Xiaoxiao HU, Dong CHENG, Kit Ian KOU, 2022. Sampling formulas for 2D quaternionic signals associated with various quaternion Fourier and linear canonical transforms. *Frontiers of Information Technology & Electronic Engineering*, 23(3):463-478. https://doi.org/10.1631/FITEE.2000499

Sampling formulas for 2D quaternionic signals associated with various quaternion Fourier and linear canonical transforms

Key words: Quaternion Fourier transforms; Quaternion linear canonical transforms; Sampling theorem; Quaternion partial and total Hilbert transforms; Generalized quaternion partial and total Hilbert; Truncation errors

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Motivation

- Sampling theory is one of the most important mathematical techniques used in communication engineering, information theory, signal analysis, image processing, and so on.
- The quaternions, which are hyper-complex numbers, have been proved to be an effective tool in quite a few applications of multidimensional signal processing analysis.
- The quaternon Fourier transforms (QFTs) and the quaternion linear canonical transforms (QLCTs) are power tools for using the quaternion modeling technique. Some important theorems associated with QFTs and QLCTs have been studied, such as the inverse theorems and convolution theorems.

Contribution

- To study the quaternionic function sampling formulas bandlimited (BL) to a rectangle that is symmetric about the origin under various QFTs;
- To study the quaternionic function sampling formulas BL to a rectangle that is not symmetric about the origin under various QFTs;
- To explore not only the sampling formulas using samples of themselves, but also samples of the partial derivatives and quaternion partial and total Hilbert transforms;
- To obtain the sampling formulas associated with various QLCTs using relationships between QFTs and QLCTs;
- To estimate the truncation errors of these sampling formulas.

Sampling theorems for QFTs

- If quaternionic function *f*(*x*, *y*) is BL to a rectangle that is symmetric about the origin, then the sampling formulas of *f* under the various types of QFTs are the same.

Theorem 2 (Sampling theorem for QFTs) Suppose that $f \in L^2(\mathbb{R}^2, \mathbb{H})$ is BL to (σ_1, σ_2) in the QFT sense. Then $f(x, y)$ can be reconstructed from its sampled values at the points $(\frac{n\pi}{\sigma_1}, \frac{m\pi}{\sigma_2})$, $(n, m) \in \mathbb{Z}^2$, via the following formula:

$$f(x,y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \Bigg[f(x_n, y_m) \cdot \frac{\sin(\sigma_1(x - x_n))}{\sigma_1(x - x_n)} \frac{\sin(\sigma_2(y - y_m))}{\sigma_2(y - y_m)} \Bigg], \tag{5}$$

where $x_n = \frac{n\pi}{\sigma_1}$ and $y_m = \frac{m\pi}{\sigma_2}$. The series converges in the L^2 norm, and it is absolutely and uniformly convergent on any compact subset of $\mathbb{R}^2$.

Sampling theorems for QFTs

Corollary 1 If $f \in L^2(\mathbb{R}^2, \mathbb{H})$, then $f(x,y)$ can be reconstructed from the samples of its quaternion partial and toltal Hilbert transforms, if one of the following conditions holds:

(1) $\mathcal{F}_{\mathrm{R}}[f](v,u) = 0$, for $|v| > \sigma_1$ or $|u| > \sigma_2$,

(2) $\mathcal{F}_{\mathrm{T}}[f](v,u) = 0$, for $|v| > \sigma_1$ or $|u| > \sigma_2$,

(3) $\mathcal{F}_{\mathrm{L}}[f](v,u) = 0$, for $|v| > \sigma_1$ or $|u| > \sigma_2$.

$$
\begin{aligned}
&f(x,y) \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left[f(\widetilde{x}_n, \widetilde{y}_m) \cos\frac{\sigma_2 \widetilde{Y}_m}{2} \cos\frac{\sigma_1 \widetilde{X}_n}{2} \right. \\
&\quad - \mathcal{H}_x[f](\widetilde{x}_n, \widetilde{y}_m) \cos\frac{\sigma_2 \widetilde{Y}_m}{2} \sin\frac{\sigma_1 \widetilde{X}_n}{2} \\
&\quad - \mathcal{H}_y[f](\widetilde{x}_n, \widetilde{y}_m) \sin\frac{\sigma_2 \widetilde{Y}_m}{2} \cos\frac{\sigma_1 \widetilde{X}_n}{2} \\
&\quad \left. + \mathcal{H}_{xy}[f](\widetilde{x}_n, \widetilde{y}_m) \sin\frac{\sigma_2 \widetilde{Y}_m}{2} \sin\frac{\sigma_1 \widetilde{X}_n}{2} \right] \\
&\quad \cdot \operatorname{sinc}\frac{\sigma_1 \widetilde{X}_n}{2\pi} \operatorname{sinc}\frac{\sigma_2 \widetilde{Y}_m}{2\pi}.
\end{aligned}
$$

