points for dispersion parts of sparse variable Y are the same as that of Y_2 in numerical example 1. The interval variable Z is represented with interval $Z\in[\bar{z}-0.6,\ \bar{z}+0.6]$. At the design point $[\bar{x},\ \bar{y},\ \bar{z}]^{\mathrm{T}}=[4.7, 1.6, 4.7]^{\mathrm{T}}$, the uncertainty representation function of sparse variable Y is the same as that of Y_2 in numerical example 1. The PDFs of distribution parameters $\boldsymbol{\theta}$ for Y are shown in Fig. 5.

The conservativeness level of failure probability considering three types of uncertainties simultaneously is calculated using the proposed method and the MCS method. In the MCS method, three sampling loops of distribution types $\boldsymbol{\zeta}$, distribution parameters $\boldsymbol{\theta}$, and uncertain design variables are implemented to represent the uncertainty of the sparse variable Y. The interval variable Z is selected randomly in every sampling

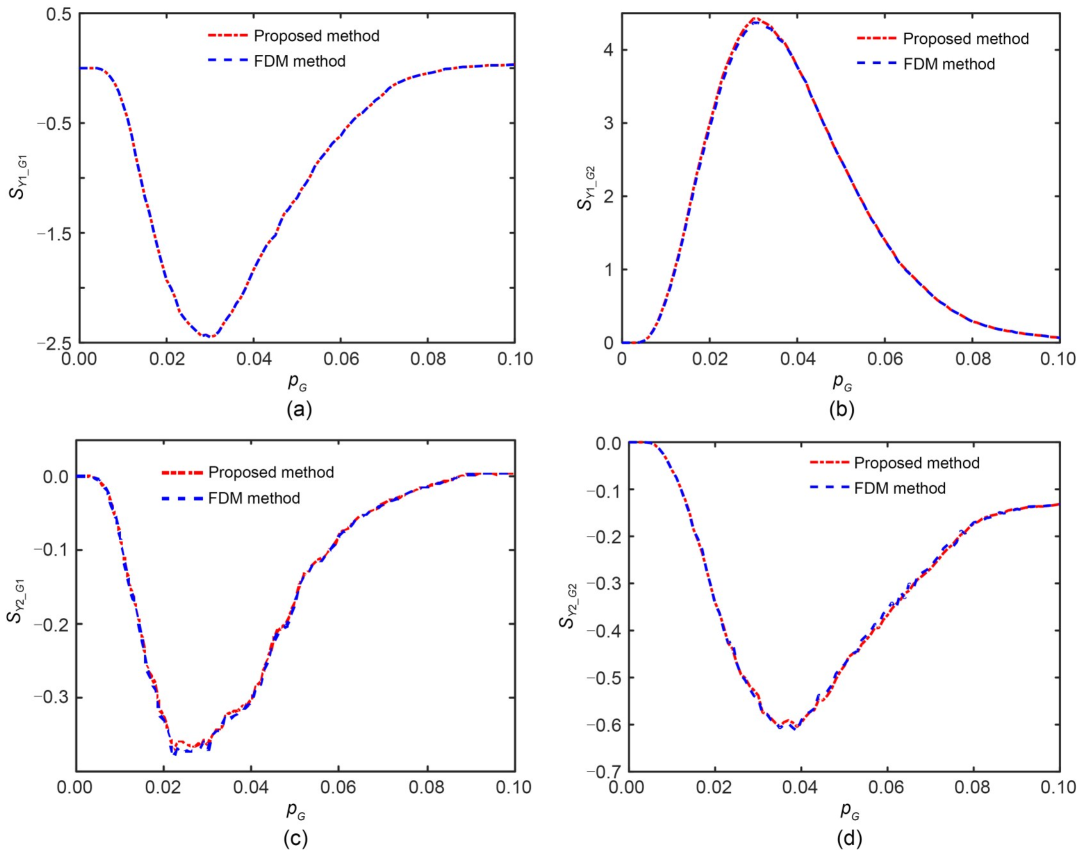

Fig. 4 Sensitivity results under different failure probabilities: (a) $S_{Y_1\_G_1}$; (b) $S_{Y_1\_G_2}$; (c) $S_{Y_2\_G_1}$; (d) $S_{Y_2\_G_2}$

