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Review:

A survey of photon mapping state-of-the-art research and future challenges*

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Abstract: Global illumination is the core part of photo-realistic rendering. The photon mapping algorithm is an effective method for computing global illumination with its obvious advantage of caustic and color bleeding rendering. It is an active research field that has been developed over the past two decades. The deficiency of precise details and efficient rendering are still the main challenges of photon mapping. This report reviews recent work and classifies it into a set of categories including radiance estimation, photon relaxation, photon tracing, progressive photon mapping, and parallel methods. The goals of our report are giving readers an overall introduction to photon mapping and motivating further research to address the limitations of existing methods.

Key words: Global illumination, Photon mapping, Radiance estimation, Photon relaxation, Progressive photon mapping
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1 Introduction

The photon mapping (PM) algorithm is an effective method for implementing global illumination, which is one of the most important parts of photo-realistic rendering. Since Jensen and Christensen (1995) proposed the photon mapping algorithm, many studies have attempted to obtain the accurate rendering result and to improve the rendering efficiency.

Photon mapping is a two-pass algorithm. The first pass is the photon tracing pass to generate the photon maps, and the second pass is the photon collection pass, in which the illumination is reconstructed via the photon maps (Fig. 1). Although the approach has advantages in many practical ways, it still faces significant challenges such as visual noise and bias problems (Fig. 2). In this paper, we analyze improved photon mapping algorithms and produce a classified summary.

Improved radiance estimation methods use more accurate radiance estimation methods. Changing the accumulation mode of photon energy (Jensen, 2001; Havran *et al.*, 2005) and improving the search bandwidth (Schjøth *et al.*, 2007; 2008) can reduce the bias and noise in the photon mapping algorithm.

Photon relaxation methods eliminate random distribution noise in the photon map. Through photon relaxation methods (Spencer and Jones, 2009), photons can be distributed uniformly in a local area.

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The noise in the distribution can be eliminated, but the illumination features may be smoothed.

Progressive photon mapping methods increase the number of photons emitted. The billions of emissions lead to the increase of memory usage, so researchers proposed a progressive photon mapping method (Hachisuka *et al.*, 2008) in which it was unnecessary to store the full photon map. The method can effectively decrease noise with little memory being used, and it can also be combined naturally with the method of bidirectional path tracing (Hachisuka *et al.*, 2012; Georgiev *et al.*, 2013).

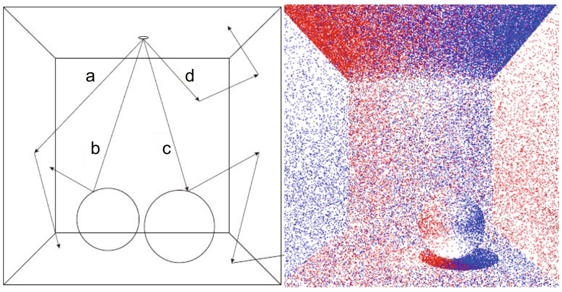


Fig. 1 A global photon map generated in the photon tracing pass (there is no photon at the glass surface)

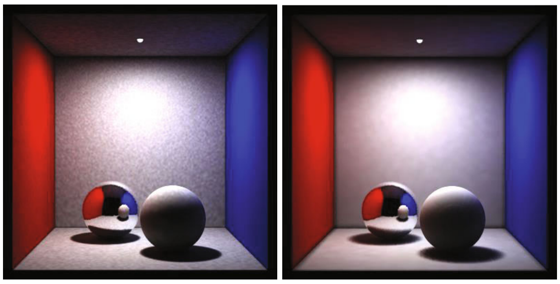


Fig. 2 A global photon map visualizes directly with 50 photons (left) and 500 photons (right) in the radiance estimate using a RenderMan render

Photon tracing methods improve the photons' emission path. Improved photon tracing methods can optimize photon distribution through the transmitting step, speed up rendering, and achieve accurate results.

Parallel methods accelerate the rendering processes by using graphics hardware. Global illumination algorithms are time-consuming in the process of photo-realistic image rendering. To take full advantage of the hardware's computing ability to accelerate, different methods use different strategies.

The analysis of the optimized methods based on three challenges, namely smoothness, features, and space-savings, is presented in Fig. 3. Fig. 3a shows the ideal solution which has good performance on the three challenges. Fig. 3b is the analysis of radiance estimation methods without considering storage completely. Fig. 3c is the analysis of photon relaxation methods which can generate smooth results with a small photon map, but it is difficult to keep the features. Fig. 3d shows that the progressive photon mapping methods can change the storage mode of photons and generate smooth results through a massive number of iterations. It is close to the ideal model. However, the progressive photon mapping methods are time-consuming as too many iterations are needed. Fig. 3e shows that the photon tracing methods can optimize the distribution focusing on the emission technique in order to improve the quality of rendering results. What is more, general

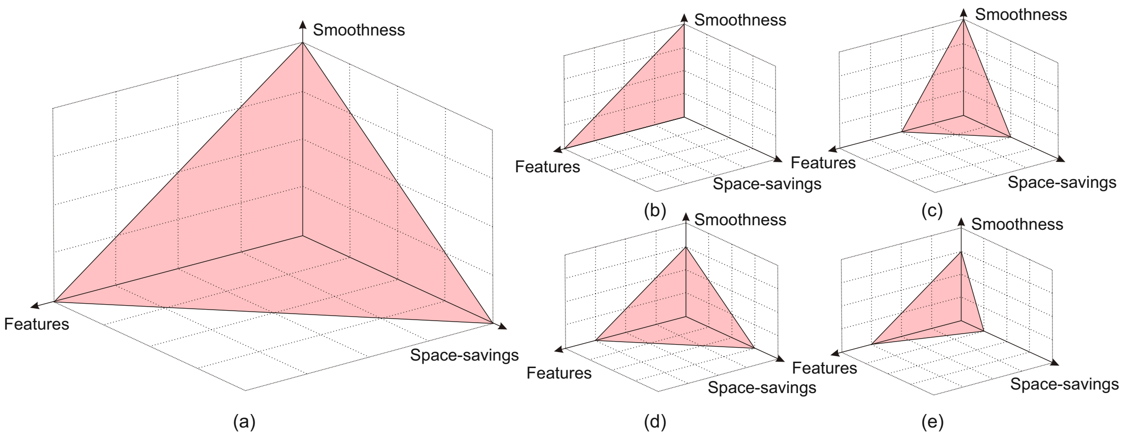


Fig. 3 Analysis of the optimized methods based on three challenges: (a) ideal solution; (b) radiance estimation methods; (c) photon relaxation methods; (d) progressive photon mapping methods; (e) photon tracing methods

methods always waste time in improving the rendering effects. We also analyze the parallel algorithms, which can accelerate rendering by using hardware acceleration units in general.

