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Analysis of the Pareto equilibrium in multi-objective games using semi-tensor product

Key words: Multi-objective game; Pareto equilibrium; Semi-tensor product; Finite-step reachability; Finite-step controllability

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Motivation

1. While most existing literature focuses on games with scalar payoffs, multi-objective games (MOGs), which involve vector payoffs, are prevalent in practical settings.
2. Equilibrium analysis is fundamental in game theory. Consequently, exploring the Pareto equilibrium of MOGs, a well-established extension of the Nash equilibrium, is both reasonable and practical. To date, no research has addressed the finite-step reachability or finite-step controllability of Pareto equilibria in MOGs.

Motivation (Cont'd)

3. Specifically, analyzing MOG Pareto equilibrium problems demands consideration of players' individual expectations, as cooperative and competitive elements are inherently intertwined in the game process.

4. The semi-tensor product (STP) of matrices has emerged as a key mathematical tool in game theory, particularly for solving scalar-payoff equilibrium problems. To our knowledge, STP has yet to be applied to Pareto equilibria in MOGs, especially regarding their existence, finite-step reachability, and finite-step controllability.

Main idea

1. A class of multi-objective games is proposed and represented in the form of multi-layer graphs.
2. Set the payoff vector and ultimate payoff expectations for every player in an MOG, where the ultimate payoff can be expressed using the weighted sum.
3. Use STP to convert the payoff function, ultimate payoff expectations, etc. into algebraic form, and then discuss the existence, finite-step reachability, and finite-step controllability of Pareto equilibrium points.
4. Base on the finite-step controllability, a backward search algorithm is provided to find the shortest evolutionary process and control sequence.

Method

1. Multi-layer graphs and STP are used to obtain the algebraic expressions of MOGs.

2. Based on the algebraic form of payoff function, the ultimate payoff expectations and Pareto equilibria are investigated.

3. A strategy updating rule is designed. It can be used to determine the number of Pareto equilibrium points, based on which some results concerning finite-step reachability and finite-step controllability are presented.

4. A backward search algorithm is presented to determine the shortest evolutionary process and control sequence.

Major results

Definition of MOGs

Definition 2 An MOG $= (N, S, \boldsymbol{a}, \boldsymbol{K}, \boldsymbol{C})$ consists of the following factors:

1. $N = \{1, 2, \ldots, n\}$ is the set of players.

2. $S = S_1 \times S_2 \times \ldots \times S_n$ is the set of strategy profiles, where S_i is the strategy set of player i and "$\times$" is the Cartesian product. $\boldsymbol{s} = (s_1, s_2, \ldots, s_n) \in S$ is a strategy profile where player i chooses strategy s_i, $i = 1, 2, \ldots, n$, $s_i \in S_i$.

3. $\boldsymbol{a} = (a_1, a_2, \ldots, a_m)$, where a_j represents the weight of the j^{th} layer, $j = 1, 2, \ldots, m$. Different layers represent the different objectives, and $m \in \mathbb{N}_+$ is the number of objectives in this game.

4. $\boldsymbol{K} = (K_1, K_2, \ldots, K_n)$, where $K_i = \{k_1^i, k_2^i, \ldots, k_{m_i}^i\}$ is the set of layers where player i resides, meaning that K_i consists of the objectives with which player i is concerned, $m_i \in \mathbb{N}_+$, $m \geq m_i \geq 1$ and $i = 1, 2, \ldots, n$.

5. $\boldsymbol{C}(\boldsymbol{s}) = (\boldsymbol{c}_1(\boldsymbol{s}), \boldsymbol{c}_2(\boldsymbol{s}), \ldots, \boldsymbol{c}_n(\boldsymbol{s}))$, where $\boldsymbol{c}_i^{\mathrm{T}}(\boldsymbol{s}) = \left[c_i^{k_1^i}(\boldsymbol{s}), c_i^{k_2^i}(\boldsymbol{s}), \ldots, c_i^{k_{m_i}^i}(\boldsymbol{s})\right]$ is the payoff vector of player i under the strategy profile $\boldsymbol{s} \in S$. Specifically, $c_i^j(\boldsymbol{s}) = \sum\limits_{l \in U_j(i)} c_{i,l}^j(s_i, s_l)$ is the payoff of the j^{th} objective for player i, and $c_{i,l}^j(s_i, s_l)$ is the payoff of player i in a game with player l in terms of the j^{th} objective, where $U_j(i)$ is the set of players who have games with player i at the j^{th} layer, $i \in N$, $j \in K_i$.

