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Solution and stability of continuous-time cross-dimensional linear systems

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Motivation

1. As an extension of classical linear systems, the cross-dimensional system is often used to study complex systems such as departure and joining of spacecraft, vehicle clutch systems, and modeling of biological systems.

2. In a continuous-time cross-dimensional linear system, its trajectory can not only express the dynamic evolution law, but also serve as an important tool to study controllability, observability, and stability.

3. Stability, as an important property of the system, is the ability to describe whether a system can operate stably.

Main idea

By transforming the solution of continuous-time cross-dimensional linear systems, a new algebraic form of the solution is obtained. Based on this solution with a new algebraic form, the stability of continuous-time cross-dimensional linear systems is analyzed. Then the controller is designed by combining the solution method of linear matrix inequality.

Method

1. The inhomogeneous linear differential equations satisfied by the solution of the system are obtained using the integral iterative method. The solution of continuous-time cross-dimensional linear systems is obtained by solving the equation.

2. Based on the stability theory of linear systems, the stability is studied using the Jordan decomposition of matrix.

3. The controller design method is given by solving the linear matrix inequality.

Major results

A continuous-time cross-dimensional linear system (CCDLS) can be described as

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A} \vec{\ltimes} \boldsymbol{x}(t), \\ \boldsymbol{A} \in \mathbb{R}^{m \times n},\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathcal{V}_{r_0} \ . \end{cases} \tag{1}$$

The corresponding control system is defined as follows:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A} \vec{\ltimes} \boldsymbol{x}(t) \vec{\vdash} \boldsymbol{B}\boldsymbol{u}(t), \\ \boldsymbol{y}(t) = \boldsymbol{C} \vec{\ltimes} \boldsymbol{x}(t) \ . \end{cases} \tag{3}$$

Major results (Cont'd)

1. The solution to CCDLS (1) is given as follows:

$$\boldsymbol{x}(t) = \mathrm{e}^{\boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]} t} (\boldsymbol{x}_0 \otimes \mathbf{1}_{\frac{\alpha}{r_0}}) + \int_0^t \mathrm{e}^{\boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]}(t-\tau)} \boldsymbol{\Xi} \boldsymbol{f}_{[s]}(\tau) \mathrm{d}\tau, \quad (6)$$

where

$$\begin{cases} \alpha = \vee_{j=0}^{s} r_j, \\ \boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]} = \dfrac{r_s}{\alpha} (\boldsymbol{A}_{r_s} \otimes \mathbf{1}_{\frac{\alpha}{r_s}}) \otimes \mathbf{1}^{\mathrm{T}}_{\frac{\alpha}{r_s}}, \\ \boldsymbol{\Xi} = \Big\{ (\boldsymbol{A} \vec{\ltimes} \boldsymbol{x}_0) \otimes \mathbf{1}_{\frac{\alpha}{r_1}} - \boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]} (\boldsymbol{x}_0 \otimes \mathbf{1}_{\frac{\alpha}{r_0}}), \\ \quad (\boldsymbol{A}^2 \vec{\ltimes} \boldsymbol{x}_0) \otimes \mathbf{1}_{\frac{\alpha}{r_2}} - \boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]} [(\boldsymbol{A} \vec{\ltimes} \boldsymbol{x}_0) \otimes \mathbf{1}_{\frac{\alpha}{r_1}}], \\ \quad \vdots \\ \quad (\boldsymbol{A}^s \vec{\ltimes} \boldsymbol{x}_0) \otimes \mathbf{1}_{\frac{\alpha}{r_s}} - \boldsymbol{\Phi}_{[r_s, \frac{\alpha}{r_s}]} [(\boldsymbol{A}^{s-1} \vec{\ltimes} \boldsymbol{x}_0) \otimes \mathbf{1}_{\frac{\alpha}{r_{s-1}}}] \Big\}. \end{cases}$$

Major results (Cont'd)