2 Standard photon mapping

In standard photon mapping (Jensen, 1996), a photon map is a set of photons including location, direction, and energy information. It is created by a particle tracing pass, in which photons are bounced and recorded around a scene using a tracing method from the light sources. Then illumination can be reconstructed using the photon map in a rendering pass through a series of collections at the shading points.

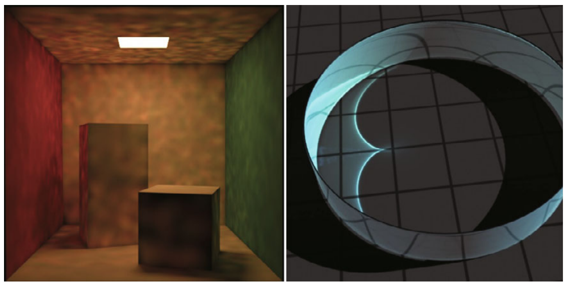
There are three kinds of photon maps. A global photon map is a set of photons that contains all collisions at the non-specular surface. A caustic photon map contains photons that have been through at least one specular reflection before hitting a diffuse surface, and a volume photon map stores the scattering photons which travel through a participating medium. The caustic photon map and volume photon map can be used independently for caustic effects and scattering in a participating medium.

The information in the photon map can be used to approximate the reflected radiance at a shading point, with location $\boldsymbol{x}$ and a given direction $\boldsymbol{w}$. This is done by integrating the reflected radiance with the surface bidirectional reflectance distribution function (BRDF). Since photons are stored with direction and energy, an approximation of the radiant power at a diffuse or slightly glossy surface can be calculated by querying the k nearest photons around the shading point. Radiance estimation at highly glossy surfaces can be treated efficiently by Monte Carlo ray tracing. Hence, the equation for the reflected radiance can be rewritten as an integration within search area Ω:

$$\boldsymbol{L}_{\mathrm{r}}(\boldsymbol{x},\boldsymbol{w}) = \int_{\Omega} f_{\mathrm{r}}(\boldsymbol{x},\boldsymbol{w},\boldsymbol{w}_i)\frac{\mathrm{d}^2\boldsymbol{\Phi}_i(\boldsymbol{x},\boldsymbol{w}_i)}{\mathrm{d}A\mathrm{d}\boldsymbol{w}_i}\mathrm{d}\boldsymbol{w}_i \approx \sum_{i=1}^{k} f_{\mathrm{r}}(\boldsymbol{x},\boldsymbol{w},\boldsymbol{w}_i)\frac{\Delta\boldsymbol{\Phi}_i(\boldsymbol{x},\boldsymbol{w}_i)}{\pi r(\boldsymbol{x})^2}, \quad (1)$$

where power $\boldsymbol{\Phi}_i$ (here we write it as a vector form since it may contain several components) is taken by the ith photon with direction $\boldsymbol{w}_i$, and f_{r} is the BRDF of the surface. The variable $r(\boldsymbol{x})$ is the radius of a sphere encompassing the k nearest photons. Therefore, $\pi r(\boldsymbol{x})^2$ is the area of a circle through the center of the sphere as an approximation of A. The radius is decided by the photon density around the shading point, and $r(\boldsymbol{x})$ is called the bandwidth in radiance estimation. The resolution of the photon map and the number of photons used in each radiance estimate can determine the degree of accuracy of the radiance estimation.

Fig. 4a shows a rendering result of standard photon mapping through direct visualization of the global photon map and Fig. 4b shows a result with caustic effects by the caustic photon map.



(a) (b)

Fig. 4 Direct visualization of the global photon map in the Cornell box (a) and a metal ring on a surface (b)

Standard photon mapping is an extension of the ray tracing method. A photon cache which preserves the illumination information makes the method more flexible than ray tracing in respect of caustic and color bleeding rendering. However, this method introduces noise and caching during rendering. Researchers have improved the photon mapping algorithm from different perspectives. Although photon mapping has been studied for many years with theoretical research reaching a certain maturity, there are still many unsolved problems in practical applications. With the improvements in computing power and the enhancements in algorithms, path tracing methods are increasingly relevant in industrial applications such as games and movies. As a mainstream rendering algorithm, photon mapping has been improved significantly after years of development. Nevertheless, new work should be constantly undertaken to help the photon mapping method adapt to the needs of industrial applications.

3 Improved radiance estimation

The photon mapping method uses k nearest neighbors to compute the outgoing radiance at a shading point. The radiance estimation in Eq. (1) can be improved by adding a filter of the distance. In addition, many studies have successfully tried to optimize the calculation results of the radiance estimation. We introduce these methods in the following subsections.

3.1 Filters

The size of the bandwidth controls the balance between variance and bias. A small bandwidth reduces the bias, but it increases the variance of the estimate, and a large bandwidth results in an increase in bias and a decrease in variance. The selection of k for the nearest neighbor searching is important. In Jensen (1996), a filter was added to weigh each photon according to its distance to the shading point. By the filter, Eq. (1) can be formulated as

$$\widehat{\boldsymbol{L}}_{\mathrm{r}}(\boldsymbol{x},\boldsymbol{w})=\frac{1}{\pi r(\boldsymbol{x})^2}\sum_{i=1}^{k} K\left(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|}{r(\boldsymbol{x})}\right) f_{\mathrm{r}}(\boldsymbol{x},\boldsymbol{w}_i,\boldsymbol{w})\boldsymbol{\Phi}_i, \tag{2}$$

where $\boldsymbol{x}_i$ is the position of the ith photon. Denoting $d=\|\boldsymbol{x}-\boldsymbol{x}_i\|$ (the distance between the photon and shading point) and r an abbreviation of $r(\boldsymbol{x})$, $K(d/r)$ is a function of the filter that computes the weight according to the distance and can be expressed as the cone filter:

$$K(d/r)=3(1-d/r), 0\le d/r\le 1. \tag{3}$$

The normalization of the filter based on a 2D distribution of the photons is 3. This filter is used to reduce the bias at the edges and obvious feature structures. In a set of photons around the shading point, photons near the point can get higher weights than those farther away.