Major results (Cont'd)

Verification for the ultimate payoffs

Theorem 1 The individual expectations $(r_1, r_2, \ldots, r_n)$ can be achieved in the MOG $= (N, S, \boldsymbol{a}, \boldsymbol{K}, C)$ if and only if there exists a positive integer j satisfying

$$\mathrm{Col}_j(\boldsymbol{\mathcal{G}}) \geq \boldsymbol{0}_n,$$

where $\boldsymbol{\mathcal{G}}$ is shown in Eq. (4).

Verification for the Pareto equilibrium

Theorem 2 For an MOG $= (N, S, \boldsymbol{a}, \boldsymbol{K}, \boldsymbol{C})$, there exists a Pareto equilibrium $\boldsymbol{s}^* = (s_1^*, s_2^*, \ldots, s_n^*)$ if and only if one of the following conclusions holds:

(i) For every $i \in N$, there exists a vector $\boldsymbol{\alpha} \in \mathbb{R}^{m_i}$ such that $\boldsymbol{H}_i^{-i} = \boldsymbol{1}_k \otimes \boldsymbol{\alpha}$ holds;

(ii) For every $i \in N$, there exists a constant $r \in S_0 \backslash \{s_i^*\}$ such that $\boldsymbol{H}_i^{-i,r} <_\exists \boldsymbol{H}_i^{-i,s_i^*}$ holds, where $-i$ satisfies $\ltimes_{p \neq i}^{n} \boldsymbol{x}_p^* = \boldsymbol{\delta}_{k^{n-1}}^{-i}$, $i = 1, 2, \ldots, n$.

Major results (Cont'd)

Design the strategy updating rule

Denote $(s_1(t), s_2(t), \ldots, s_n(t))$ as the strategy profile at time t; its algebraic form is described as $(\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_n(t))$. Let $\boldsymbol{x}_{-i}(t) = (\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_{i-1}(t), \boldsymbol{x}_{i+1}(t), \ldots, \boldsymbol{x}_n(t))$. Given $\boldsymbol{x}_i \in \Delta_k \backslash \{\boldsymbol{x}_i(t)\}$ and $\boldsymbol{X}_{-i}(t) = \ltimes_{j\neq i}^{n} \boldsymbol{x}_j(t) \in \Delta_{k^{n-1}}$, set $f_i(t, \boldsymbol{x}_i) = \boldsymbol{c}_i(\boldsymbol{x}_i, \boldsymbol{X}_{-i}(t)) - \boldsymbol{c}_i(\boldsymbol{x}_i(t), \boldsymbol{X}_{-i}(t))$, $P_i(t) = \{\boldsymbol{x}_i | f_i(t, \boldsymbol{x}_i) \in \mathbb{R}_+^{k_i} \backslash \{\boldsymbol{0}_{k_i}\}\}$, and $Q_i(t) = \{\boldsymbol{x}_i | -f_i(t, \boldsymbol{x}_i) \in \mathbb{R}_+^{k_i} \backslash \{\boldsymbol{0}_{k_i}\}\}$. Thus, the strategy updating rule is designed as

$$O_i(t) = \begin{cases} \{\boldsymbol{x}_i(t)\}, & |Q_i(t)| = k-1, \\ P_i(t), & |Q_i(t)| + |P_i(t)| = k-1 \\ & \quad \text{and } |Q_i(t)| < k-1, \\ \Delta_k \backslash Q_i(t), & \text{otherwise.} \end{cases} \tag{16}$$

Here, $O_i(t)$ is the set of strategies that can be chosen at time $t+1$ for player i. Particularly, $O_i(t)$ with $\boldsymbol{X}_{-i}(t) = \delta_{k^{n-1}}^{l}$ is designated as O_i^l, and it is assumed that player i chooses the strategies in O_i^l with equal probability, $i = 1, 2, \ldots, n$, $l = 1, 2, \ldots, k^{n-1}$.

