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A novel multiple-outlier-robust Kalman filter

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Key words: Kalman filtering; Multiple statistical similarity measure; Multiple outliers; Fixed-point iteration; State estimate

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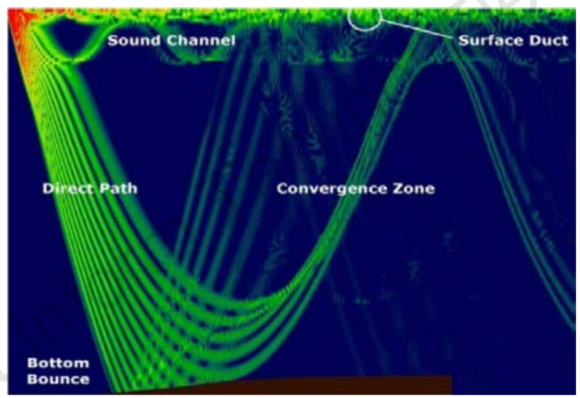
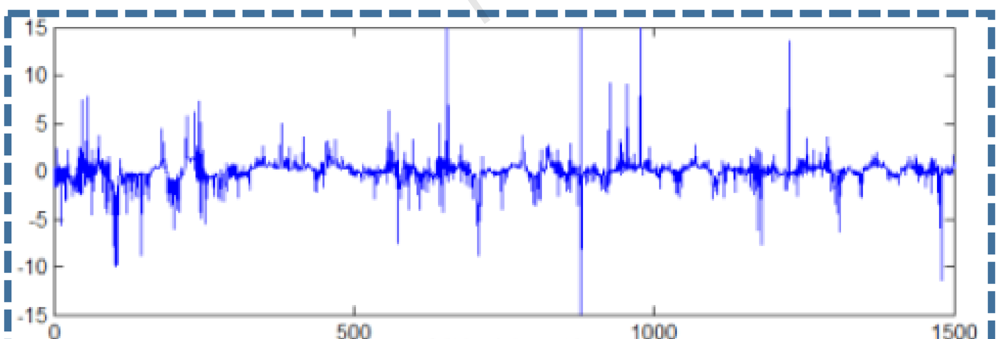
Motivation

Outliers are common in underwater applications because of the changing marine environment, sound velocity variation, and severe maneuver of vehicles.





Ranging error samples including outliers

Motivation (Cont'd)

1. The outliers occurring in different state and measurement dimensions may possess different statistical properties in practical applications.

2. The newly emerging statistical similarity measure based Kalman filtering (SSMKF) is incapable of addressing multiple outliers because it was designed based on an assumption that the outliers occurring in different state and measurement dimensions have the same statistical properties.

3. The existing M-estimator can to some extent reduce the effects of multiple outliers, but the randomness inherent in the state vector is neglected, which limits its estimation accuracy.

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Main idea

1. Proposing a new multiple statistical similarity measure (MSSM) to evaluate the similarity between two random vectors from dimension to dimension.

2. Maximizing an MSSM-based cost function by the fixed-iteration method to develop a multiple-outlier-robust estimator.

3. Convergence is guaranteed under mild conditions. The similarity function selection is also presented.

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Method

1. To evaluate the similarity between two random vectors from dimension to dimension, we define a novel MSSM as

$$s(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{i=1}^{p} \int \int f((\alpha_i - \beta_i)^2) p(\boldsymbol{\alpha}, \boldsymbol{\beta}) \mathrm{d}\boldsymbol{\alpha} \mathrm{d}\boldsymbol{\beta},$$

where $f(\cdot)$ represents the similarity function satisfying:

- $f(\cdot)$ is continuous and differentiable on $[0,+\infty)$;
- $\dot{f}(l) < 0$ on $[0,+\infty)$;
- $\ddot{f}(l) \geq 0$ on $[0,+\infty)$.