Another useful filter is the Epanechnikov kernel, which can reduce the mean integrated square error of the kernel density estimation. In photon mapping, Roland (2003) employed it and achieved smooth results. In addition, there are many other filtering methods such as the quartic filter and the Gaussian filter. All of these filtering methods can be used in photon mapping. We summarize some filtering methods which can be used in photon mapping generally (Table 1).

Table 1 Function of several filters

Filter	Expression	Constraint
Kernel	$K(d/r)$	–
Cone	$3(1-d/r)$	$0\le d/r\le 1$
Epanechnikov	$6\left(1-(d/r)^2\right)$	$0\le d/r\le 1$
Quartic	$3\left(1-(d/r)^2\right)^2$	$0\le d/r\le 1$
Gaussian	$\pi r^2\mathrm{e}^{-1/2(d/r)^2}$	$0\le d/r\le 1$

García *et al.* (2012) presented an analysis of the nearest neighbour density estimator commonly used in the photon mapping algorithm. They found that the overestimated correct value is related to the number of photons in a query, and proposed a correct way to perform photon density estimation. Later, García *et al.* (2014) analyzed different filtering kernels in the context of photon mapping density estimation by using the distribution of order statistics and produced new and consistent estimators through the improved filtering kernels.

3.2 Adaptive searching methods

Jensen and Christensen (1995) proposed an extended method to improve the balance between bias and variance called differential checking. They pointed out that the bias comes mainly from the bandwidth which may cross boundaries of distinct lighting features. If the radiance varies severely as more photons are used, then the iterative process stops and an optimal bandwidth is chosen.

A similar method proposed by Myszkowski (1997) computes a bias and noise error estimate for density estimation at the shading point. It is called the enhanced nearest neighbor (ENN) method. By comparing the variance of different bandwidths, the method selects one that minimizes the error. Myszkowski (1997) was one of the first to thoroughly investigate bias in density estimation. Roland (2003) used the same error estimate in density estimation, following up Myszkowski (1997)'s work, but Roland optimized it by using a binary search to select the optimal bandwidth.

3.3 Adaptive kernel shape methods

Schjøth *et al.* (2007; 2008) and Schjøth (2009) summarized an approach using an anisotropic estimator to reduce the topological bias and noise at radiance discontinuities. Schjøth *et al.* (2008) suggested an approach inspired by a shape-adaptive structure tensor in density estimation to reduce the bias along edges and structures. The structure tensor is constructed in an in-between pass after photon tracing, and computed from the first order structure of the photon map. Another accurate method was presented for reconstruction of illumination with photons (Schjøth *et al.*, 2007). The method uses ray differentials (Igehy, 1999) during the photon tracing pass on every photon and calculates footprints of every photon instead of using classic density estimation with a series of a specified number of photons and their total area. Then the method stores the irradiance of the photon and changes Eq. (1) using only the irradiance without the area. In this way, indirect illumination can achieve a high accuracy reconstruction. We can see the changes in the kernel in Fig. 5. Photon differentials can obtain an accurate rendering especially in the area of feature and edge (Schjøth *et al.*, 2007).

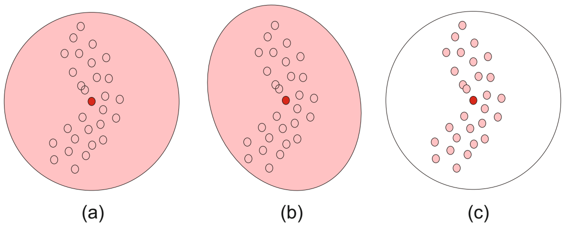


Fig. 5 The changes in the kernel: (a) in regular photon mapping, the search area is circular; (b) in the diffusion based photon mapping method, the search area changes into a narrow shape; (c) in the photon differentials method, the search area is according to the distribution of photons

Ray map (Havran *et al.*, 2005) is a novel data structure of photons. It extends the concept of photon maps by storing the whole photon paths as well as the spatial distribution of photons. This method can use the direction information in ray maps to eliminate boundary bias and reduce topological bias in density estimation in global illumination.

3.4 Summary

All of the methods summarized above have introduced the special calculation on radiance estimation. These methods improve photon mapping in calculation to balance bias and noise. However, there is a general problem that noise affects the choice, and on a light smooth surface it is difficult to remove noise. Most of the improvements in this section are concerned with the details. It is still difficult to directly remove the influence of noise in the estimation. At the same time, improving the photon tracing process is a good idea.

Most of the improved radiance estimation methods take more computational time than the standard photon mapping because of the accurate computation, e.g., the error estimate for density estimation at the shading point. In the ray map method, the density estimation with ray maps is 2.1–4.7 times slower than the direct visualization from photon maps. The diffusion based photon mapping method (Schjøth *et al.*, 2008) uses 1.1–1.5 times computational time totally. The photon differentials method (Schjøth *et al.*, 2007) costs more time in ray differentials' tracing and it demands 24 extra bytes per photon differential.

As the core part of the photon mapping method, radiance estimation is worthy of further enhancement, and in recent years researchers have proposed to analyze the kernels used in photon density estimation.

4 Photon relaxation

The radiance estimation problem in photon mapping has been the target of extensive research, in which obtaining smooth results is dependent on several factors including the number of emitted photons, the kernel's shape, and bandwidth. Photon relaxation is a method that optimizes the properties of the photon distribution to directly reduce noise and bias from the distribution rather than to focus on the radiance estimation kernel.