Major results (Cont'd)

Verification for the number of Pareto equilibria

Theorem 3 Consider the EMOG with strategy updating rule (16) and all players simultaneously update their strategies. The number of Pareto equilibria n_{p} satisfies

$$n_{\mathrm{p}} = \mathrm{trace}(\boldsymbol{J}).$$

Criterion for the finite-step reachability of Pareto equilibria

Theorem 4 Consider the EMOG with the one-step evolutionary equation (19). There are $J^s_{p_*,p_\mathrm{o}}$ possible processes to reach the strategy profile $\boldsymbol{\delta}^{p_*}_{k^n}$ from $\boldsymbol{\delta}^{p_\mathrm{o}}_{k^n}$ at step s with a positive probability, $s \in \mathbb{N}_+$.

Criterion for the finite-step controllability of Pareto equilibria

Theorem 5 Consider the EMOG with the controlled one-step evolutionary equation (20). The initial strategy profile $\boldsymbol{X}(0) = \boldsymbol{\delta}^{p_\mathrm{o}}_{k^n}$ can be controlled to the target strategy profile $\boldsymbol{X}(s) = \boldsymbol{\delta}^{p_*}_{k^n}$ at step s with a positive probability if and only if $(\bar{\boldsymbol{J}}^s_{\mathrm{u}})_{p_*,p_\mathrm{o}} > 0$.

Major results (Cont'd)

Algorithm 1 Backward search method (to find the shortest evolutionary process and control sequence)

Input: $p_{\rm o},\ p_*,\ \boldsymbol{J}_{{\rm u},i},\ i=1,2,\ldots,k^m$

Output: $t,\ \boldsymbol{P},\ \boldsymbol{Q}$

Initialize $t=0,\ \boldsymbol{q}=0^{\rm T}_{k^n}$;
while $q_{p_{\rm o}} \neq 1$ **do**
    $t=t+1$;
    for $j=1:k^m$ **do**
        for $r=1:k^n$ **do**
            $\boldsymbol{\alpha}=0^{\rm T}_{k^n}$;
            if $t==1$ and $(\boldsymbol{J}_{{\rm u},j})_{p_*,r}==1$ **then**
                $\alpha_r=1$;
            else if $t \neq 1$ and $\{l|(\boldsymbol{J}_{{\rm u},j})_{l,r}=1\} \subset \{l|Q_{t-1,l}=1\}$ **then**
                $\alpha_r=1$;
            end if
        end for
        if $j==1$ **then**
            $\boldsymbol{p}=\boldsymbol{\alpha}$;
        else
            $\boldsymbol{p}=[\boldsymbol{p};\boldsymbol{\alpha}]$;
        end if
    end for
    for $r=1:k^n$ **do**
        $q_r=p_{1,r} \vee p_{2,r} \vee \ldots \vee p_{k^m,r}$;
    end for
    if $t==1$ **then**
        $\boldsymbol{P}=\boldsymbol{p}$;
        $\boldsymbol{Q}=\boldsymbol{q}$;
    else
        $\boldsymbol{P}=[\boldsymbol{P};\ \boldsymbol{p}]$;
        $\boldsymbol{Q}=[\boldsymbol{Q};\ \boldsymbol{q}]$;
    end if
end while

Major results (Cont'd)

A numerical example

Example 2 Consider an MOG $= (N, S, \boldsymbol{a}, \boldsymbol{K}, \boldsymbol{C})$; the three-layer graph is shown in Fig. 2. Here, we set $N = \{1,2,3,4\}$, $S_1 = S_2 = S_3 = S_4 = S_0 = \{1,2\}$, $\boldsymbol{a} = (3,2,2)$, $\boldsymbol{K} = (\{1\},\{1,3\},\{1,2,3\},\{2\})$. $\boldsymbol{C}(\boldsymbol{s}) = (c_1(\boldsymbol{s}), c_2(\boldsymbol{s}), c_3(\boldsymbol{s}), c_4(\boldsymbol{s}))$ is shown in Table 1, and the payoff expectations of players 1–4 are $r_1 = 21$, $r_2 = 28$, $r_3 = 36$, and $r_4 = 12$.