$f(l)$	$\dot{f}(l)$	$\ddot{f}(l)$
$-0.5l$	-0.5	0
$\sigma^2 \exp\left(\frac{1-l}{2\sigma^2}\right)$	$-0.5 \exp\left(\frac{1-l}{2\sigma^2}\right)$	$\frac{1}{4\sigma^2} \exp\left(\frac{1-l}{2\sigma^2}\right)$
$-0.5(\nu+1)\log(1+\frac{l}{\nu})$	$-0.5\frac{\nu+1}{\nu+l}$	$0.5\frac{\nu+1}{(\nu+l)^2}$
$-\sqrt{(\omega+1)(\omega+l)}$	$-0.5\sqrt{\frac{\omega+1}{\omega+l}}$	$0.25\frac{\sqrt{\omega+1}}{\sqrt[3]{\omega+l}}$

σ: kernel size; ν and ω: degree-of-freedom (DOF) parameters

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Method (Cont'd)

2. Construct an MSSM-based cost function. The aim is to look for an optimal posterior probability density function (PDF):

$$q^*(\boldsymbol{x}_k) = \arg\max_{q(\boldsymbol{x}_k)} \left\{ s\left(\boldsymbol{S}_{k|k-1}^{-1}\boldsymbol{x}_k, \boldsymbol{S}_{k|k-1}^{-1}\hat{\boldsymbol{x}}_{k|k-1}\right) + s\left(\boldsymbol{S}_{\boldsymbol{R}_k}^{-1}\boldsymbol{z}_k, \boldsymbol{S}_{\boldsymbol{R}_k}^{-1}\boldsymbol{H}_k\boldsymbol{x}_k\right) \right\}$$

$$\boldsymbol{P}_{k|k-1} = \boldsymbol{S}_{k|k-1}\boldsymbol{S}_{k|k-1}^{\mathrm{T}}, \ \boldsymbol{R}_k = \boldsymbol{S}_{\boldsymbol{R}_k}\boldsymbol{S}_{\boldsymbol{R}_k}^{\mathrm{T}}.$$

$$q^*(\boldsymbol{x}_k) = \arg\max_{q(\boldsymbol{x}_k)} \left\{ \sum_{i=1}^{n} \int f_x([\boldsymbol{T}_{ki}(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k|k-1})]^2) \cdot q(\boldsymbol{x}_k)\mathrm{d}\boldsymbol{x}_k + \sum_{j=1}^{m} \int f_z([\boldsymbol{U}_{kj}(\boldsymbol{z}_k - \boldsymbol{H}_k\boldsymbol{x}_k)]^2) q(\boldsymbol{x}_k)\mathrm{d}\boldsymbol{x}_k \right\}$$

Compared with the existing M-estimators, the proposed method considers the randomness of the state.

- The explicit form of the posterior PDF is unavailable
- The integrals above cannot be solved analytically

$$\boldsymbol{S}_{k|k-1}^{-1} = \left[\boldsymbol{T}_{k1}^{\mathrm{T}}, \boldsymbol{T}_{k2}^{\mathrm{T}}, \ldots, \boldsymbol{T}_{kn}^{\mathrm{T}}\right]^{\mathrm{T}},$$

$$\boldsymbol{S}_{\boldsymbol{R}_k}^{-1} = \left[\boldsymbol{U}_{k1}^{\mathrm{T}}, \boldsymbol{U}_{k2}^{\mathrm{T}}, \ldots, \boldsymbol{U}_{km}^{\mathrm{T}}\right]^{\mathrm{T}}.$$

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Method (Cont'd)

3. Two essential approximations are made to address the optimization problem above:

➢ Approximate the posterior PDF as Gaussian:

$$q(\boldsymbol{x}_k) \approx N(\boldsymbol{x}_k; \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$$

