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Robust and accurate optimal transportation map by self-adaptive sampling

Key words: Optimal transportation; Monge-Ampère equation; Self-adaptive sampling

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Motivation

1. Optimal transportation plays an important role in many fields, such as parameterization in graphics, surface registration in vision, and generative models in deep learning.
2. Optimal transportation is equivalent to solving the Monge-Ampère equation, which is reduced to a geometric variational problem to find the Brenier potential.
3. For optimal transportation maps between probability distributions with complicated supports and large variation densities, the existing methods are unstable and inaccurate.

Main idea

1. The robustness of the algorithm (stability and accuracy) is influenced by the uniformity of the sampling pattern.
2. A novel self-adaptive sampling algorithm is proposed, such that the sampling density is consistent with the target probability density, which ensures the stability of the combinatorial structure of the Brenier potential and keeps the search path inside the admissible space.
3. The algorithm is based on the adaptive mesh generation method in computational geometric algorithms and has theoretical guarantees.

Method

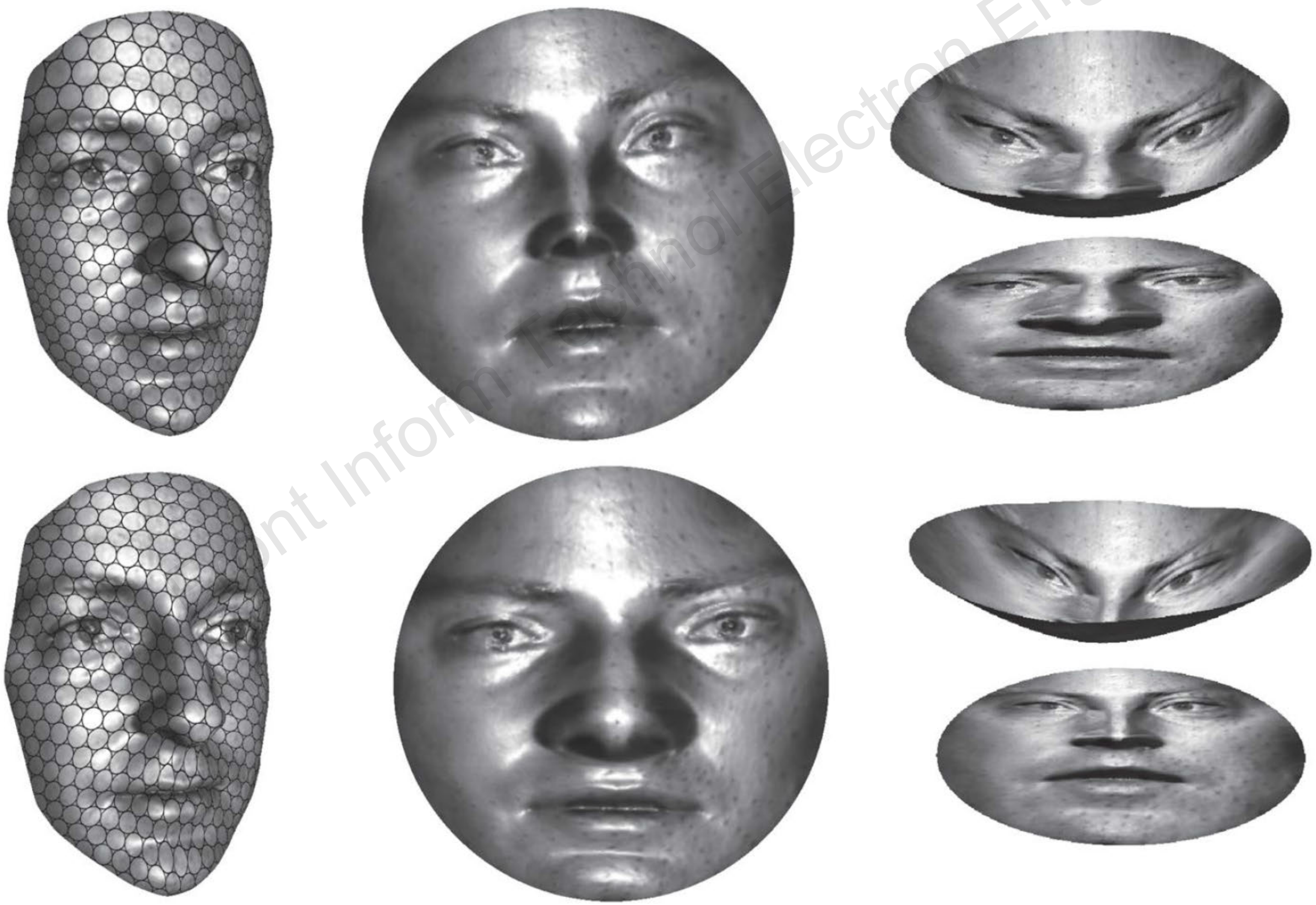
1. The Brenier potential is represented by the upper envelope of a support plane, whose gradients equal to the samples and the heights are solved by a convex optimization using damped Newton's method.
2. The self-adaptive sampling algorithm is based on the weighted Delaunay refinement algorithm. It starts with uniform random samples, and in each iteration, it selects the worst triangle with a small angle or a big weighted circum-radius, and then inserts its circum-center to the samples. It stops when all the faces are with uniform sizes and good shapes.

Major results

1. The method guarantees the quality of the samplings:
   - All the inner angles are greater than or equal to 30°;
   - The weighted area of the faces is uniform.
2. The computational complexity for each iteration is $O(n\log n)$, where n is the number of samples.
3. The convergence rate of Newton's method for the optimal transportation is linear.
4. The total time complexity is $O(-\ln(n)\log n)$.

Major results

Example on area preserving parameteriztion:



Major results (Cont'd)

Statistics for different tests:

Table 1 Number of iterations and final relative error of different models

Model	Vert	Edge	Face	Fig.	Iter	Error
Buddha	32 765	97 897	65 133	12	11	4.305e-11
Brain	25 909	77 602	51 699	6, 7	35	5.425e-11
Hand	3706	11 072	7367	8, 9	28	6.212e-10
Male 1	24 848	74 032	49 185	1	17	7.563e-11
Male 2	24 977	74 418	49 442	4	54	6.555e-11
Oldman	25 627	76 554	50 928	10	16	3.963e-11
Female	24 979	74 458	49 480	11	54	9.310e-11

Vert: number of vertices; Edge: number of edges; Face: number of faces; Iter: number of iterations; Error: relative error

Major results (Cont'd)

The challenging hand model showing the robustness:

Conclusions

In this study, we proposed a novel self-adaptive sampling method which is uniform with respect to the density of the target measure, and this greatly improved the robustness and accuracy of the geometric variational algorithm for computing optimal transportation maps. Experimental results demonstrated the effectiveness of this method. In the future, we will generalize this method to higher-dimensional situations and to compute the optimal transportation maps with more general cost functions.

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