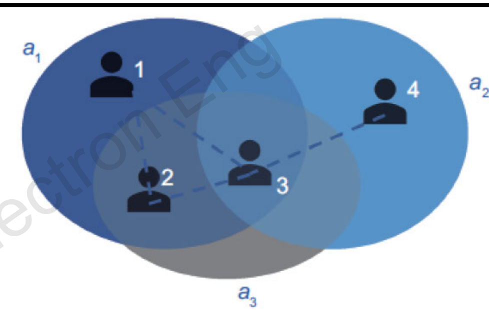

Fig. 2 Three-layer graph representation of the MOG in Example 2

Table 1 Payoffs of Example 2

	$c_i(s)$							
	s_1=(1,1,1,1)	s_2=(1,1,1,2)	s_3=(1,1,2,1)	s_4=(1,1,2,2)	s_5=(1,2,1,1)	s_6=(1,2,1,2)	s_7=(1,2,2,1)	s_8=(1,2,2,2)
c_1	7	7	1	9	8	8	2	4
c_2	$[5,4]^{\mathrm{T}}$	$[3,2]^{\mathrm{T}}$	$[7,4]^{\mathrm{T}}$	$[10,4]^{\mathrm{T}}$	$[3,9]^{\mathrm{T}}$	$[6,1]^{\mathrm{T}}$	$[9,2]^{\mathrm{T}}$	$[9,6]^{\mathrm{T}}$
c_3	$[6,1,2]^{\mathrm{T}}$	$[3,10,2]^{\mathrm{T}}$	$[6,1,1]^{\mathrm{T}}$	$[7,4,5]^{\mathrm{T}}$	$[3,7,7]^{\mathrm{T}}$	$[0,1,3]^{\mathrm{T}}$	$[5,2,3]^{\mathrm{T}}$	$[5,4,2]^{\mathrm{T}}$
c_4	9	2	5	8	10	7	5	5

	$c_i(s)$							
	s_9=(2,1,1,1)	s_{10}=(2,1,1,2)	s_{11}=(2,1,2,1)	s_{12}=(2,1,2,2)	s_{13}=(2,2,1,1)	s_{14}=(2,2,1,2)	s_{15}=(2,2,2,1)	s_{16}=(2,2,2,2)
c_1	3	9	7	6	5	12	8	3
c_2	$[5,3]^{\mathrm{T}}$	$[6,2]^{\mathrm{T}}$	$[3,5]^{\mathrm{T}}$	$[5,2]^{\mathrm{T}}$	$[1,3]^{\mathrm{T}}$	$[8,4]^{\mathrm{T}}$	$[5,7]^{\mathrm{T}}$	$[3,6]^{\mathrm{T}}$
c_3	$[6,1,9]^{\mathrm{T}}$	$[3,10,4]^{\mathrm{T}}$	$[7,7,2]^{\mathrm{T}}$	$[3,1,5]^{\mathrm{T}}$	$[4,6,3]^{\mathrm{T}}$	$[8,5,8]^{\mathrm{T}}$	$[5,6,2]^{\mathrm{T}}$	$[3,3,4]^{\mathrm{T}}$
c_4	2	3	7	3	7	12	6	4

The four values of $\boldsymbol{s}_i$ in the brackets represent the corresponding selected strategies of the four players

Major results (Cont'd)

Static analysis

Furthermore, based on Eq. (4), $\mathcal{G}$ can be calculated as Eq. (6) (at the top of the next page). Thus, it can be verified that $j = 4$ and $j = 14$ satisfy Theorem 1; that is, payoff vectors of all players are in line with their expectations under the strategy profiles $\delta_{16}^4 \sim (1,1,2,2)$ and $\delta_{16}^{14} \sim (2,2,1,2)$.

Therefore, it can be checked that Theorem 2 holds for every $i \in N$ as $l = 1,4,5,8,14$, and 15, meaning that δ_{16}^1, δ_{16}^4, δ_{16}^5, δ_{16}^8, δ_{16}^{14}, and δ_{16}^{15} are Pareto equilibria. Thus, combining with the result in Example 2, we find that strategy profiles $\delta_{16}^4 \sim (1,1,2,2)$ and $\delta_{16}^{14} \sim (2,2,1,2)$ are Pareto equilibria, and all players can meet their expectations under these strategy profiles.