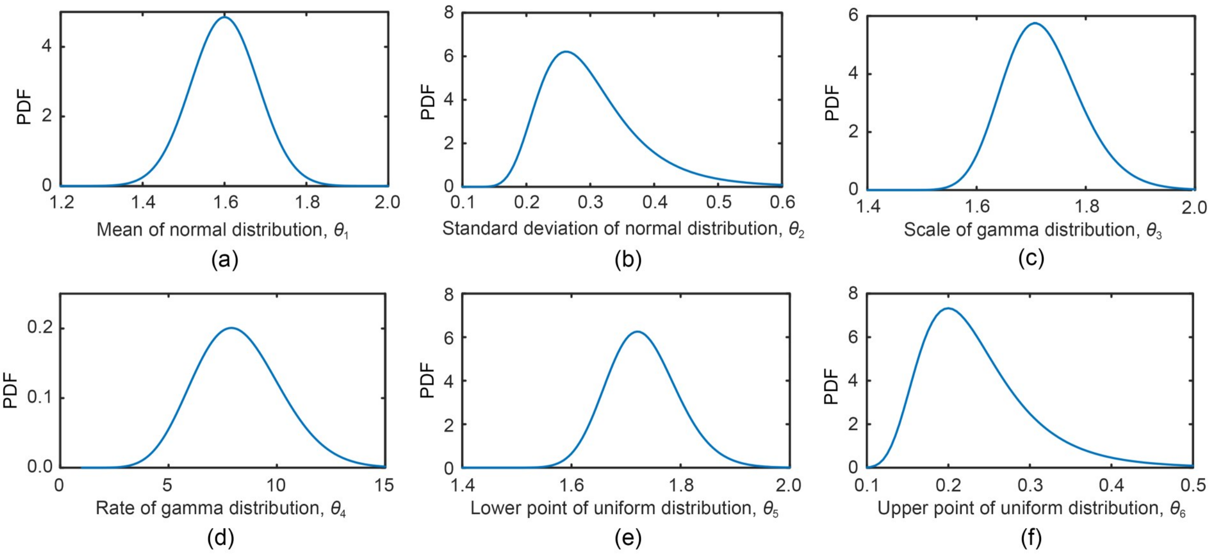

Fig. 5 PDFs of distribution parameters for sparse variable Y: (a) θ_1; (b) θ_2; (c) θ_3; (d) θ_4; (e) θ_5; (f) θ_6

loop of uncertain variables according to the sampling value of auxiliary variable ψ_Z, and the statistical variable X is sampled according to its distribution type and distribution parameters. The total sampling number, which is also the calculation number of performance function $G(X, Y, Z)$, is 175 million, which contains seven sampling points of distribution types in the first loop, 5000 sampling points for distribution parameters in the second loop, and 5000 sampling points for uncertain variables (the statistical variable X, sparse variable Y, and interval variable Z) in the third loop. Through using the proposed method, the sparse variable Y is represented with weight summation of multiple distribution types. Only two sampling loops for distribution parameters and uncertainty variables are implemented and the total sampling number is 9 million (3000 for distribution parameters and 3000 for uncertainty variables). The results are shown in Fig. 6 and the proposed method can obtain the accurate conservativeness level of failure probability; however, only 5.14% of the MCS sample points are used for the developed method.

The design sensitivity for p_G=2.275% at $[\bar{x}, \bar{y}, \bar{z}]$=[4.7, 1.6, 4.7] is computed using the proposed method and the FDM, and the results are listed in Table 1. In