Sampling theorems for QFTs

Theorem 7 If $f \in L^2 \cap C^2(\mathbb{R}^2, \mathbb{H})$ is BL to (σ_1, σ_2), and $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y} \in L^2(\mathbb{R}^2, \mathbb{H})$, then the following sampling formula holds:

$$
\begin{aligned}
& f(x, y) \\
& = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left[f(\widetilde{x}_n, \widetilde{y}_m) + (x - \widetilde{x}_n) \frac{\partial f}{\partial x}(\widetilde{x}_n, \widetilde{y}_m) \right. \\
& \quad + (y - \widetilde{y}_m) \frac{\partial f}{\partial y}(\widetilde{x}_n, \widetilde{y}_m) + (x - \widetilde{x}_n)(y - \widetilde{y}_m) \\
& \quad \left. \cdot \frac{\partial^2 f}{\partial x \partial y}(\widetilde{x}_n, \widetilde{y}_m) \right] \left(\operatorname{sinc}\left(\frac{\sigma_1 x}{2\pi} - n \right) \right)^2 \\
& \quad \cdot \left(\operatorname{sinc}\left(\frac{\sigma_2 y}{2\pi} - m \right) \right)^2 .
\end{aligned}
$$

Sampling theorems for QFTs

If $f(x,y)$ is BL to a rectangle that is not symmetric about the origin, then the sampling formula is as follows:

Theorem 3 (Sampling theorem for two-sided QFT) If $f \in L^2(\mathbb{R}^2, \mathbb{H})$ and $\mathcal{F}_{\mathrm{T}}[f](v,u) = 0$, for $|v - v_0| > \sigma_1$ or $|u - u_0| > \sigma_2$, we have

$$f(x,y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left[\mathrm{e}^{\mathrm{i}v_0(x-x_n)} f(x_n, y_m) \mathrm{e}^{\mathrm{j}u_0(y-y_m)} \cdot \mathrm{sinc}\frac{\sigma_1(x-x_n)}{\pi} \mathrm{sinc}\frac{\sigma_2(y-y_m)}{\pi} \right].$$

The series converges in the L^2 norm, and it is absolutely and uniformly convergent on any compact subset of $\mathbb{R}^2$.

Sampling theorems for QFTs

Theorem 4 (Sampling theorem for right-sided QFT) If $f \in L^2(\mathbb{R}^2, \mathbb{H})$ and $\mathcal{F}_{\mathrm{R}}[f](v,u) = 0$, for $|v - v_0| > \sigma_1$ or $|u - u_0| > \sigma_2$, we have

$$\begin{aligned} & f(x,y) \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left[f(x_n, y_m) \mathrm{e}^{\mathrm{i}v_0(x-x_n)} \cos(u_0(y-y_m)) \right. \\ & \quad \left. + f(-x_n, y_m) \mathrm{e}^{-\mathrm{i}v_0(x-x_n)} \sin(u_0(y-y_m)) j \right] \\ & \quad \cdot \operatorname{sinc} \frac{\sigma_1(x-x_n)}{\pi} \operatorname{sinc} \frac{\sigma_2(y-y_m)}{\pi}. \end{aligned}$$

The series converges in the L^2 norm. Moreover, it converges absolutely and uniformly on any compact subset of $\mathbb{R}^2$.

Sampling theorems for QFTs

Theorem 5 (Sampling theorem for left-sided QFT) If $f \in L^2(\mathbb{R}^2, \mathbb{H})$ and $\mathcal{F}_{\mathrm{L}}[f](v,u) = 0$, for $|v - v_0| > \sigma_1$ or $|u - u_0| > \sigma_2$, we have

$$
\begin{aligned}
& f(x,y) \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \Big(\mathrm{e}^{\mathrm{j} u_0 (y - y_m)} f(x_n, y_m) \cos(v_0(x - x_n)) \\
&\quad + \mathrm{e}^{-\mathrm{j} u_0 (y - y_m)} \sin(v_0(x - x_n)) \mathrm{i} f(x_n, -y_m) \Big) \\
&\quad \cdot \operatorname{sinc} \frac{\sigma_1 (x - x_n)}{\pi} \operatorname{sinc} \frac{\sigma_2 (y - y_m)}{\pi}.
\end{aligned}
$$

The series converges in the L^2 norm. Moreover, it converges absolutely and uniformly on any compact subset of $\mathbb{R}^2$.

Sampling theorems for QLCTs

Theorem 9 If $f \in L^2(\mathbb{R}^2, \mathbb{H})$ is BL to (σ_1, σ_2) in the two-sided QLCT sense, then the following sampling formula for f holds:

$$f(x,y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\frac{a_1 s_n^2 - a_1 x^2}{2b_1}} f(s_n, t_m) \mathrm{e}^{\mathrm{j}\frac{a_2 t_m^2 - a_2 y^2}{2b_2}} \cdot \operatorname{sinc}\frac{\sigma_1(x - s_n)}{b_1 \pi} \operatorname{sinc}\frac{\sigma_2(y - t_m)}{b_2 \pi},$$

where $s_n = \frac{n b_1 \pi}{\sigma_1}$ and $t_m = \frac{m b_2 \pi}{\sigma_2}$. The series converges in the L^2 norm. Moreover, it converges absolutely and uniformly on any compact subset of $\mathbb{R}^2$.