$$\mathcal{G} = \begin{bmatrix} 14 & 14 & -4 & 20 & 17 & 17 & -1 & 5 & 2 & 20 & 14 & 11 & 8 & 29 & 17 & 2 \\ -5 & -15 & 1 & 10 & -1 & -8 & 3 & 11 & -7 & -6 & -9 & -9 & -19 & 4 & 1 & -7 \\ -12 & -3 & -14 & 3 & 1 & -28 & -11 & -9 & 2 & 1 & 3 & -15 & -6 & 14 & -5 & -13 \\ 12 & -2 & 4 & 10 & 14 & 8 & 4 & 4 & -2 & 0 & 8 & 0 & 8 & 18 & 6 & 2 \end{bmatrix}. \quad (6)$$

$$\boldsymbol{H}_1^l = \mathrm{Col}_l\left(\begin{bmatrix} 7 & 7 & 1 & 9 & 8 & 8 & 2 & 4 \\ 3 & 9 & 7 & 6 & 5 & 12 & 8 & 3 \end{bmatrix}\right),$$

$$\boldsymbol{H}_2^l = \mathrm{Col}_l\left(\begin{bmatrix} 5 & 3 & 7 & 10 & 5 & 6 & 3 & 5 \\ 4 & 2 & 4 & 4 & 3 & 2 & 5 & 2 \\ 3 & 6 & 9 & 9 & 1 & 8 & 5 & 3 \\ 9 & 1 & 2 & 6 & 3 & 4 & 7 & 6 \end{bmatrix}\right),$$

$$\boldsymbol{H}_3^l = \mathrm{Col}_l\left(\begin{bmatrix} 6 & 3 & 3 & 0 & 6 & 3 & 4 & 8 \\ 1 & 10 & 7 & 1 & 1 & 10 & 6 & 5 \\ 2 & 2 & 7 & 3 & 9 & 4 & 3 & 8 \\ 6 & 7 & 5 & 5 & 7 & 3 & 5 & 3 \\ 1 & 4 & 2 & 4 & 7 & 1 & 6 & 3 \\ 1 & 5 & 3 & 2 & 2 & 5 & 2 & 4 \end{bmatrix}\right),$$

$$\boldsymbol{H}_4^l = \mathrm{Col}_l\left(\begin{bmatrix} 9 & 5 & 10 & 5 & 2 & 7 & 7 & 6 \\ 2 & 8 & 7 & 5 & 3 & 3 & 12 & 4 \end{bmatrix}\right).$$

Major results (Cont'd)

Dynamic analysis

According to Eqs. (16) and (17), it can be determined that

$$\bar{x}_i(t+1) = \boldsymbol{R}_i \boldsymbol{X}_{-i}(t), \quad i = 1, 2, 3, 4,$$

where

$$\boldsymbol{R}_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix},$$

$$\boldsymbol{R}_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix},$$

$$\boldsymbol{R}_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix},$$

$$\boldsymbol{R}_4 = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

$$\boldsymbol{J} = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

Based on Theorem 3, we find that the number of Pareto equilibria is trace($\boldsymbol{J}$)=6, which is in agreement with Example 4. In this example, we set the length of the evolution process to 3 and $\boldsymbol{X}(0) = \delta_{16}^2 \sim (1, 1, 1, 2)$. By calculation, we find that there are $J_{4,2}^3 = 4$ and $J_{14,2}^3 = 0$ paths to the strategy profiles $\delta_{16}^4 \sim (1, 1, 2, 2)$ and $\delta_{16}^{14} \sim (2, 2, 1, 2)$ with a positive probability, respectively.

Major results (Cont'd)

Dynamic analysis

Then, it can be calculated that $J^3_{2,4} = 0$, meaning that the EMOG cannot naturally evolve to $\delta_8^2 \sim (1,1,2)$ at step 3. Thus, taking player 3 in Example 2 as a pseudo-player, the controlled one-step evolutionary equation is

$$\bar{\boldsymbol{X}}_{\mathrm{u}}(t+1) = \boldsymbol{J}_{\mathrm{u}}\boldsymbol{u}(t)\boldsymbol{X}_{\mathrm{u}}(t), \tag{23}$$

where

$$\boldsymbol{J}_{\mathrm{u}} = \begin{bmatrix} 1&0&1&0&0&0&0&0&0&0&0&1&0&1&0&1 \\ 0&0&0&0&1&0&1&0&0&1&0&1&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0&0&0&1&0&1&0&1 \\ 0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0 \\ 0&1&0&1&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0 \\ 0&1&0&1&0&0&0&0&0&0&1&0&1&0&1&0 \\ 0&0&0&0&0&1&0&1&1&0&1&0&0&0&0&0 \end{bmatrix}.$$