➢ Substitute the original cost function by its lower bound:

$$\int f_x([\boldsymbol{T}_{ki}(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k|k-1})]^2) q(\boldsymbol{x}_k) \mathrm{d}\boldsymbol{x}_k \geq f_x\left(\int [\boldsymbol{T}_{ki}(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k|k-1})]^2 q(\boldsymbol{x}_k) \mathrm{d}\boldsymbol{x}_k\right)$$

$$\int f_z([\boldsymbol{U}_{kj}(\boldsymbol{z}_k - \boldsymbol{H}_k \boldsymbol{x}_k)]^2) q(\boldsymbol{x}_k) \mathrm{d}\boldsymbol{x}_k \geq f_z\left(\int [\boldsymbol{U}_{kj}(\boldsymbol{z}_k - \boldsymbol{H}_k \boldsymbol{x}_k)]^2 q(\boldsymbol{x}_k) \mathrm{d}\boldsymbol{x}_k\right)$$

$$\{\boldsymbol{\mu}_k^*, \boldsymbol{\Sigma}_k^*\} \approx \arg \max_{\{\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k\}} J(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k) \quad \text{s.t.} \quad \boldsymbol{\Sigma}_k > 0$$

$$J(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k) = \sum_{i=1}^{n} f_x\left(\boldsymbol{T}_{ki} \boldsymbol{A}_k \boldsymbol{T}_{ki}^{\mathrm{T}}\right) + \sum_{j=1}^{m} f_z\left(\boldsymbol{U}_{kj} \boldsymbol{B}_k \boldsymbol{U}_{kj}^{\mathrm{T}}\right)$$

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Theoretical result

4. Modified Kalman-like update:

$$\tilde{\boldsymbol{K}}_k^* = \tilde{\boldsymbol{P}}_{k|k-1}^* \boldsymbol{H}_k^{\mathrm{T}} \left(\boldsymbol{H}_k \tilde{\boldsymbol{P}}_{k|k-1}^* \boldsymbol{H}_k^{\mathrm{T}} + \tilde{\boldsymbol{R}}_k^* \right)^{-1},$$
$$\boldsymbol{\mu}_k^* = \hat{\boldsymbol{x}}_{k|k-1} + \tilde{\boldsymbol{K}}_k^* (\boldsymbol{z}_k - \boldsymbol{H}_k \hat{\boldsymbol{x}}_{k|k-1}),$$

$$\tilde{\boldsymbol{P}}_{k|k-1}^* = \boldsymbol{S}_{k|k-1} \left(\boldsymbol{\varPsi}_{\boldsymbol{x}_k}^* \right)^{-1} \boldsymbol{S}_{k|k-1}^{\mathrm{T}},$$
$$\tilde{\boldsymbol{R}}_k^* = \boldsymbol{S}_{\boldsymbol{R}_k} \left(\boldsymbol{\varPsi}_{\boldsymbol{z}_k}^* \right)^{-1} \boldsymbol{S}_{\boldsymbol{R}_k}^{\mathrm{T}}.$$

with the auxiliary matrices calculated by

$$\begin{aligned}\boldsymbol{\varPsi}_{\boldsymbol{x}_k}^* =& \mathrm{diag}\Big(-2\dot{f}_x(\boldsymbol{T}_{k1}\boldsymbol{A}_k^*\boldsymbol{T}_{k1}^{\mathrm{T}}), -2\dot{f}_x(\boldsymbol{T}_{k2}\boldsymbol{A}_k^*\boldsymbol{T}_{k2}^{\mathrm{T}}),\\ &\ldots, -2\dot{f}_x(\boldsymbol{T}_{kn}\boldsymbol{A}_k^*\boldsymbol{T}_{kn}^{\mathrm{T}})\Big),\\ \boldsymbol{\varPsi}_{\boldsymbol{z}_k}^* =& \mathrm{diag}\Big(-2\dot{f}_z(\boldsymbol{U}_{k1}\boldsymbol{B}_k^*\boldsymbol{U}_{k1}^{\mathrm{T}}), -2\dot{f}_z(\boldsymbol{U}_{k2}\boldsymbol{B}_k^*\boldsymbol{U}_{k2}^{\mathrm{T}}),\\ &\ldots, -2\dot{f}_z(\boldsymbol{U}_{km}\boldsymbol{B}_k^*\boldsymbol{U}_{km}^{\mathrm{T}})\Big).\end{aligned}$$

$$\boldsymbol{A}_k^* = \boldsymbol{\varSigma}_k^* + (\boldsymbol{\mu}_k^* - \hat{\boldsymbol{x}}_{k|k-1})(\boldsymbol{\mu}_k^* - \hat{\boldsymbol{x}}_{k|k-1})^{\mathrm{T}},$$
$$\boldsymbol{B}_k^* = (\boldsymbol{z}_k - \boldsymbol{H}_k\boldsymbol{\mu}_k^*)(\boldsymbol{z}_k - \boldsymbol{H}_k\boldsymbol{\mu}_k^*)^{\mathrm{T}} + \boldsymbol{H}_k\boldsymbol{\varSigma}_k^*\boldsymbol{H}_k^{\mathrm{T}}.$$