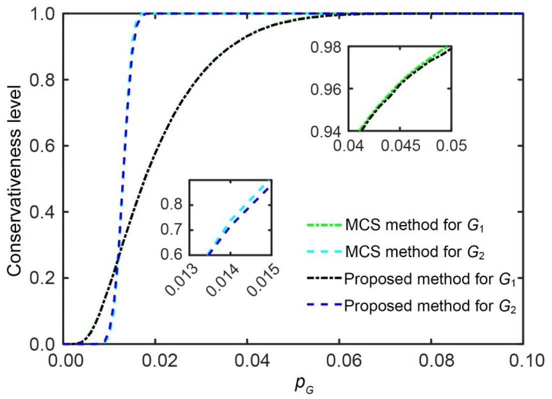

Fig. 6 Conservativeness level of failure probability of proposed method and MCS method

the FDM method, the additional calculation of failure probability is evaluated through forward and backward perturbed design for every uncertainty variable, and the total sampling number of MCS method is 12.25 billion for design sensitivity analysis. However, using the proposed method, the sensitivity indices can be obtained with little additional computational burden in the reliability measure procedure; the total sampling number is still 9 million. Through using fewer sampling points and computational time, the agreement between the developed sensitivity indices and results from the FDM method varies from 96.07% to 100.20%, which indicates that the proposed method can obtain accurate sensitivity results under mixture uncertainties.

6.3 Numerical example 3

To demonstrate the effectiveness of the proposed method in the reliability measure for multiple types of epistemic uncertainties, the numerical example 2 is extended to analysis reliability indices and sensitivity indices considering p-box variables, multi-modal variables, and evidence theory variables.

6.3.1 Reliability measure considering p-box variable

The sparse variable Y of the 3D functions G_1 and G_2 in Eqs. (35) and (36) is changed to p-box variable U, whose distribution type is normal distribution, the mean is the design point $\bar{u}$, and the standard deviation is also a normal distribution function with mean 0.3 and standard deviation 0.1.

Compared with sparse variable Y, the p-box variable U is represented using a single determinate distribution type. Two sampling loops of distribution parameters and design variables are implemented in the calculation of failure probability and sensitivity indices. In the first loop, 3000 sampling points for distribution parameters $\boldsymbol{\theta}$ of p-box variable U and auxiliary variable ψ of interval variable Z are randomly selected

Table 1 Design sensitivity of conservativeness level in example 2

Method	Sensitivity index						Time (s)
	G_1_X	G_2_X	G_1_Y	G_2_Y	G_1_Z	G_2_Z	
Proposed method	0.512	−1.246	2.858	2.328	0.713	1.370	145.39
FDM	0.511	−1.283	2.861	2.344	0.739	1.426	610.14
Agreement degree*	100.20%	97.12%	99.90%	99.32%	96.48%	96.07%	–

* Agreement degree = $\dfrac{\text{Sensitivty index calculated by the proposed method}}{\text{Sensitivty index calculated by FDM method}}$

according to their PDFs; in the second loop, 3000 sampling points for uncertainty variables X, U, and Z are selected according to the sampling values of distribution parameters $\boldsymbol{\theta}$ and auxiliary variable ψ. The conservation level of failure probability for G_1 and G_2 is calculated and shown in Fig. 7.

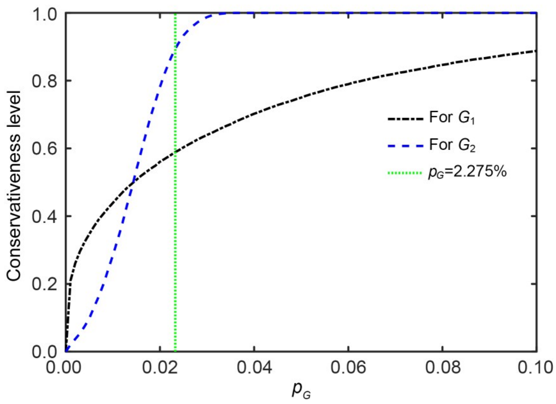

Fig. 7 Reliability result considering *p*-box variable

The sensitivity results of the proposed method are compared with the results of the FDM method, which are listed in Table 2. The computational complexity is decreased. In the FDM method, additional six times of failure probability computation are implemented through forward and backward perturbed design for three uncertainty variables. However, the sensitivity index can be calculated using Eq. (16) for X, Eq. (24) for U, and Eq. (21) for Z through semi-analytical calculation without additional sampling points. The calculation complexity is reduced, but the agreement degrees of the proposed method vary from 98.97% to 101.53% compared to the FDM method, which demonstrates the effectiveness of the reliability measure method considering p-box variables.