4.1 Photon relaxation based on k-means

The photon relaxation method (Spencer and Jones, 2009) is intended to improve caustic rendering by directly removing noise from the photon distribution. With photon relaxation, background noise can be reduced dramatically, and the uniform illumination can be calculated after reconstruction. The algorithm adds an intermediate pass between photon tracing and illumination reconstruction, i.e., iterative point repulsion which uses vector computing through the position information of neighbors (Fig. 6). Feature detection should be analyzed in advance to isolate those points that lie near boundaries and other important visual cues. After a sufficient number of iterations, noise is diffused away in the photon distribution, and high-frequency details are kept by feature detection and inhibiting motion in the direction of migration when photons relax. The resulting distribution has a blue noise spectral signature, which has been shown to be an optimal sample pattern in many areas of computer graphics (Ulichney, 1988), including photon mapping (Spencer and Jones, 2009).

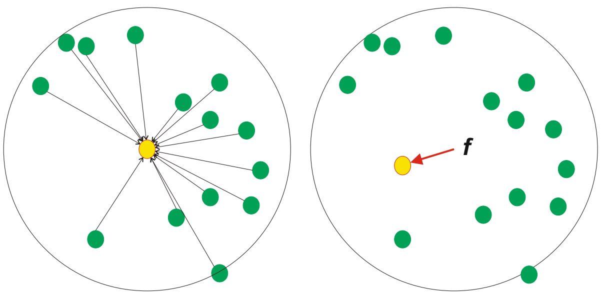


Fig. 6 Calculation of displacement vector *f*

Chen *et al.* (2013) proposed an improved blue noise sampling which can be applied in photon relaxation. Arbitrary attributes of samples can be considered as a similarity measure like sample position. For photon mapping, photon flux and photon direction as properties decide the displacement vector during the relaxation pass, together with photon position. What is more, the new bilateral blue noise sampling strategy enables photon relaxation to produce a better performance in keeping sharp features.

4.2 Photon relaxation based on Voronoi diagram

Spencer and Jones (2013b) developed a progressive method for photon relaxation. They used a conventional photon tracing pass to contribute to the Voronoi cells for every photon. Then the Voronoi pass is followed by successive passes of photon tracing, in which the new photons' fluxes are stored over the Voronoi cells and the center positions are moved according to the flux distribution. During the iterative process, a capacity constraint equation can be easily obtained from cell areas and accurate estimation made of flux density per photon.

4.3 Photon relaxation based on new parameters

Spencer and Jones (2013a) proposed another more robust approach in photon relaxation. Each photon's initial trajectory is encoded to build a high-dimensional KD-tree. Using these new parameters can minimize detail degradation in the pass of photon relaxation. First, they explored a new k-nearest neighbor (k-NN) query with the higher-dimensional photon KD-tree. Then they identified an anisotropic structure within the photon distribution for the kernel and feature detection. Finally, they defined the force with neighbors and applied it to every photon in each iteration pass. The method with the photon's original trajectory is effective in isolating overlapping or interfering illumination.

Spencer *et al.* (2015) designed an effective parameterization using a visualization tool with close user scrutiny and interaction. The researchers developed a visualization tool of photon data allowing different supposed coordinates testing and parameter adjustment in the algorithm. They proved that the use of the tool to explore high-dimensional photon map data can develop and optimize photon map noise removal methods.

4.4 Summary

Photon relaxation as a pre-pass between photon tracing and illumination reconstruction removes noise directly rather than improves radiance estimation. This method uses a small number of emissions in photon tracing and a low bandwidth, which will lead to a significant reduction in rendering cost. However, the feature detection and maintenance are difficult, making this method currently available and suitable only for the caustic photon map with obvious features. The bilateral method (Chen *et al.*, 2013) can render global illumination to a certain extent by adding arbitrary attributes to the relaxation. Fig. 7 shows the comparison with the bilateral blue noise method (Chen *et al.*, 2013), which adds color attributes. The results of the parameterized method are shown in Fig. 8. However, the parameters are difficult to organize in global illumination. It should have a great capacity for having a great development in the aspects of feature keeping and global illumination.

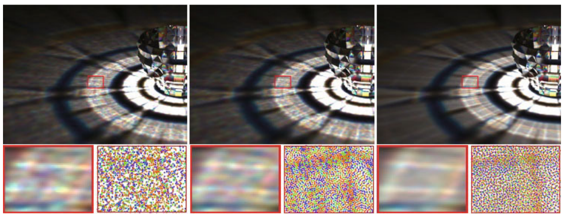


Fig. 7 Comparison of the results: (a) standard PM; (b) original photon relaxation; (c) bilateral blue noise sampling (Chen *et al.*, 2013). Reprinted from Chen *et al.* (2013), Copyright 2013, with permission from Association for Computing Machinery, Inc.

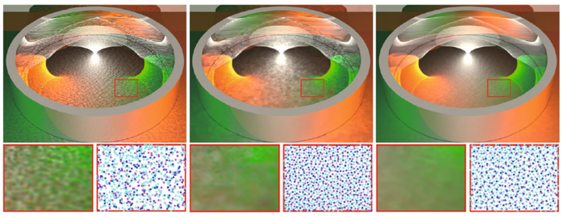


Fig. 8 Comparison of the results: (a) standard PM; (b) robust photon relaxation (Spencer and Jones, 2013a); (c) bilateral blue noise sampling (Chen *et al.*, 2013). Reprinted from Chen *et al.* (2013), Copyright 2013, with permission from Association for Computing Machinery, Inc.

At the same time, the methods increase calculation in the relaxation step. Otherwise, calculation on radiance estimation is simple due to the small photon collection. The relaxation takes 1.5–5.7 times the time that the rendering step requires in the original method. The progressive photon relaxation can render a similarly accurate result by using the same amount of time as the progressive photon mapping in the relaxation step, i.e., about 1 h. However, the rendering time is less than 2 min. Therefore, the photon relaxation method is extremely suitable for a roaming scenario.

