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An improved ROF denoising model based on time-fractional derivative

Key words: Improved ROF denoising model; Time-fractional derivative; Caputo derivative; Image denoising

Corresponding author: Xing-ran LIAO
E-mail: xrliao_scu@163.com
ORCID: https://orcid.org/0000-0003-3721-403X

Motivation

- Due to the demand of high quality denosing images in the engineering field, it is important to reduce noise and keep as more edge and texture information as possible.
- The traditional Rudin-Osher-Fatemi (ROF) model cannot effectively maintain the weak edge and texture of images, and its denosing results tend to be smooth block by block, which can not be effectively used in astronomy and medical fields. Using the memory effect of time fraction derivative, we can effectively solve this problem.

Main idea

1. Consider additive noise and introduce the regular term

$$u_0(x,y) = u(x,y) + \eta(x,y) \quad \mathrm{TV}[u(x,y)] = \iint |\nabla u|\, \mathrm{d}x\mathrm{d}y$$

2. Build the optimization model

$$\min_u TV \triangleq \iint_\Omega |\nabla u| dxdy$$

$$s.t. \quad \iint_\Omega (u-u_0)^2 = \sigma^2,$$

3. Functional minimization and Euler-Lagrange equation:

$$E_\beta(\lambda,u) = \iint_\Omega \sqrt{|\nabla u|^2 + \beta}\, dxdy + \frac{\lambda}{2}\left(\iint_\Omega (u-u_0)^2 dxdy - \sigma^2\right)$$

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{|\nabla u|^2 + \beta}}\right) + \lambda(u-u_0) = 0, \quad \text{in } \Omega \subset R^n,$$

4. Gradient downflow method:

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{|\nabla u|^2 + \beta}}\right) + \lambda(u-u_0) = -{}_0^{c}D_t^{\alpha} u(X,t)$$

Method: Caputo difference

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \ (n-1 \le \alpha < n, t > a) $$

1. Discrete integral: divide it into cells with length τ and then sum all parts up.

2. Discrete time derivative: forward difference.

3. Truncation error: Taylor expansion.

$$ {}_0^cD_t^{\alpha} u(X,t) = \frac{1}{\Gamma(1-\alpha)} \sum_{n=0}^{k} \frac{u(X,t_{n+1}) - u(X,t_n)}{\tau} \int_{t_n}^{t_{n+1}} (t_{k+1} - \eta)^{-\alpha} d\eta + \tilde{R}_{k+1} $$

$$ \left| \tilde{R}_{k+1} \right| \le C_3 \tau^{2-\alpha} $$

Method: space difference

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{|\nabla u|^2+\beta}}\right)+\lambda(u-u_0)=-{}_0^cD_t^\alpha u(X,t)$$

1. Discretization method: center difference
2. Discrete left first term $\mathrm{div}\left(\frac{\nabla u(X,t)}{\sqrt{|\nabla u(X,t)|^2+\beta}}\right)=\frac{(u_x^2+\beta)u_{yy}+(u_y^2+\beta)u_{xx}-2u_{xy}u_xu_y}{(u_x^2+u_y^2+\beta)^{3/2}}$
3. Using the θ method to build the explicit and implicit scheme

$$u^{k+1}-\theta\mu\left[\mathrm{div}\left(\frac{\nabla u^{k+1}}{\sqrt{|\nabla u^{k+1}|^2+\beta}}\right)+\lambda(u^{k+1}-u_0)\right]-(1-\theta)\mu\left[\mathrm{div}\left(\frac{\nabla u^{k}}{\sqrt{|\nabla u^{k}|^2+\beta}}\right)+\lambda(u^{k}-u_0)\right]$$

$$=u^k-\sum_{n=1}^{k}b_n\left(u^{k-n+1}-u^{k-n}\right)+R_{k+1}$$

Major results


Gauss filter


ROF model


Original image


Noisy image

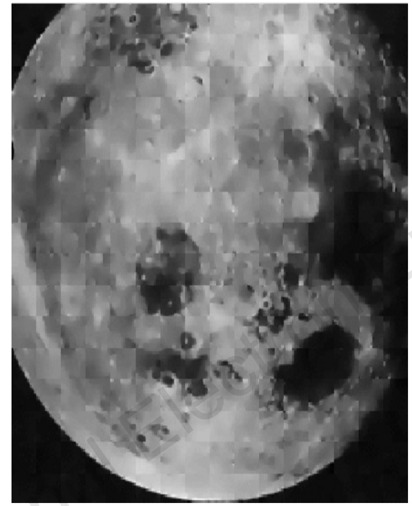

PM model


New model

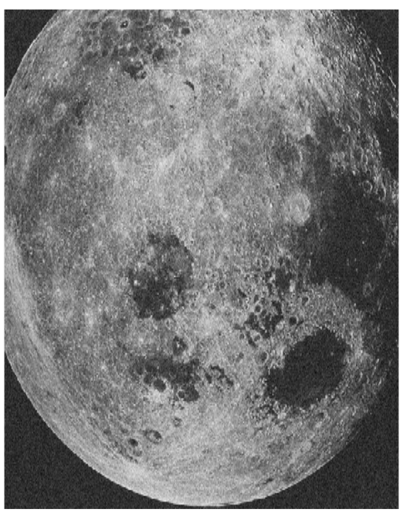
Major results

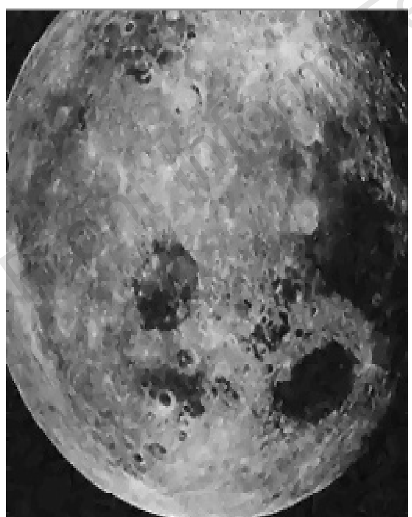

Gauss filter


Explicit ROF


Implicit ROF


Moon origin


Lena noise


New model θ=0


New model θ=1/2


New model θ=α/2

Conclusions

- The new model shows strong denosing ability in both theory and practice, and can be nicely used in the engineering field. The stability and convergence conditions of the numerical scheme are given, and tests on the image are closely related to practical applications.
- The new model has demonstrated its potential. Future work on the fractional derivative should lead the traditional ROF model to a whole new area to achieve better denoising results.