6.3.2 Reliability measures considering multi-modal variables

The sparse variable Y of the 3D functions G_1 and G_2 in Eqs. (35) and (36) is changed to the multi-modal variable Q, which is represented with weight summation of normal distribution ζ_1 and Weibull distribution ζ_2 in Eq. (37). The distribution parameters $\theta_1 \sim N(0.3, 0.1)$ and $\theta_2 \sim N(7.9, 0.1)$ are also uncertainty variables.

$$Q \sim 0.5\zeta_1(1.6, \theta_1) + 0.5\zeta_2(1.71, \theta_2). \tag{37}$$

Compared with sparse variable Y, the weight ratios and distribution types of multi-modal variable Q are determinate. The failure probability can be calculated using a two-level sampling method for uncertain distribution parameters and design variables, as shown in Fig. 8, which demonstrates that the proposed method can be effectively extended to calculate reliability index considering multi-modal variables.

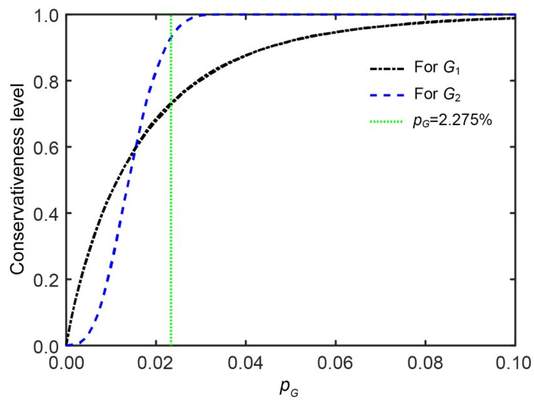

Fig. 8 Reliability result considering multi-modal variables

The sensitivity indices of statistical variable X, multi-modal variable Q, and interval variable Z are calculated using the proposed algorithm and the FDM method, respectively. The results are listed in Table 3. The agreement of sensitivity indices varies from 98.20% to 101.03%. The proposed method can obtain accuracy sensitivity results with little computation complexity, which demonstrates the effectiveness of the proposed sensitivity calculation method considering multi-modal variables.

Table 2 Design sensitivity of conservativeness level considering the *p*-box variable

Method	Sensitivity index						Time (s)
	G_1_X	G_2_X	G_1_U	G_2_U	G_1_Z	G_2_Z	
Proposed method	0.464	−1.446	0.931	1.363	0.272	−7.119	148.64
FDM	0.467	−1.461	0.917	1.372	0.272	−7.028	663.97
Agreement degree	99.36%	98.97%	101.53%	99.34%	100.00%	101.29%	–

Table 3 Design sensitivity of conservativeness level considering the multi-modal variable

Method	Sensitivity index						Time (s)
	G_1_X	G_2_X	G_1_Q	G_2_Q	G_1_Z	G_2_Z	
Proposed method	0.353	−1.889	1.255	1.435	0.977	−8.223	129.63
FDM	0.356	−1.900	1.278	1.428	0.967	−8.333	614.74
Agreement degree	99.16%	99.42%	98.20%	100.49%	101.03%	98.68%	–

6.3.3 Reliability measure considering evidence theory variable

The uncertainty information of interval variable Z is unavailable, which is represented by using a single uncertainty interval. When there is little uncertainty information, the interval variable Z can be extended to evidence theory variable R, which is represented using some subintervals with BPAs. Therefore, the interval variable Z of the 3D functions G_1 and G_2 in Eqs. (35) and (36) is changed to evidence theory variable R, which is represented using subintervals $[4.1, 4.7]$ and $[4.7, 5.3]$, whose BPAs are 0.6 and 0.4, respectively.

In the reliability measure and sensitivity calculation, the uncertainty analysis of evidence theory variable R can be calculated using two auxiliary variables ψ_1 and ψ_2. Every subinterval is treated as an interval variable, which is calculated using the auxiliary variable method for interval variable Z. Using the two-level sampling method and semi-analytical sensitivity calculation method, the reliability indices of G_1 and G_2 considering X, Y, and R are shown in Fig. 9, and the corresponding sensitivity indices are listed in Table 4. Results indicate that the proposed method can measure the reliability considering the effectiveness of evidence theory variables.