Sampling theorems for QLCTs

Theorem 10 If $f \in L^2 \cap C^2(\mathbb{R}^2, \mathbb{H})$ is BL to (σ_1, σ_2) in the two-sided QLCT sense, and $xf, yf, \frac{\partial f}{\partial x}, y\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, x\frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y} \in L^2(\mathbb{R}^2, \mathbb{H})$, then the following sampling series for f holds:

$$\begin{aligned} f(x,y) &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \Bigg[\mathrm{e}^{\mathrm{i}\frac{a_1 \widetilde{s}_n^2}{2b_1}} f(\widetilde{s}_n, \widetilde{t}_m) \mathrm{e}^{\mathrm{j}\frac{a_2 \widetilde{t}_m^2}{2b_2}} + (x - \widetilde{s}_n) \\ &\quad \cdot \Bigg[\mathrm{e}^{\mathrm{i}\frac{a_1 \widetilde{s}_n^2}{2b_1}} \frac{\partial f}{\partial x}(\widetilde{s}_n, \widetilde{t}_m) \mathrm{e}^{\mathrm{j}\frac{a_2 \widetilde{t}_m^2}{2b_2}} + \mathrm{i}\frac{a_1 \widetilde{s}_n}{b_1} \mathrm{e}^{\mathrm{i}\frac{a_1 \widetilde{s}_n^2}{2b_1}} \\ &\quad \cdot f(\widetilde{s}_n, \widetilde{t}_m) \mathrm{e}^{\mathrm{j}\frac{a_2 \widetilde{t}_m^2}{2b_2}} \Bigg] + (y - \widetilde{t}_m) \Bigg[\mathrm{e}^{\mathrm{i}\frac{a_1 \widetilde{s}_n^2}{2b_1}} \frac{\partial f}{\partial y}(\widetilde{s}_n, \widetilde{t}_m) \\ &\quad \cdot \mathrm{e}^{\mathrm{j}\frac{a_2 \widetilde{t}_m^2}{2b_2}} + \mathrm{e}^{\mathrm{i}\frac{a_1 \widetilde{s}_n^2}{2b_1}} f(\widetilde{s}_n, \widetilde{t}_m) \mathrm{e}^{\mathrm{j}\frac{a_2 \widetilde{t}_m^2}{2b_2}} \mathrm{j}\frac{a_2 \widetilde{t}_m}{b_2} \Bigg] \\ &\quad + (x - \widetilde{s}_n)(y - \widetilde{t}_m) \Bigg[\mathrm{e}^{\mathrm{i}\frac{a_1 \widetilde{s}_n^2}{2b_1}} \frac{\partial^2 f}{\partial x \partial y}(\widetilde{s}_n, \widetilde{t}_m) \mathrm{e}^{\mathrm{j}\frac{a_2 \widetilde{t}_m^2}{2b_2}} \\ &\quad + \mathrm{i}\frac{a_1 \widetilde{s}_n}{b_1} \mathrm{e}^{\mathrm{i}\frac{a_1 \widetilde{s}_n^2}{2b_1}} \frac{\partial f}{\partial y}(\widetilde{s}_n, \widetilde{t}_m) \mathrm{e}^{\mathrm{j}\frac{a_2 \widetilde{t}_m^2}{2b_2}} \\ &\quad + \mathrm{e}^{\mathrm{i}\frac{a_1 \widetilde{s}_n^2}{2b_1}} \frac{\partial f}{\partial x}(\widetilde{s}_n, \widetilde{t}_m) \mathrm{e}^{\mathrm{j}\frac{a_2 \widetilde{t}_m^2}{2b_2}} \mathrm{j}\frac{a_2 \widetilde{t}_m}{b_2} \\ &\quad + \mathrm{i}\frac{a_1 \widetilde{s}_n}{b_1} \mathrm{e}^{\mathrm{i}\frac{a_1 \widetilde{s}_n^2}{2b_1}} f(\widetilde{s}_n, \widetilde{t}_m) \mathrm{e}^{\mathrm{j}\frac{a_2 \widetilde{t}_m^2}{2b_2}} \mathrm{j}\frac{a_2 \widetilde{t}_m}{b_2} \Bigg] \Bigg] \\ &\quad \cdot \left(\mathrm{sinc}\left(\frac{\sigma_1 x}{2b_1 \pi} - n \right) \right)^2 \left(\mathrm{sinc}\left(\frac{\sigma_2 y}{b_2 2\pi} - m \right) \right)^2, \end{aligned}$$

where $\widetilde{s}_n = \frac{2nb_1\pi}{\sigma_1}$ and $\widetilde{t}_m = \frac{2mb_2\pi}{\sigma_2}$. The series converges in the L^2 norm. Moreover, it converges absolutely and uniformly on any compact subset of $\mathbb{R}^2$.