5 Photon splatting

The first splatting method (Stürzlinger and Bastos, 1997) was proposed as a substitute for the k-NN searching, and it realizes the accumulation of photon contribution by rendering the kernel texture for every photon's triangle to obtain results at interactive frame rates. The scalable photon splatting method (Lavignotte and Paulin, 2003) can control the accuracy of the final results and render them by splatting each photon using a quad scaled for the visible surface identifier per pixel.

Herzog *et al.* (2007) presented the photon ray splatting method. They performed direct splatting of photon rays to the generated shading points by no longer storing the photons in a map. Photon splatting can obtain high-quality results with simple search data structures and with a slight increase in computation time. Another improvement related to splatting comes from Frisvad *et al.* (2014), and this is focused on the caustic effects. The authors combined photon differentials and splatting to determine the size and shape of the splats, instead of heuristic splat size. The size of splat gives rise to adaptive anisotropic flux density estimation.

The photon splatting method was proposed initially to simulate the global illumination in hardware and to gain efficiency at the cost of accuracy in density estimation. With the improvement proposed, photon ray splatting can estimate the irradiance without storing the photons in a map. What the method needs to store are the eye path hit-points. This method has better density estimation than the standard photon mapping (Fig. 9).

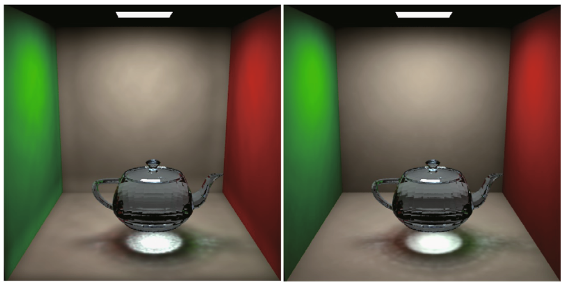


Fig. 9 Comparing photon mapping using k-NN density estimation (a) with photon ray splatting method (b) using the Cornell Box scene

6 Progressive photon mapping

To obtain an acceptable rendering result, billions of photons need to be emitted and stored, which entails a high memory overhead. Progressive photon mapping (PPM) is very important in the development of the photon mapping algorithm. Because of certain visual noise in the photon mapping algorithm, while reconstructing indirect lighting, the global photon maps are not generally directly visualized, but rather as a result of secondary bounce, in addition to the caustic photon map, until PPM appears.

Hachisuka *et al.* (2008) introduced progressive photon mapping, a hybrid global illumination solution that solves memory problems when billions of photons are emitted and stored. PPM transforms the items in the map into the shading points which are obtained from the first tracing pass from eyes with an array of primary and specular rays. Then photons are emitted repeatedly. After each launch, the photons' information is collected and saved immediately using the method in standard photon mapping at the shading points which are computed from the first step, and then the points' radius decreases with the iterations. After an iterative photon tracing process, a shading point map with illumination information can be obtained and the approach applies this map to render the final image (Fig. 10).

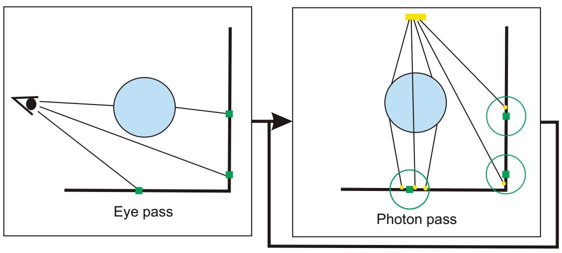


Fig. 10 The process of progressive photon mapping

6.1 Improved shading point generation

Hachisuka and Jensen (2009) presented a new formulation of progressive photon mapping, called stochastic progressive photon mapping (SPPM), which makes it possible to calculate the average radiance value over a region and render distributed ray tracing effects. The difference between PPM and SPPM is that SPPM adds a new distributed ray tracing pass after each photon tracing pass. Through the distributed ray tracing pass, the algorithm can randomly generate hit-points for the average radiance value over a region of a pixel. Fig. 11 shows the process of SPPM.

Knaus and Zwicker (2011) also proposed a probabilistic derivation of PPM as an expansion of SPPM. This approach does not require the maintenance of local statistics. It uses a probability value of the eye path in the first step to accumulate the illumination reconstruction to the pixel value which is computed in the same way as standard photon mapping algorithms. They presented a probabilistic analysis based on Monte Carlo sampling. This method can obtain a better result at glossy surfaces than SPPM.

Improved shading point generation is a significant improvement in PPM methods. With these improvements, PPM can handle rendering an anisotropic surface, such as glossy surfaces. Fig. 12 shows the comparison of PPM and SPPM.

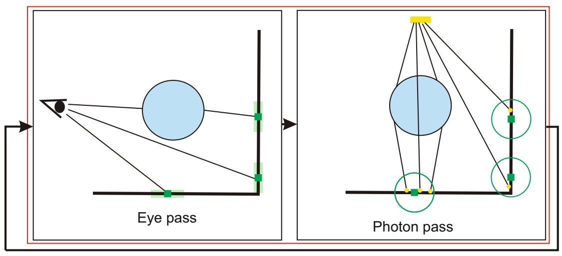


Fig. 11 The process of stochastic progressive photon mapping

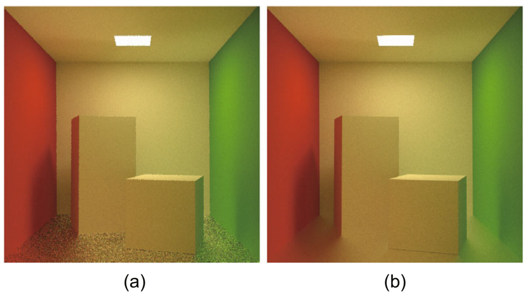


Fig. 12 Comparison of progressive photon mapping (a) and stochastic progressive photon mapping (b) with the same number of iterations

6.2 Radiance estimation in PPM

Hachisuka *et al.* (2010) used bias and variance estimation when describing an error estimation framework for adaptive rendering. The error estimation is the sum of constructing a bias estimator and noise estimator constructed for PPM.