Furthermore, from Eq. (22) we have

$$\bar{\boldsymbol{J}}_{\mathrm{u}} = \begin{bmatrix} 1&0&1&1&0&1&0&1 \\ 0&1&0&1&1&0&1&0 \\ 1&0&1&1&0&1&0&1 \\ 0&1&0&1&0&0&0&0 \\ 0&1&1&1&0&0&0&0 \\ 1&0&1&0&0&0&0&0 \\ 0&1&1&1&1&0&1&0 \\ 1&0&1&0&0&1&0&1 \end{bmatrix}.$$

Based on Theorem 5, it can be verified that $(\bar{\boldsymbol{J}}^3_{\mathrm{u}})_{2,4} = 4$, meaning that there are four evolutionary processes that can be controlled from δ_8^4 to δ_8^2 at step 3 with a positive probability.

Major results (Cont'd)

Dynamic analysis

Next, we find the shortest evolutionary process and control sequence from δ_8^4 to δ_8^2 with probability 1. According to Algorithm 1, the backward search process is shown in Fig. 4. Therefore, the evolutionary processes and control sequences are $\delta_8^4 \xrightarrow{\boldsymbol{u}(0)=\boldsymbol{\delta}_2^1} \delta_8^5 \xrightarrow{\boldsymbol{u}(1)=\boldsymbol{\delta}_2^1} \delta_8^2$ and $\delta_8^4 \xrightarrow{\boldsymbol{u}(0)=\boldsymbol{\delta}_2^1} \delta_8^7 \xrightarrow{\boldsymbol{u}(1)=\boldsymbol{\delta}_2^1} \delta_8^2$.

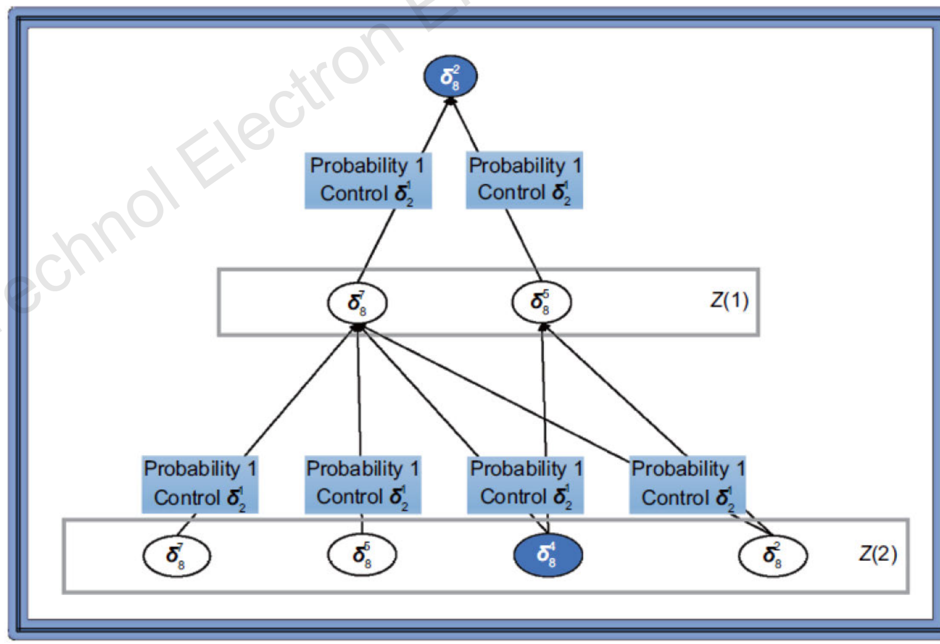

Fig. 4 The backward search process of Example 6

Conclusions

1. The algebraic expressions of MOGs are obtained using multi-layer graphs and STP.

2. Two necessary and sufficient conditions are proposed to verify whether all players can meet their expectations and to verify the existence of the Pareto equilibrium, separately.

3. A new strategy updating rule is designed, and evolutionary MOGs are considered in algebraic form, based on which some results concerning finite-step reachability and finite-step controllability are presented.

4. A backward search algorithm is presented to determine the shortest evolutionary process and control sequence.