Sampling theorems for QLCTs

Theorem 11 If $f \in L^2(\mathbb{R}^2, \mathbb{H})$ is BL to (σ_1, σ_2) in the two-sided QLCT sense, then the following sampling series for f holds:

$$
\begin{aligned}
&f(x,y) \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\Bigg[\mathrm{e}^{\mathrm{i}\frac{a_1(s_n^2-x^2)}{2b_1}} f(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2(t_m^2-y^2)}{2b_2}} \\
&\quad \cdot \cos\frac{\sigma_2\widetilde{T}_m}{2b_2}\cos\frac{\sigma_1\widetilde{s}_n}{2b_1} - \mathrm{e}^{\mathrm{i}\frac{a_1(s_n^2-x^2)}{2b_1}}\mathcal{H}^{x}_{\boldsymbol{A}_1}[f](\widetilde{s}_n,\widetilde{t}_m) \\
&\quad \cdot \mathrm{e}^{\mathrm{j}\frac{a_2(t_m^2-y^2)}{2b_2}}\cos\frac{\sigma_2\widetilde{T}_m}{2b_2}\sin\frac{\sigma_1\widetilde{S}_n}{2b_1} - \mathrm{e}^{\mathrm{i}\frac{a_1(s_n^2-x^2)}{2b_1}} \\
&\quad \cdot \mathcal{H}^{y}_{\boldsymbol{A}_2}[f](\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2(t_m^2-y^2)}{2b_2}}\sin\frac{\sigma_2\widetilde{T}_m}{2b_2}\cos\frac{\sigma_1\widetilde{S}_n}{2b_1} \\
&\quad + \mathrm{e}^{\mathrm{i}\frac{a_1(s_n^2-x^2)}{2b_1}}\mathcal{H}^{xy}_{\boldsymbol{A}_1\boldsymbol{A}_2}[f](\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2(t_m^2-y^2)}{2b_2}} \\
&\quad \cdot \sin\frac{\sigma_2\widetilde{T}_m}{2b_2}\sin\frac{\sigma_1\widetilde{S}_n}{2b_1}\Bigg]\mathrm{sinc}\frac{\sigma_1\widetilde{S}_n}{2b_1\pi}\mathrm{sinc}\frac{\sigma_2\widetilde{T}_m}{2b_2\pi},
\end{aligned}
$$

where $\widetilde{S}_n = x - \widetilde{s}_n$ and $\widetilde{T}_m = y - \widetilde{t}_m$.

Truncation errors for QFT

Truncation error occurs naturally in applications, because only a finite number of samples are given in practice. For $f \in L^2(\mathbb{R}^2, \mathbb{H})$ which is BL to (σ_1, σ_2) in the QFT sense, let $\mathrm{R1}_{N,M}, \mathrm{R2}_{N,M}$, and $\mathrm{R3}_{N,M}$ denote the truncation errors of $f(x, y)$:

$$\begin{aligned} &\mathrm{R1}_{N,M}(x,y) \\ &:= f(x,y) - \sum_{n=-N}^{N} \sum_{m=-M}^{M} f(nh_1, mh_2) \\ &\quad \cdot \frac{\sin(\sigma_1(x-nh_1))}{\sigma_1(x-nh_1)} \frac{\sin(\sigma_2(y-mh_2))}{\sigma_2(y-mh_2)}, \end{aligned} \tag{22}$$

where $h_1 := \pi/\sigma_1$ and $h_2 := \pi/\sigma_2$,

$$\begin{aligned} &\mathrm{R2}_{N,M}(x,y) \\ &:= f(x,y) - \sum_{n=-N}^{N} \sum_{m=-M}^{M} \Big[f(\widetilde{x}_n, \widetilde{y}_m) \cos\frac{\sigma_2 \widetilde{Y}_m}{2} \\ &\quad \cdot \cos\frac{\sigma_1 \widetilde{X}_n}{2} - \mathcal{H}_x[f](\widetilde{x}_n, \widetilde{y}_m) \cos\frac{\sigma_2 \widetilde{Y}_m}{2} \sin\frac{\sigma_1 \widetilde{X}_n}{2} \\ &\quad - \mathcal{H}_y[f](\widetilde{x}_n, \widetilde{y}_m) \sin\frac{\sigma_2 \widetilde{Y}_m}{2} \cos\frac{\sigma_1 \widetilde{X}_n}{2} \\ &\quad + \mathcal{H}_{xy}[f](\widetilde{x}_n, \widetilde{y}_m) \sin\frac{\sigma_2 \widetilde{Y}_m}{2} \sin\frac{\sigma_1 \widetilde{X}_n}{2} \Big] \\ &\quad \cdot \operatorname{sinc}\frac{\sigma_1 \widetilde{X}_n}{2\pi} \operatorname{sinc}\frac{\sigma_2 \widetilde{Y}_m}{2\pi}, \end{aligned} \tag{23}$$

$$\begin{aligned} &\mathrm{R3}_{N,M}(x,y) \\ &:= f(x,y) - \sum_{n=K_1(x)-N}^{K_1(x)+N} \sum_{m=K_2(y)-M}^{K_2(y)+M} \Big[f(n_2h_1, m_2h_2) \\ &\quad + (x - n_2h_1) \frac{\partial f}{\partial x}(n_2h_1, m_2h_2) \\ &\quad + (y - m_2h_2) \frac{\partial f}{\partial y}(n_2h_1, m_2h_2) \\ &\quad + (x - n_2h_1)(y - \widetilde{y}_m) \frac{\partial^2 f}{\partial x \partial y}(n_2h_1, m_2h_2) \Big] \\ &\quad \cdot \left(\operatorname{sinc}\left(\frac{x}{2h_1} - n \right) \right)^2 \left(\operatorname{sinc}\left(\frac{y}{2h_2} - m \right) \right)^2, \end{aligned} \tag{24}$$