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Simulation study

Noise settings:

Case 1:

$$\begin{cases} \boldsymbol{w}_k \sim \begin{cases} N(\mathbf{0}, \boldsymbol{Q}), & \text{w.p. } 0.97, \\ N(\mathbf{0}, U_1\boldsymbol{Q}), & \text{w.p. } 0.03, \end{cases} \\ \boldsymbol{v}_k \sim \begin{cases} N(0, \boldsymbol{R}), & \text{w.p. } 0.97, \\ N(0, U_3\boldsymbol{R}), & \text{w.p. } 0.03, \end{cases} \end{cases}$$

$$\begin{cases} [w_{1,k},\ w_{3,k}]^{\mathrm{T}} \sim \begin{cases} N(\mathbf{0}, \boldsymbol{Q}_1), & \text{w.p. } 0.97, \\ N(\mathbf{0}, U_1\boldsymbol{Q}_1), & \text{w.p. } 0.03, \end{cases} \\ [w_{2,k},\ w_{4,k}]^{\mathrm{T}} \sim \begin{cases} N(\mathbf{0}, \boldsymbol{Q}_2), & \text{w.p. } 0.90, \\ N(\mathbf{0}, U_2\boldsymbol{Q}_2), & \text{w.p. } 0.10, \end{cases} \\ v_{1,k} \sim \begin{cases} N(0, r), & \text{w.p. } 0.97, \\ N(0, U_3 r), & \text{w.p. } 0.03, \end{cases} \\ v_{2,k} \sim \begin{cases} N(0, r), & \text{w.p. } 0.90, \\ N(0, U_4 r), & \text{w.p. } 0.10, \end{cases} \end{cases}$$

First stage (1–500 s)

Second stage (501–1000 s)

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Simulation study (Cont'd)