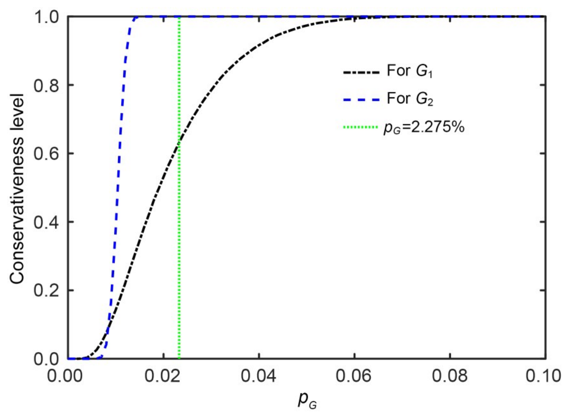

Fig. 9 Reliability measure result considering the evidence theory variable

6.4 Engineering example: forging hydraulic press

The forging hydraulic press is a large piece of equipment, which uses liquid as its working medium, and transfers energy to the forging process, as shown in Fig. 10.

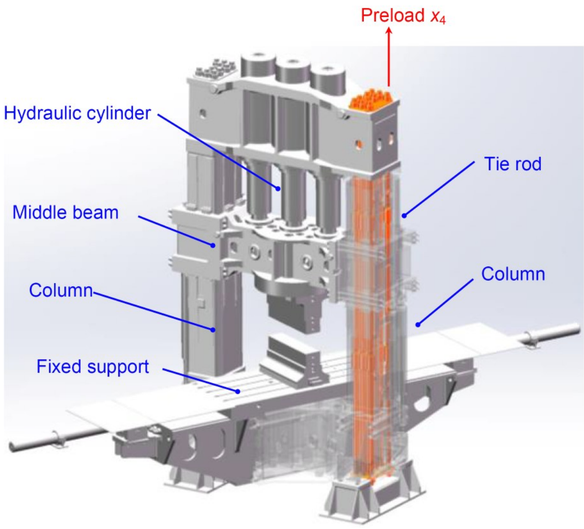

Fig. 10 Forging hydraulic press

Due to the enormous forging force, the tie rods will be stretched, which will lead to the separation of the upper beam and the column and will also reduce the forging accuracy. Therefore, the deformation of tie rods ΔL is an important index reflecting the performance of the forging hydraulic press and is chosen as the performance function. A reliability analysis is necessary.

The deformation of the tie rods ΔL is affected by several uncertain variables. The load of the hydraulic press x_1 is an interval variable which can be represented by $[\bar{x}_1-5, \bar{x}_1+5]\times10^7$ N. The diameter of tie rods x_2 is a statistical variable with normal distribution type, whose mean value is $\bar{x}_2=0.25$ m and standard deviation is 0.001 m; Young's modulus of tie rods x_3 is also a statistical variable with normal distribution type, whose mean value is $\bar{x}_3=2\times10^{11}$ Pa and standard deviation is 1×10^9 Pa. The preload acting on tie

Table 4 Design sensitivity of conservativeness level considering the evidence theory variable

Method	Sensitivity index						Time (s)
	G_1_X	G_2_X	G_1_Y	G_2_Y	G_1_R	G_2_R	
Proposed method	0.909	−0.663	0.453	0.777	0.653	−3.070	124.73
FDM	0.928	−0.676	0.444	0.783	0.656	−3.074	643.82
Agreement degree	97.95%	98.08%	102.03%	99.23%	99.54%	99.87%	–

rods x_4 is a sparse variable, whose available information for dispersion part is 20 random points of normal distribution function $N(1.625\times10^8,\ 8\times10^6)$ N. ΔL can be obtained according to the deformation results from the well-known finite element analysis (FEA) software ANSYS. One of the results under some specific sampling points of $\{x_1, x_2, x_3, x_4\}$ is shown in Fig. 11.