Belcour and Soler (2011) extended PPM with frequency analysis of light transport to adaptively choose bandwidth in the radiance estimation kernels. They divided photons into ordinary photons and frequency photons. In addition, they traced frequency photons and accumulated frequency information which was used to update the shading points' kernel bandwidth.

Kaplanyan and Dachsbacher (2013) combined PPM with an optimized computing kernel, through a better bandwidth selection, which can optimally balance noise and bias and speed up the convergence for progressive rendering. This method conducts an analysis of the radiance estimation and error estimation to obtain the optimal estimation parameters such as the bandwidth and an attenuation ratio of the radius.

Liu and Zheng (2014b) proposed an anisotropic method to accelerate the convergence in PPM. At each rendering pass, they used the anisotropic method to compute the radiance of each shading point.

Radiance estimation strategies make global results more accurate than the original PPM. The results of adaptive progressive photon mapping are shown in Fig. 13.

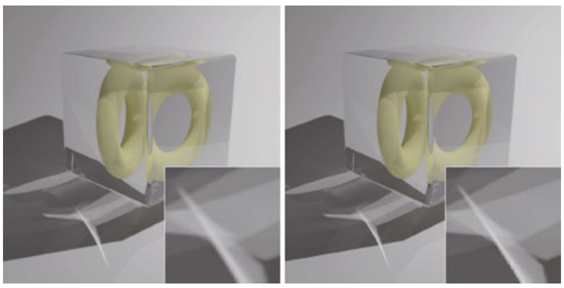


Fig. 13 Comparison of progressive photon mapping (a) and adaptive progressive photon mapping (b) with the torus scene

6.3 Combination with bidirectional path tracing

Path tracing is a very general simulation of global illumination (Kajiya, 1986), and bidirectional path tracing (BPT) (Lafortune and Willems, 1993) performs well on a wide range of illumination and scene configurations. However, there is no global illumination method which can simulate all types of light transport efficiently. Jensen (1995) simply used the photon mapping method in improved path tracing with a new importance sampling strategy through a rough estimate of photon irradiance. Until SPPM was proposed to combine photon density estimation and distributed ray tracing, a new idea about combination had been the research focus. Two efficient combinations between BPT and PM were presented by Hachisuka *et al.* (2012) and Georgiev *et al.* (2013), and the two methods combine the best of both algorithms.

Georgiev *et al.* (2013) proposed a vertex connection and merging (VCM) algorithm using different derivations concurrently. They considered the last step in photon mapping as establishing a regular vertex connection between the eye path and the light path. This technique is called vertex merging, as it can be intuitively thought to weld the endpoints of the two sub-paths if they lie close to each other. The first GPU implementation of vertex connection and merging algorithm was presented by Davidovič *et al.* (2014). In the implementation, a full hash grid was used to store the light path vertices, and each eye path point was connected to a predetermined number of light path vertices.

Hachisuka *et al.* (2012) considered an extension of BPT that samples paths in the higher-dimensional space of PM, under the name unified path sampling. This was done by considering a random perturbation of the eye path vertex in an r-neighborhood. The combinations were even extended to the participating media in Křivánek *et al.* (2014).

The recent development of combining photon mapping with bidirectional path tracing is significant, as this kind of method can handle the specular-diffuse-specular (SDS) light transport efficiently. What is more, the path tracing method is going back to the mainstream of the way that movies are rendered (Keller *et al.*, 2015), which can handle gigantic amounts of geometry, textures, and light sources. Combining with bidirectional path tracing allows the photon mapping technique to make a contribution to larger-scale industrial rendering.

6.4 Summary

Progressive photon mapping can increase the number of photon emissions by iteration without any increase in memory pressure. During the iteration process, the search radius is gradually attenuated, which can lead to noise reduction in photon distribution and bias elimination in radiance estimation. Although global illumination can be reconstructed by using the photon map directly in the PPM algorithm, thousands of iterations should be considered via iterative calculation to obtain an accurate result. The improvement on each step is conducive to reduce the number of iterations effectively. Improved PPM methods render a smooth noise-free result at the same time.

The relation of the intermediate data is also very important in the progressive process. PPM is a variation of the photon mapping algorithm, and all aspects of the improvement on the photon mapping algorithm can be applied into the PPM algorithm.

7 Adaptive photon tracing

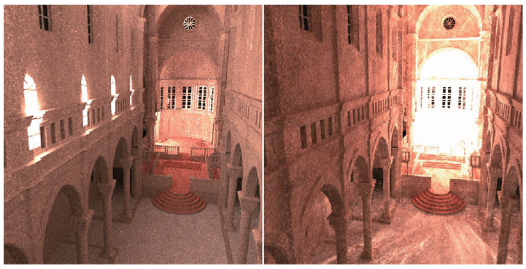
Another method to reduce the number of photons is photon tracing. A better photon distribution can be obtained through recording the viewpoint relevant region in the scene and then selectively shooting photons or storing photons is executed based on the region.

7.1 Photon tracing methods

The improvements on photon tracing are originally proposed for standard photon mapping. Density controlling for photon maps (Suykens and Willems, 2000) is a method which can store photons selectively to keep the density of photons. When the density reaches a certain level, photon flux can be distributed to neighbors. Keller and Wald (2000) introduced another importance driven photon map generation, in which photon paths are directed into the areas with high visual importance.

Fan *et al.* (2005) applied adaptive photon tracing to progressive photon mapping. Later, Hachisuka and Jensen (2011) proposed a method which is appropriate due to the visible information being obtained directly. The authors established a function about importance sampling based on the visibility of each photon path. Then they employed this important sampling for choosing light sources and direction while emitting photons. They also used adaptive Markov chain Monte Carlo methods and replica exchange to render and sample, which can adjust the sampling parameters. Fig. 14 shows the results for Sibenik cathedral using photon path visibility and making the image visually plausible with replica exchange.

Liu and Zheng (2014a) proposed an adaptive method by building cumulative distribution functions on the surfaces of the scene to apply the importance photon shooting technique. The adaptive cumulative distribution function was built in each node of the surface KD-tree based on their neighbors' error values, and was used to select the reflection direction of photon when a photon hits the surface in the scene. The method can accelerate the convergence through an optimized distribution.