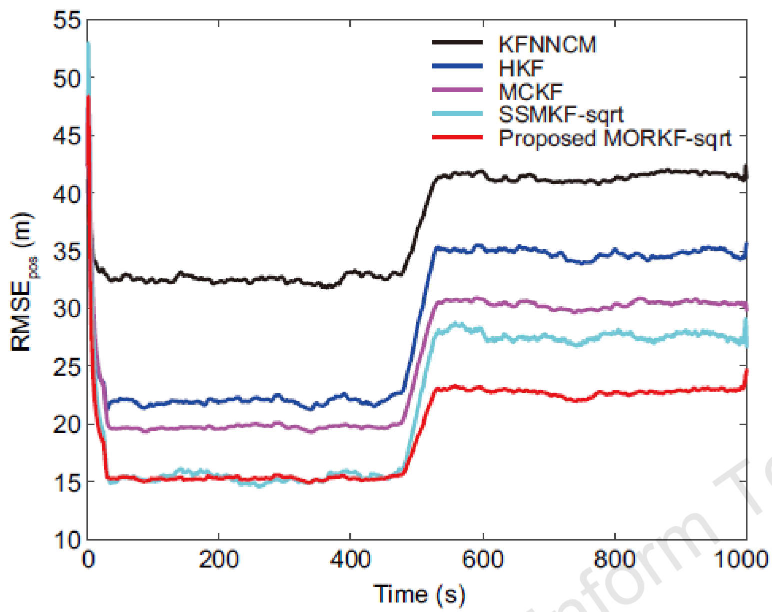

Fig. 1 RMSEs of position from all filters in case 1

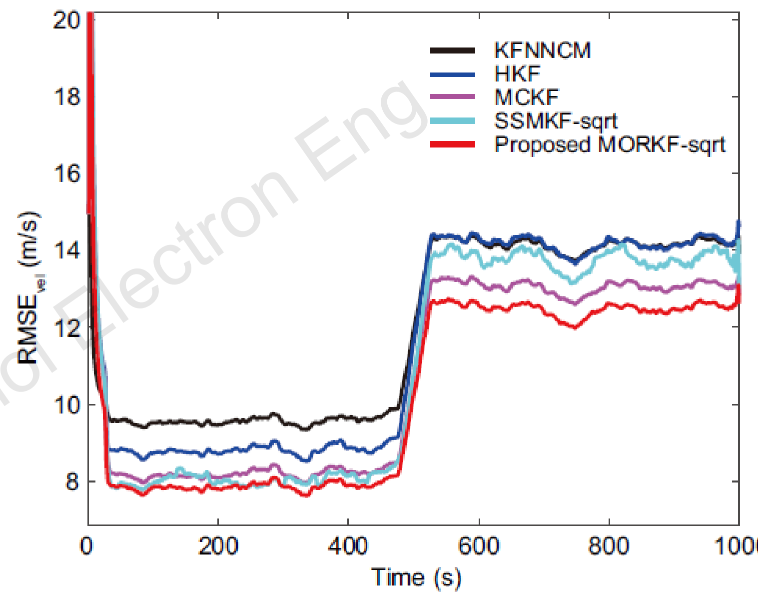

Fig. 2 RMSEs of velocity from all filters in case 1

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Comparisons of root mean square errors (RMSEs) and average RMSEs (ARMSE)

Table 6 Steady-state ARMSEs during 600–1000 s and runtime in a single step in case 1

Filter	$ARMSE_{pos}$ (m)	$ARMSE_{vel}$ (m/s)	Time (ms)
KFNNCM	41.61	14.13	0.049
HKF	34.80	14.15	1.397
MCKF	30.52	13.03	1.185
SSMKF-sqrt	28.21	13.94	1.819
MORKF-sqrt	22.72	12.43	2.023

ARMSE: average root mean square error

Simulation study (Cont’d)

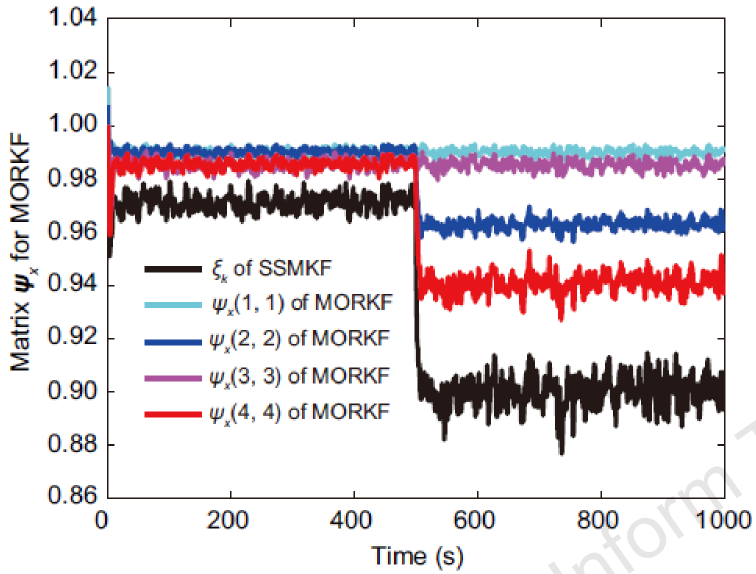

Fig. 3 Comparisons of scalar-scale factor ξ_k from SSMKF and diagonal elements of Ψ_{x_k} from MORKF in case 1