where $\widetilde{x}_n = n_2h_1, \widetilde{y}_m = m_2h_2, \widetilde{X}_n = x - n_2h_1, \widetilde{Y}_m = y - m_2h_2, \frac{1}{2h_1} - \frac{1}{2} < K_1(x) \le \frac{1}{2h_1} + \frac{1}{2}, \frac{1}{2h_2} - \frac{1}{2} < K_2(y) \le \frac{1}{2h_2} + \frac{1}{2}$, and $(K_1(x), K_2(y))$ are integer points nearest to the truncation error observation point (x, y). The following lemmas are used to prove the estimation of truncation errors $\mathrm{R1}_{N,M}$ and $\mathrm{R2}_{N,M}$.

Truncation errors for QFT

Theorem 13 For $f \in L^2(\mathbb{R}^2, \mathbb{H})$ which is BL to (σ_1, σ_2) in the QFT sense, let $\mathrm{R1}_{N,M}$ be defined by Eq. (22), where $|x| < Nh_1$, $|y| < Mh_2$, $N \geq 1$, $M \geq 1$,

$$K_N = \left(h_1^2 \sum_{|n|>N} \sum_{m=-\infty}^{+\infty} |f(nh_1, mh_2)|^2 \right)^{\frac{1}{2}},$$

$$L_M = \left(h_2^2 \sum_{|m|>M} \sum_{n=-\infty}^{+\infty} |f(nh_1, mh_2)|^2 \right)^{\frac{1}{2}},$$

$$J_{N,M} = \left(h_1^2 h_2^2 \sum_{|m|>M} \sum_{|n|>N} |f(nh_1, mh_2)|^2 \right)^{\frac{1}{2}}.$$

Then

$$|\mathrm{R1}_{N,M}(x,y)| \leq I_1(x,y,N) + I_2(x,y,M) + \frac{2J_{N,M}\sqrt{MN}}{\pi^2\sqrt{((Nh_1)^2 - x^2)((Mh_2)^2 - y^2)}},$$

where

$$I_1(x,y,N) := \frac{2\sqrt{2N}}{\pi} |\sin(\sigma_1 x)| \frac{K_N}{\sqrt{(Nh_1)^2 - x^2}},$$

$$I_2(x,y,M) := \frac{2\sqrt{2M}}{\pi} |\sin(\sigma_2 y)| \frac{L_M}{\sqrt{(Mh_2)^2 - y^2}}.$$

Truncation errors for QFT

Theorem 14 Let $\mathrm{R2}_{N,M}$ be defined by Eq. (24), $|x| < Nh_1, |y| < Mh_2, N \geq 1, M \geq 1$,

$$\mathrm{K}i_N = \frac{1}{2}\left(h_1^2 \sum_{|n|>N} \sum_{m=-\infty}^{+\infty} |f_i(2nh_1, 2mh_2)|^2\right)^{\frac{1}{2}},$$

$$\mathrm{L}i_M = \frac{1}{2}\left(h_2^2 \sum_{|m|>M} \sum_{n=-\infty}^{+\infty} |f_i(2nh_1, 2mh_2)|^2\right)^{\frac{1}{2}},$$

$$\mathrm{J}i_{N,M} = \frac{1}{4}\left(h_1^2 h_2^2 \sum_{|m|>M} \sum_{|n|>N} |f_i(2nh_1, 2mh_2)|^2\right)^{\frac{1}{2}},$$

where $i = 2, 3, 4, 5$, $f_2 = f$, $f_3 = \mathcal{H}_x[f]$, $f_4 = \mathcal{H}_y[f]$, and $f_5 = \mathcal{H}_{xy}[f]$. Then

$$|\mathrm{R2}_{N,M}(x, y)| \leq S_2(x, y) + S_3(x, y) + S_4(x, y) + S_5(x, y),$$

where

$$S_j(x, y, N) = \frac{2\sqrt{2N}}{\pi}\left|\sin\left(\frac{\sigma_1 x}{2}\right)\right| \frac{\mathrm{K}j_N}{\sqrt{(\frac{Nh_1}{2})^2 - x^2}} + \frac{2\sqrt{2M}}{\pi}\left|\sin\left(\frac{\sigma_2 y}{2}\right)\right| \frac{\mathrm{L}j_M}{\sqrt{(Mh_2)^2 - y^2}} + \frac{2\sqrt{MN}}{\pi^2} \frac{\mathrm{J}j_{N,M}}{\sqrt{((\frac{Nh_1}{2})^2 - x^2)\,((\frac{Mh_2}{2})^2 - y^2)}}.$$

Herein, $j = 2, 3, 4, 5$.