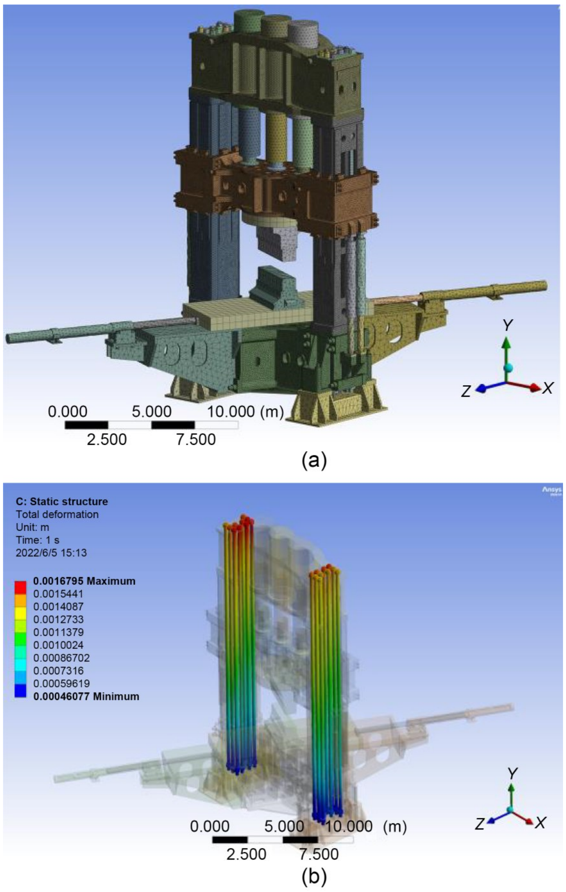

Fig. 11 FEA of the forging hydraulic press: (a) mesh; (b) deformation of the tie rods

The limit state function is given as $g=0.018-\Delta L$. At the design point $\bar{x}_1=1.25\times10^8$ N, $\bar{x}_2=0.25$ m, $\bar{x}_3=2\times10^{11}$ Pa, and $\bar{x}_4=1.625\times10^8$ N, the uncertainty of sparse variable x_4 is represented first. The selected distribution types for x_4 are normal and extreme value types, and the corresponding weight ratios are 0.5981 and 0.4019, respectively. The PDFs of corresponding distribution parameters $\boldsymbol{\theta}$ for x_4 are shown in Fig. 12.

The conservativeness level of failure probability considering hybrid uncertainties is calculated using the proposed method and the MCS method, as shown in Fig. 13. The sensitivity results computed using the proposed method and the FDM are listed in Table 5. These results indicate the proposed method can obtain accurate reliability index and sensitivity results.

7 Conclusions

In this study, a reliability measure approach considering mixture uncertainties under insufficient input data is proposed. First, the sparse variable is represented using weight summation of multiple distribution types based on AIC method under insufficient input data. Second, the failure probability under mixture uncertainties is calculated using the proposed two-level sampling method. Then, a semi-analytical method is proposed to calculate the sensitivity indices of mixture uncertainty variables. Finally, the proposed reliability measure method is extended to deal with p-box variables, multi-modal variables, and evidence theory variables.

From the results of three numerical examples and two engineering examples, the proposed method can obtain accuracy reliability measure results with higher computational efficiency compared with the MCS and FDM methods. Some conclusions are obtained: (i) The proposed method can obtain accuracy reliability measure results with less computational times. The traditional three-level sampling loop for sparse variables is decreased to a two-level sampling loop, which decreases the computation complexity for the reliability measure. (ii) The semi-analytical sensitivity calculation method based on an auxiliary variable method decreases the computational burden, and can be integrated into the uncertainty optimization