(a) (b)

Fig. 14 The scene of Sibenik cathedral with only a directional light source: (a) with replica exchange; (b) without replica exchange, where samples get stuck within a small region (Hachisuka and Jensen, 2011). Reprinted from Hachisuka and Jensen (2011), Copyright 2011, with permission from Association for Computing Machinery, Inc.

7.2 Summary

The adaptive photon tracing algorithm can handle illumination reconstruction in the scenes such as outdoor scenes, close-ups of a small part of an illuminated region, and illumination coming through a small gap. Selectively emitted photon can obtain better photon distribution for an accurate result. This kind of method not only expands the application field but also optimizes the rendering results. However, the enhanced tracing process introduces the cost of time and space. Because of the demand of the data for guiding photon tracing, the method generally includes multiple photon emissions. It is more convenient to apply it to the progressive photon mapping method.

8 Parallel photon mapping methods

8.1 Parallel photon mapping

Photon mapping is improved on result optimization but at the same time, the efficiency of the algorithm is very important. Some algorithms can improve the rendering efficiency, e.g., the method proposed by Kaplanyan and Dachsbacher (2013). However, the most direct way is parallel acceleration by using graphics hardware.

Early works on parallel photon mapping are based on the optimized data structures which are adapted to the GPU. Ma and McCool (2002) used spatial hashing to replace the nearest-neighbour queries required in KD-tree in the density estimation pass, which fits better to parallel GPUs. In Purcell *et al.* (2003), the photons were directly stored in a grid-based photon map which was constructed and used on graphics hardware. They all presented efficient methods for locating the nearest photons in the new data structures.

Hardware implementation of a tree-based algorithm is difficult because it would require a large cache to avoid a high latency on average. To use an efficient KD-tree algorithm, Zhou *et al.* (2008) presented an efficient KD-tree building method, which is used both for tracing rays and for density estimation. This is the first real-time KD-tree algorithm on GPU. They divided the contribution process to exploit the fine-grained parallelism of GPUs.

Some researchers focused on the existing improved algorithms, and used a parallel compute unified device architecture (CUDA) implementation to accelerate the rendering. Fabianowski and Dingliana (2009) presented an interactive photon mapping method, which improves the photon differentials method (Schjøth *et al.*, 2007) on GPU and extends the original algorithm to global photon mapping.

Hachisuka and Jensen (2010) proposed the first GPU implementation of progressive photon mapping based on a new data structure. They avoided using irregular spatial data structures, which do not fit well to the balanced workload required for GPUs, in the storage of photons. In their method, the costly ray tracing-based photon shooting is not accelerated. For adaptive progressive photon mapping, Kaplanyan and Dachsbacher (2013) suggested that their method can be parallelized easily. NVIDIA OptiX (Parker *et al.*, 2010) is a GPU-based ray tracing rendering engine, and with the parallel ray tracing there is an example implementation of progressive photon mapping.

Mara *et al.* (2013) introduced an efficient GPU density estimation method, which uses an explored ray tracing method and the novel shading method focusing on density estimation. The researchers proposed a method for sampling and photon gathering with vector processors. Frolov *et al.* (2014) introduced an algorithm for constructing multiple reference octrees on a GPU in photon mapping techniques. They successfully ignored octrees' hierarchical property and presented parallel append and parallel sort primitives.

In 2010, Intel issued the Xeon Phi coprocessor which is based on a many integrated core architecture (MIC) with the KNC instructions. A hybrid parallel ray tracing algorithm (Benthin *et al.*, 2012) has been achieved based on this hardware. Various hardware architectures are worthy of attention for hardware implementation. Kang *et al.* (2015) presented the first photon mapping method on the MIC. They decomposed the collection step according to the spatial coherence, and applied the vector calculation unit to accelerate. However, Singh and Faloutsos (2007) firstly applied the SIMD packet techniques for photon mapping. Their solution is to use sample-point density estimation instead of k-NN density estimation, which contains the overheads that make SIMD instructions impractical. Wang *et al.* (2009) applied the coherence of raytraced shading points and interpolation to reduce the final gather cost, and this is a GPU-based method.

Some studies focus on interactive frames rendering. In Dmitriev *et al.* (2002), photons were divided into groups and re-emitted selectively based on dynamic objects collision. Larsen and Christensen (2004) proposed a new method for selectively redistributing photons and different elements in global illumination being calculated individually to demonstrate interactive frame rates.

There are also some distributed parallel methods. Günther *et al.* (2004) achieved a real-time distributed photon mapping for caustic rendering. Tamura *et al.* (2008) presented a parallel algorithm based on photon map partitioning. The photon map was assigned to each processing element to reduce memory requirement, and the algorithm worked by using MPI. Following Fradin *et al.* (2005), Günther and Grosch (2014) presented a distributed out-of-core stochastic progressive photon mapping method. The method was implemented to trace photons and eye rays in parallel programs in a portal-based system by automatically subdividing the scene geometry. Fallahpour *et al.* (2014) proposed another parallel photon-mapping rendering on a homogeneous multiprocessor SoC (MPSoC) platform which considers load balancing, less memory usage, and effective communication.

There are several references on real-time image space photon mapping, such as McGuire and Luebke (2009) and Yao *et al.* (2010). Image space photon mapping instead scatters the photon distribution to the image space pixels they contribute to. These methods do not support area light sources or participating media, and they cannot simulate the full global illumination result.

8.2 Summary

Global illumination algorithms are time-consuming, and the most effective solution is parallel implementations on graphics hardware. The photon mapping algorithm has made some progress in the parallel field. However, as a two-pass method, ray tracing-based photon shooting and radiance estimation should be equally important parts in photon mapping. With the continuous development of computing hardware, the parallel efficiency of the algorithm has been further enhanced. At the moment, many methods are at interactive frame rates, and some ways have achieved the real-time rates. The grid-based method (Purcell *et al.*, 2003) has a 1.4–1.9 times speedup with Stencil Routing, while the KD-tree method (Zhou *et al.*, 2008) has a speedup of 8.9–11.1 times on the GPU. The interactive global photon mapping (Fabianowski and Dingliana, 2009) maintains 3–4 frames/s. The distributed caustic photon mapping (Günther *et al.*, 2004) achieves frame rates of up to 22 frames/s at video resolution (640×480).