Truncation errors for QFT

Theorem 15 If quaternionic function $f(x, y)$ is BL to $(r_1\sigma_1, r_2\sigma_2)$ in the QFT sense, where $0 \leq r_i < 1$ $(i = 1, 2)$ and $|f(x, y)| \leq C$, for $(x, y) \in \mathbb{R}^2$ and $C > 0$, then an upper bound for the truncation error $\mathrm{R3}_{N,M}$ at point (x, y) is given by

$$|\mathrm{R3}_{N,M}(x,y)| \leq \frac{2C\left|\sin\left(\frac{y\pi}{2h_2}\right)\right|^2}{\frac{\pi M \sin(\pi r_2)}{r_2}} + \frac{2C\left|\sin\left(\frac{x\pi}{2h_1}\right)\right|^2}{\frac{\pi N \sin(\pi r_1)}{r_1}} + \frac{4C\left|\sin\left(\frac{x\pi}{2h_1}\right)\sin\left(\frac{y\pi}{2h_2}\right)\right|^2}{\frac{\pi N \sin(\pi r_1)}{r_1}\frac{\pi M \sin(\pi r_2)}{r_2}}.$$

Truncation errors for QLCT

If $f(x,y)$ is BL to (σ_1,σ_2) in the two-sided QLCT sense, Lemma 8 implies that $\mathrm{e}^{\mathrm{i}a_1x^2/(2b_1)}f(x,y)\mathrm{e}^{\mathrm{j}a_2y^2/(2b_2)}$ is BL to $(-\frac{\sigma_1}{b_1},\frac{\sigma_2}{b_2})$ in the two-sided QFT sense. So, we have the following truncation errors in the QLCT sense from Theorems 13–15. Let $\breve{\mathrm{R}}1_{N,M}$, $\breve{\mathrm{R}}2_{N,M}$, and $\breve{\mathrm{R}}3_{N,M}$ denote the truncation errors of $f(x,y)$ as follows (Assume $\breve{h}_1 := b_1\pi/\sigma_1$ and $\breve{h}_2 = b_2\pi/\sigma_2$) :

$$\begin{aligned}&\breve{\mathrm{R}}1_{N,M}(x,y)\\ =&f(x,y)-\sum_{n=-N}^{N}\sum_{m=-M}^{M}\mathrm{e}^{\mathrm{i}\frac{a_1(s_n^2-x^2)}{2b_1}}f(s_n,t_m)\\ &\cdot\mathrm{e}^{\mathrm{j}\frac{a_2(t_m^2-y^2)}{2b_2}}\mathrm{sinc}\frac{\sigma_1(x-s_n)}{b_1\pi}\mathrm{sinc}\frac{\sigma_2(y-t_m)}{b_2\pi},\end{aligned} \tag{26}$$

$$\begin{aligned}&\breve{\mathrm{R}}2_{N,M}(x,y)\\ =&f(x,y)-\sum_{n=-N}^{N}\sum_{m=-M}^{M}\Big[\mathrm{e}^{\mathrm{i}\frac{a_1(s_n^2-x^2)}{2b_1}}f(\widetilde{s}_n,\widetilde{t}_m)\\ &\cdot\mathrm{e}^{\mathrm{j}\frac{a_2(t_m^2-y^2)}{2b_2}}\cos\frac{\sigma_2\widetilde{T}_m}{2b_2}\cos\frac{\sigma_1\widetilde{s}_n}{2b_1}-\mathrm{e}^{\mathrm{i}\frac{a_1(s_n^2-x^2)}{2b_1}}\\ &\cdot\mathcal{H}^x_{\boldsymbol{A}_1}[f](\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2(t_m^2-y^2)}{2b_2}}\cos\frac{\sigma_2\widetilde{T}_m}{2b_2}\sin\frac{\sigma_1\widetilde{S}_n}{2b_1}\\ &-\mathrm{e}^{\mathrm{i}\frac{a_1(s_n^2-x^2)}{2b_1}}\mathcal{H}^y_{\boldsymbol{A}_2}[f](\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2(t_m^2-y^2)}{2b_2}}\\ &\cdot\sin\frac{\sigma_2\widetilde{T}_m}{2b_2}\cos\frac{\sigma_1\widetilde{S}_n}{2b_1}+\mathrm{e}^{\mathrm{i}\frac{a_1(s_n^2-x^2)}{2b_1}}\mathcal{H}^{xy}_{\boldsymbol{A}_1\boldsymbol{A}_2}[f](\widetilde{s}_n,\widetilde{t}_m)\\ &\cdot\mathrm{e}^{\mathrm{j}\frac{a_2(t_m^2-y^2)}{2b_2}}\sin\frac{\sigma_2\widetilde{T}_m}{2b_2}\sin\frac{\sigma_1\widetilde{S}_n}{2b_1}\Big]\\ &\cdot\mathrm{sinc}\frac{\sigma_1\widetilde{S}_n}{2b_1\pi}\mathrm{sinc}\frac{\sigma_2\widetilde{T}_m}{2b_2\pi},\end{aligned} \tag{27}$$