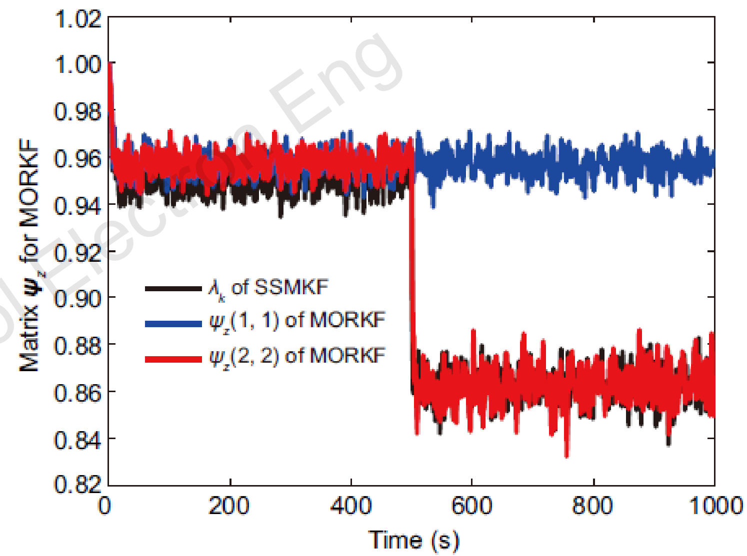

Fig. 4 Comparisons of scalar-scale factor λ_k from SSMKF and diagonal elements of Ψ_{z_k} from MORKF in case 1

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In the second stage, only the second and fourth dimensions of Ψ_{x_k} and the second dimension of Ψ_{z_k} are reduced to accommodate the suddenly increased outliers, which benefits the performance of the proposed method.

Simulation study (Cont'd)

Noise settings:

Case 2:

$$\begin{cases} \boldsymbol{w}_k \sim \begin{cases} N(\boldsymbol{0}, \boldsymbol{Q}), & \text{w.p. } 0.97, \\ N(\boldsymbol{0}, U_1\boldsymbol{Q}), & \text{w.p. } 0.03, \end{cases} \\ \boldsymbol{v}_k \sim \begin{cases} N(0, \boldsymbol{R}), & \text{w.p. } 0.97, \\ N(0, U_3\boldsymbol{R}), & \text{w.p. } 0.03, \end{cases} \end{cases}$$

$$\begin{cases} [w_{1,k},\ w_{3,k}]^{\mathrm{T}} \sim \begin{cases} N(\boldsymbol{0}, \boldsymbol{Q}_1), & \text{w.p. } 0.97, \\ N(\boldsymbol{0}, U_1\boldsymbol{Q}_1), & \text{w.p. } 0.03, \end{cases} \\ [w_{2,k},\ w_{4,k}]^{\mathrm{T}} \sim \begin{cases} N(\boldsymbol{0}, \boldsymbol{Q}_2), & \text{w.p. } 0.90, \\ U(w_{2,k}; -100, 100), & \text{w.p. } 0.10, \\ U(w_{4,k}; -200, 200), & \text{w.p. } 0.10, \end{cases} \\ v_{1,k} \sim \begin{cases} N(0, r), & \text{w.p. } 0.97, \\ N(0, U_3 r), & \text{w.p. } 0.03, \end{cases} \\ v_{2,k} \sim \begin{cases} N(0, r), & \text{w.p. } 0.90, \\ U(v_{2,k}; -600, 600), & \text{w.p. } 0.10, \end{cases} \end{cases}$$

First stage (1–500 s) → **Second stage (501–1000 s)**

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Simulation study (Cont'd)