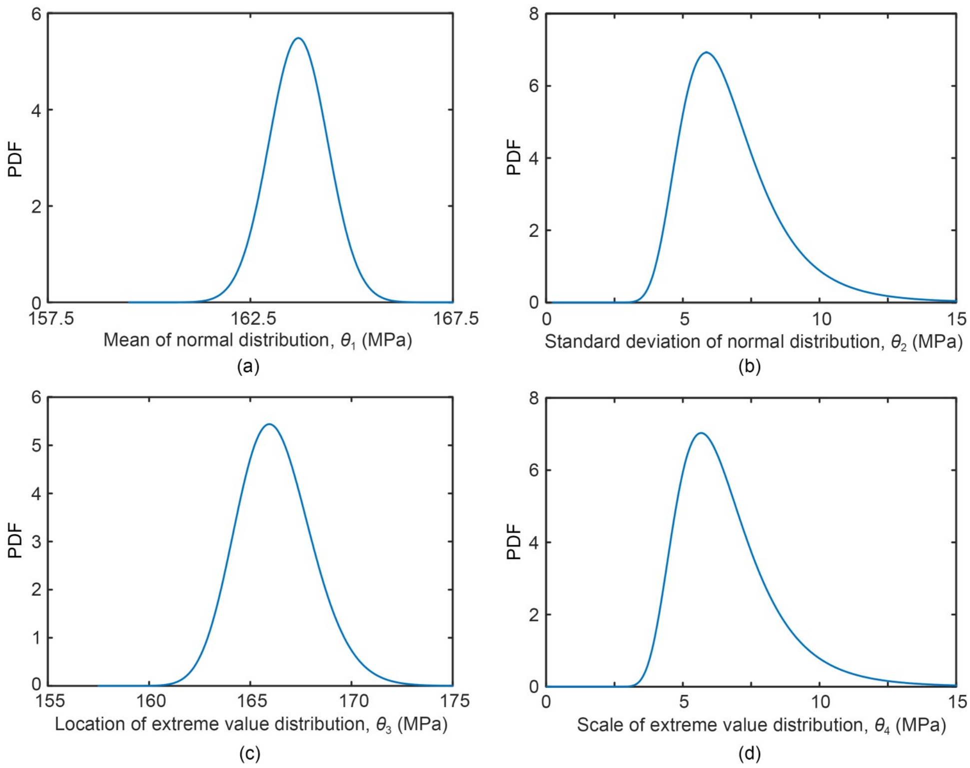

Fig. 12 PDFs of distribution parameters for sparse variable x_4: (a) θ_1; (b) θ_2; (c) θ_3; (d) θ_4

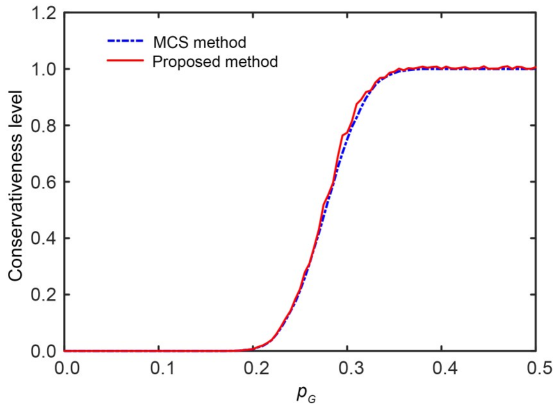

Fig. 13 Reliability result of forging hydraulic press

Table 5 Design sensitivity of conservativeness level for forging hydraulic press

Method	Sensitivity index				Time (s)
	x_1	x_2	x_3	x_4	
Proposed method	1.812	−5.374	−3.626	7.241	206.63
FDM	1.836	−5.393	−3.583	7.163	747.94
Agreement degree	98.69%	99.65%	101.20%	101.09%	–

method with little extra calculation. (iii) The proposed method has been extended to analyze p-box variables, multi-modal variables, and evidence theory variables, which can be extended to measure reliability index and sensitivity indices considering more uncertainty types, which is useful for mixture uncertainty optimization design.

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Author contributions

Zhenyu LIU designed the research. Yufeng LYU and Guodong SA derived the mathematical formulas and analyzed the experimental and simulation cases. Yufeng LYU wrote the first draft of the manuscript. Guodong SA was in charge of the whole project. Jianrong TAN gave the theoretical guidance on the whole work.

Conflict of interest

Zhenyu LIU, Yufeng LYU, Guodong SA, and Jianrong TAN declare that they have no conflict of interest.

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Electronic supplementary materials

Sections S1–S4