Truncation errors for QLCT

$$
\begin{aligned}
&\breve{\mathrm{R}}3_{N,M} \\
=&f(x,y)-\sum_{|n|\le N}\sum_{|m|\le M}\Bigg[\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}f(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}} \\
&+(x-\widetilde{s}_n)\Bigg[\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}\frac{\partial f}{\partial x}(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}} \\
&+\mathrm{i}\frac{a_1\widetilde{s}_n}{b_1}\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}f(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}}\Bigg] \\
&+(y-\widetilde{t}_m)\Bigg[\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}\frac{\partial f}{\partial y}(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}} \\
&+\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}f(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}}\mathrm{j}\frac{a_2\widetilde{t}_m}{b_2}\Bigg] \\
&+(x-\widetilde{s}_n)(y-\widetilde{t}_m)\Bigg[\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}\frac{\partial^2 f}{\partial x\partial y}(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}} \\
&+\mathrm{i}\frac{a_1\widetilde{s}_n}{b_1}\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}\frac{\partial f}{\partial y}(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}} \\
&+\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}\frac{\partial f}{\partial x}(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}}\mathrm{j}\frac{a_2\widetilde{t}_m}{b_2} \\
&+\mathrm{i}\frac{a_1\widetilde{s}_n}{b_1}\mathrm{e}^{\mathrm{i}\frac{a_1\widetilde{s}_n^2}{2b_1}}f(\widetilde{s}_n,\widetilde{t}_m)\mathrm{e}^{\mathrm{j}\frac{a_2\widetilde{t}_m^2}{2b_2}}\mathrm{j}\frac{a_2\widetilde{t}_m}{b_2}\Bigg]\Bigg] \\
&\cdot\left(\operatorname{sinc}\left(\frac{\sigma_1 x}{2b_1\pi}-n\right)\right)^2\left(\operatorname{sinc}\left(\frac{\sigma_2 y}{b_2 2\pi}-m\right)\right)^2.
\end{aligned}
$$

Truncation errors for QLCT

Corollary 2 Suppose that $f \in L^2(\mathbb{R}^2, \mathbb{H})$ is BL to (σ_1, σ_2) in the two-sided QLCT sense. Let $\breve{\mathrm{R}}1_{N,M}$ be defined by Eq. (26), where $|x| < N\breve{h}_1, |y| < M\breve{h}_2, N \geq 1, M \geq 1,$

$$\breve{K}_N = \left(\breve{h}_1^2 \sum_{|n|>N} \sum_{m=-\infty}^{+\infty} |f(n\breve{h}_1, m\breve{h}_2)|^2 \right)^{\frac{1}{2}},$$

$$\breve{L}_M = \left(\breve{h}_2^2 \sum_{|m|>M} \sum_{n=-\infty}^{+\infty} |f(n\breve{h}_1, m\breve{h}_2)|^2 \right)^{\frac{1}{2}},$$

$$\breve{J}_{N,M} = \left(\breve{h}_1^2 \breve{h}_2^2 \sum_{|m|>M} \sum_{|n|>N} |f(n\breve{h}_1, m\breve{h}_2)|^2 \right)^{\frac{1}{2}}.$$

Then

$$|\breve{\mathrm{R}}1_{N,M}(x,y)| \leq \breve{I}_1(x,y,N) + \breve{I}_2(x,y,M) + \frac{2\breve{J}_{N,M}\sqrt{MN}}{\pi^2 \sqrt{((N\breve{h}_1)^2 - x^2)((M\breve{h}_2)^2 - y^2)}},$$

where

$$\breve{I}_1(x,y,N) := \frac{2\sqrt{2N}}{\pi} \left| \sin\left(\frac{\sigma_1 x}{b_1}\right) \right| \frac{\breve{K}_N}{\sqrt{(N\breve{h}_1)^2 - x^2}},$$

$$\breve{I}_2(x,y,M) := \frac{2\sqrt{2M}}{\pi} \left| \sin\left(\frac{\sigma_2 y}{b_2}\right) \right| \frac{\breve{L}_M}{\sqrt{(M\breve{h}_2)^2 - y^2}}.$$

Truncation errors for QLCT

Corollary 3 Let $\breve{\mathrm{R}}2_{N,M}$ be defined by Eq. (27), $|x| < N\breve{h}_1, |y| < M\breve{h}_2, N \geq 1, M \geq 1,$

$$\breve{\mathrm{K}}i_N = \frac{1}{2}\left(\breve{h}_1^2 \sum_{|n|>N} \sum_{m=-\infty}^{+\infty} |f_i(2n\breve{h}_1, 2m\breve{h}_2)|^2\right)^{\frac{1}{2}},$$

$$\breve{\mathrm{L}}i_M = \frac{1}{2}\left(\breve{h}_2^2 \sum_{|m|>M} \sum_{n=-\infty}^{+\infty} |f_i(2n\breve{h}_1, 2m\breve{h}_2)|^2\right)^{\frac{1}{2}},$$

$$\breve{\mathrm{J}}i_{N,M} = \frac{1}{4}\left(\breve{h}_1^2\breve{h}_2^2 \sum_{|m|>M} \sum_{|n|>N} |f_i(2n\breve{h}_1, 2m\breve{h}_2)|^2\right)^{\frac{1}{2}},$$