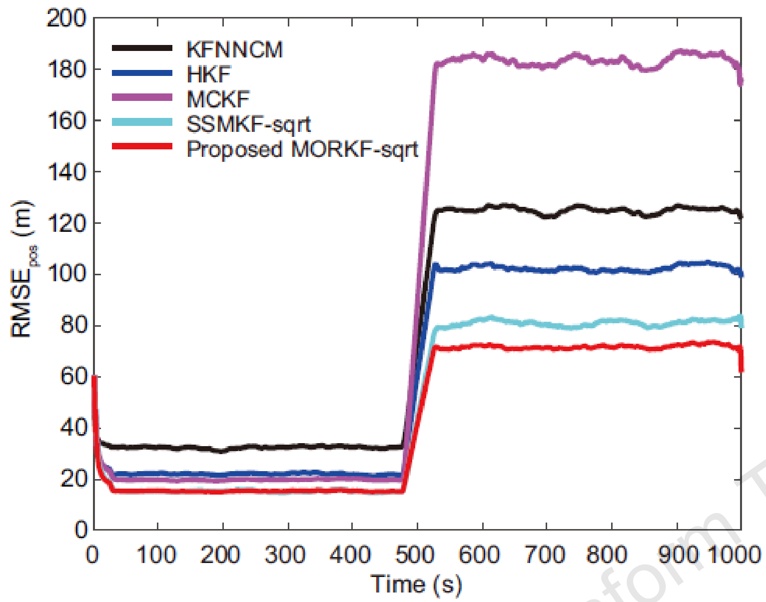

Fig. 5 RMSEs of position from all filters in case 2

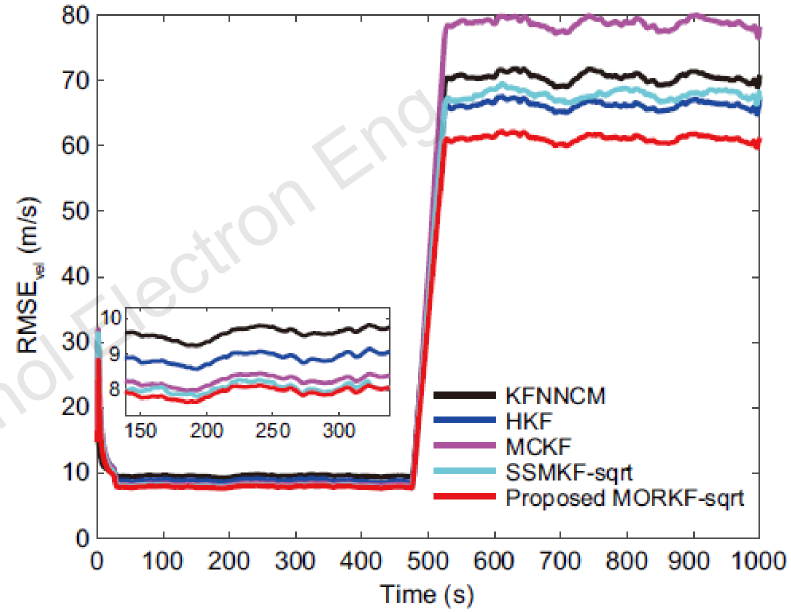

Fig. 6 RMSEs of velocity from all filters in case 2

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Comparisons of root mean square errors (RMSEs) and average RMSEs (ARMSE)

Table 7 Steady-state ARMSEs during 600–1000 s in case 2

Filter	$ARMSE_{pos}$ (m)	$ARMSE_{vel}$ (m/s)
KFNNCM	125.12	70.50
HKF	102.36	66.27
MCKF	183.59	78.74
SSMKF-sqrt	81.11	67.90
MORKF-sqrt	71.73	61.14

ARMSE: average root mean square error

Conclusions

1. To evaluate the similarity between two random vectors from dimension to dimension, a new MSSM was first introduced.

2. The MORKF was derived by maximizing an MSSM-based cost function.

3. Simulation results demonstrated that the developed MORKF outperforms the existing cutting-edge robust KFs in linear systems with multiple outliers.

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Yulong HUANG received his BS degree in automation and PhD degree in control science and engineering from the College of Automation, Harbin Engineering University (HEU), Harbin, China, in 2012 and 2018, respectively. His current research interests include state estimation, intelligent information fusion, and their applications in navigation technology, such as inertial navigation, integrated navigation, intelligent navigation, and cooperative navigation.

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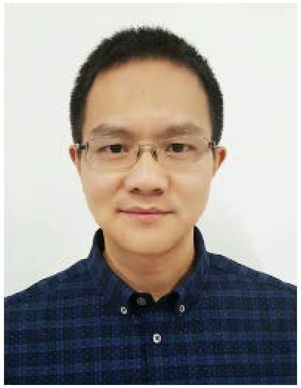

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