The appearance of different processing units and the use of clusters have diversified the methods. In particular, the vector calculation units should be considered to ensure the maximum utilization of the hardware's computing power.

9 Discussion and future work

This paper summarizes the development of photon mapping since the algorithm was first proposed. Up to now, there have been more and more new studies proposed to improve the rendering results of photon mapping, so that this algorithm can be further applied to industrial rendering. Through the summary above, we find that progressive photon mapping is a hot-spot among existing studies. The improvements on the regular photon mapping algorithm focus mostly on details of caustics, and global photon mapping cannot be directly visualized unless it is adapted to the progressive photon mapping algorithm. As the progressive photon mapping algorithm requires a massive number of iterations to obtain accurate results, this algorithm has been improved in the aspect of radiance estimation.

Photon relaxation is a novel approach which removes noise through moving photons, but it is difficult in feature detection and feature keeping especially when global illumination is applied. Photon tracing can obtain an improved result to some extent. Photon relaxation and photon tracing methods are similar from the aspect that both of them change the photon distribution to reduce noise. In addition to the methods summarized in this paper, there are many other aspects of development in photon mapping, for example, photon mapping in media applications.

Table 2 shows our comparison of different methods in speed, quality, scalability, and parallelism. The more solid points are painted, the higher score is obtained. Speed represents the performance of a method. However, there are many photon mapping methods focused on result optimization. Quality can be used as a measure for this important indicator. The score of scalability is high if a method can handle complex geometry, light, and special effects. Parallelism is important with regard to future hardware and applications.

As for future work, photon mapping is expected to obtain more accurate results and play a more important role in industrial rendering. How to obtain an ideal global illumination result with limited time and memory space remains as one of the priorities of the present study. Photon relaxation, photon splatting, and photon tracing are local optimization methods. Future work on these kinds of methods can start from the parallel algorithm and new parameters. What is more, relaxation and dart throwing are two implements in blue noise. We think that the combination of dart throwing or relaxation with progressive photon mapping is a developed trend. Progressive photon mapping combined with bidirectional path tracing methods indicate the new direction of photon mapping that can integrate into the

Table 2 Comparison of different methods

Method	Speed	Quality	Scalability	Parallelism
Improved radiance estimation				
Jensen (1996)	●●○○○	●●○○○	●●●○○	●●●○○
Myszkowski (1997)	●●○○○	●●●○○	●●●○○	●●○○○
Roland (2003)	●●●○○	●●●○○	●●●○○	●●○○○
García *et al.* (2012)	●●○○○	●●●●○	●●●○○	●●○○○
García *et al.* (2014)	●●○○○	●●●●●	●●●○○	●●○○○
Schjøth *et al.* (2008)	●●○○○	●●●●○	●●●○○	●●○○○
Schjøth *et al.* (2007)	●●○○○	●●●●●	●●●●○	●●●○○
Havran *et al.* (2005)	●●○○○	●●●○○	●●○○○	●●○○○
Photon relaxation				
Spencer and Jones (2009)	●●●○○	●●●●○	●●○○○	●●○○○
Chen *et al.* (2013)	●●●○○	●●●●●	●●●○○	●●○○○
Spencer and Jones (2013b)	●●●○○	●●●●○	●●○○○	●●○○○
Spencer and Jones (2013a)	●●●○○	●●●●●	●●○○○	●●○○○
Spencer *et al.* (2015)	●●●○○	●●●●●	●●○○○	●●○○○
Photon splatting				
Stürzlinger and Bastos (1997)	●●●○○	●●○○○	●●●○○	●●●●○
Lavignotte and Paulin (2003)	●●●○○	●●●○○	●●●○○	●●●●○
Herzog *et al.* (2007)	●●○○○	●●●●○	●●●○○	●●●○○
Frisvad *et al.* (2014)	●●○○○	●●●●●	●●●○○	●●○○○
Progressive photon mapping				
Hachisuka *et al.* (2008)	●●●○○	●●●●○	●●●○○	●●●○○
Hachisuka and Jensen (2009)	●●●○○	●●●●○	●●●●○	●●●○○
Knaus and Zwicker (2011)	●●●○○	●●●●●	●●●●○	●●○○○
Hachisuka *et al.* (2010)	●●●○○	●●●●●	●●●○○	●●○○○
Benthin *et al.* (2012)	●●●○○	●●●●●	●●●○○	●●○○○
Kaplanyan and Dachsbacher (2013)	●●●●○	●●●●●	●●●○○	●●●○○
Liu and Zheng (2014b)	●●●○○	●●●●●	●●●○○	●●○○○
Georgiev *et al.* (2013)	●●●○○	●●●●●	●●●●●	●●●○○
Davidovič *et al.* (2014)	●●●●○	●●●●●	●●●●●	●●●●○
Hachisuka *et al.* (2012)	●●●○○	●●●●●	●●●●●	●●●○○
Photon tracing				
Suykens and Willems (2000)	●●○○○	●●●○○	●●●○○	●●●○○
Fan *et al.* (2005)	●●○○○	●●●○○	●●●○○	●●○○○
Hachisuka and Jensen (2011)	●●○○○	●●●●○	●●●●●	●●○○○
Liu and Zheng (2014a)	●●●○○	●●●●○	●●●○○	●●●○○

mainstream while retaining its own advantages in some light transmission simulation. However, parallel methods are essential in real industrial applications, and real-time progressive photon mapping is still a shallow field, which will become a hot research topic.

In this paper, we have presented an extensive survey of the improvements on the photon mapping algorithm based on rendering effects. We analyze three challenges in the photon mapping algorithm, smoothness, features, and storage, and divide previous studies into five aspects of improvement. We believe that this survey of the photon mapping algorithm with an extensive literature review can give valuable insight into this important research topic and encourage new research.

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