where $i = 2,3,4,5$, $f_2 = f$, $f_3 = \mathcal{H}^x_{\boldsymbol{A}_1}[f]$, $f_4 = \mathcal{H}^y_{\boldsymbol{A}_2}[f]$, and $f_5 = \mathcal{H}^{xy}_{\boldsymbol{A}_1\boldsymbol{A}_2}[f]$. Then we have

$$|\breve{\mathrm{R}}2_{N,M}(x,y)| \leq \breve{S}_2(x,y) + \breve{S}_3(x,y) + \breve{S}_4(x,y) + \breve{S}_5(x,y),$$

where

$$\breve{S}_j(x,y,N) = \frac{2\sqrt{2N}}{\pi}\left|\sin\left(\frac{\sigma_1 x}{2b_1}\right)\right| \frac{\breve{\mathrm{K}}j_N}{\sqrt{(\frac{N\breve{h}_1}{2})^2 - x^2}} + \frac{2\sqrt{2M}}{\pi}\left|\sin\left(\frac{\sigma_2 y}{2b_2}\right)\right| \frac{\breve{\mathrm{L}}j_M}{\sqrt{(Mh_2)^2 - y^2}} + \frac{2\sqrt{MN}}{\pi^2} \frac{\breve{\mathrm{J}}j_{N,M}}{\sqrt{((\frac{N\breve{h}_1}{2})^2 - x^2)((\frac{M\breve{h}_2}{2})^2 - y^2)}}.$$

Herein, $j = 2,3,4,5$.

Truncation errors for QLCT

Corollary 4 If quaternionic function $f(x,y)$ is BL to $(r_1\sigma_1/b_1, r_2\sigma_2/b_2)$ in the QFT sense, where $0 \le r_i < 1$ $(i=1,2)$ and $|f(x,y)| \le C$, for $(x,y) \in \mathbb{R}^2$ and $C > 0$, then an upper bound for the truncation error $\breve{R}3_{N,M}$ at point (x,y) is given by

$$|\breve{R}3_{N,M}(x,y)| \le \frac{2C\left|\sin\left(\frac{y\pi}{2\breve{h}_2}\right)\right|^2}{\frac{\pi M \sin(\pi r_2)}{r_2}} + \frac{2C\left|\sin\left(\frac{x\pi}{2\breve{h}_1}\right)\right|^2}{\frac{\pi N \sin(\pi r_1)}{r_1}}$$

$$+ \frac{4C\left|\sin\left(\frac{x\pi}{2\breve{h}_1}\right)\sin\left(\frac{y\pi}{2\breve{h}_2}\right)\right|^2}{\frac{\pi N \sin(\pi r_1)}{r_1}\frac{\pi M \sin(\pi r_2)}{r_2}}.$$

Simulation results

Algorithm 1 Image reconstruction

1: Input the test color image $f(t_1, t_2)$ and convert the color image into the quaternion form.
2: The test image is downsampled by factor 2.
3: Generate a high-resolution (HR) image from the down-sampled image by Eq. (8).
4: Compute the SSIM and FSIM to evaluate the quality of the generated HR image.

Table 1 SSIM and FSIM values of the reconstructed images

Image	SSIM	FSIM
Lena	0.9440	0.8912
Flower	0.9554	0.9495
Bird	0.9462	0.9353
House	0.9357	0.8822
Pepper	0.9559	0.8953
Horse	0.8531	0.8751

Fig. 2 Reconstructed images for Lena, flower, and bird by Algorithm 1

The first row shows the original images. The second row shows the degraded images with the resolution of 128×128. The third row shows the reconstructed images

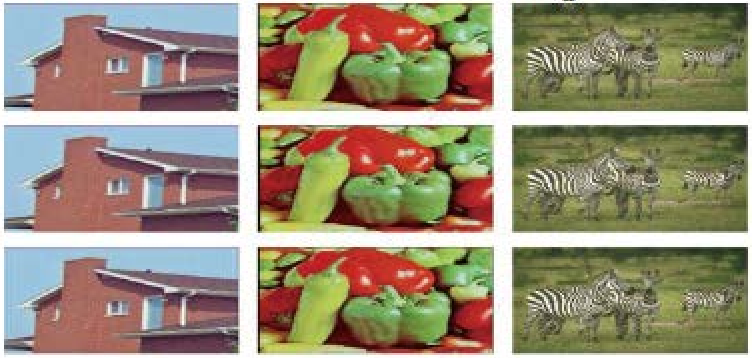

Fig. 3 Reconstructed images for house, pepper, and horse by Algorithm 1

The first row shows the original images. The second row shows the degraded images with the resolution of 128×128. The third row shows the reconstructed images

Conclusions

- If the quaternionic function is bandlimited to a rectangle that is symmetric about the origin, then the sampling formula associated with various QFTs is identical.
- If the quaternionic function is bandlimited to a rectangle that is not symmetric about the origin, then the sampling formulas associated with various QFTs are different.
- We obtain not only the sampling formulas using the samples, but also the sampling series using samples of the partial derivatives and quaternion partial and total Hilbert transforms.
- The sampling formulas associated with various QLCTs are obtained by the relationships of QFTs and QLCTs.
- The truncation errors of those sampling formulas are derived.
- By Algorithm 1, the sampling formula is applied to color image reconstruction.

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