2. A necessary and sufficient condition for stability is given.

If the initial state of CCDLS (1) is $\boldsymbol{x}_0$, then CCDLS (1) is asymptotically stable with $\boldsymbol{x}_0$ if and only if the following conditions hold:

$$\boldsymbol{\Xi} = 0_{\alpha\times s},\ \boldsymbol{\Gamma}\boldsymbol{x}_0 = 0_{\alpha-r_s},\ \text{and}\ \boldsymbol{\Upsilon}\boldsymbol{x}_0 = 0_{\max(\gamma_2,1)},$$

where

$$\begin{cases} \boldsymbol{\Gamma} = [0_{(\alpha-r_s)\times r_s}, \boldsymbol{I}_{\alpha-r_s}]\boldsymbol{P}_{[r_s,\frac{\alpha}{r_s}]}(\boldsymbol{I}_{r_0}\otimes \boldsymbol{1}_{\frac{\alpha}{r_0}}), \\ \boldsymbol{\Upsilon} = [0_{\max(\gamma_2,1)\times\gamma_1}, \boldsymbol{I}_{\gamma_2}]\boldsymbol{N}[\boldsymbol{I}_{r_s}, 0_{r_s\times(\alpha-r_s)}] \\ \qquad \times\boldsymbol{P}_{[r_s,\frac{\alpha}{r_s}]}(\boldsymbol{I}_{r_0}\otimes \boldsymbol{1}_{\frac{\alpha}{r_0}}). \end{cases}$$

Major results (Cont'd)

3. An algorithm for designing the corresponding controller ***K*** is obtained.

Algorithm 1 Iterative algorithm for computing the controller $\boldsymbol{K}$ satisfying $\mathcal{M}_{[\boldsymbol{K},r_0]} = \hat{\mathcal{F}}_{r_0}$

1: Let $\bar{\boldsymbol{A}} = (\boldsymbol{A}\vec{\ltimes}\boldsymbol{I}_\alpha)(\boldsymbol{I}_\beta \otimes \boldsymbol{1}_{\frac{\alpha}{\beta}})$ and $\bar{\boldsymbol{B}} = \boldsymbol{B} \otimes \boldsymbol{1}_{\frac{\beta}{m}}$

2: Find $\hat{\boldsymbol{K}}$ and a positive definite matrix $\boldsymbol{W}$ to satisfy $\boldsymbol{W}\bar{\boldsymbol{A}}^{\mathrm{T}} + \bar{\boldsymbol{A}}\boldsymbol{W} + \hat{\boldsymbol{K}}^{\mathrm{T}}\bar{\boldsymbol{B}}^{\mathrm{T}} + \bar{\boldsymbol{B}}\hat{\boldsymbol{K}} < 0_{\beta\times\beta}$

3: **for** $i = 1$ to β **do**

4: Find $\boldsymbol{K}_i \in \mathbb{R}^{p\times\frac{\alpha}{\beta}}$ satisfying $\boldsymbol{K}_i\boldsymbol{1}_{\frac{\alpha}{\beta}} = \mathrm{Col}_i(\hat{\boldsymbol{K}}\boldsymbol{W}^{-1})$

5: **end for**

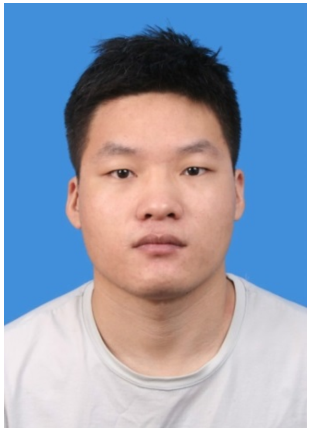
6: Set $\boldsymbol{K} = [\boldsymbol{K}_1,\ \boldsymbol{K}_2, \ldots,\ \boldsymbol{K}_\beta]$

Conclusions

1. The solution to continuous-time cross-dimensional linear systems has been proposed through integral iteration. This solution has a simpler algebraic form than the existing one.

2. An equivalent condition has been obtained when a continuous-time cross-dimensional linear system is asymptotically stable with a given initial value.

3. We have analyzed which initial state can be stabilized, and provided a method to calculate the corresponding controller.



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