Frontiers of Information Technology & Electronic Engineering
www.zju.edu.cn/jzus; engineering.cae.cn; www.springerlink.com
ISSN 2095-9184 (print); ISSN 2095-9230 (online)
E-mail: jzus@zju.edu.cn

Optimization of formation for multi-agent systems based on LQR*

Chang-bin YU[1,2], Yin-qiu WANG[‡1], Jin-liang SHAO[3]
([1]School of Automation, Hangzhou Dianzi University, Hangzhou 310018, China)
([2]Research School of Engineering, The Australian National University, Canberra ACT 0200, Australia)
([3]School of Mathematics Science, University of Electronic Science and Technology of China, Chengdu 611731, China)
E-mail: brad.yu@anu.edu.au; wh6508@gmail.com; jinliangshao@126.com

Received Dec. 30, 2015; Revision accepted Jan. 20, 2016; Crosschecked Jan. 22, 2016

Abstract: In this paper, three optimal linear formation control algorithms are proposed for first-order linear multi-agent systems from a linear quadratic regulator (LQR) perspective with cost functions consisting of both interaction energy cost and individual energy cost, because both the collective object (such as formation or consensus) and the individual goal of each agent are very important for the overall system. First, we propose the optimal formation algorithm for first-order multi-agent systems without initial physical couplings. The optimal control parameter matrix of the algorithm is the solution to an algebraic Riccati equation (ARE). It is shown that the matrix is the sum of a Laplacian matrix and a positive definite diagonal matrix. Next, for physically interconnected multi-agent systems, the optimal formation algorithm is presented, and the corresponding parameter matrix is given from the solution to a group of quadratic equations with one unknown. Finally, if the communication topology between agents is fixed, the local feedback gain is obtained from the solution to a quadratic equation with one unknown. The equation is derived from the derivative of the cost function with respect to the local feedback gain. Numerical examples are provided to validate the effectiveness of the proposed approaches and to illustrate the geometrical performances of multi-agent systems.

Key words: Linear quadratic regulator (LQR), Formation control, Algebraic Riccati equation (ARE), Optimal control, Multi-agent systems
http://dx.doi.org/10.1631/FITEE.1500490 **CLC number:** TP273

1 Introduction

The study of formations for a group of agents is inspired by the behaviors of various animal species in nature. For instance, fish, birds, and ants always work in a cooperative manner so as to accomplish tasks that are beyond the capability of individuals. The formation for agents has its wide range of applications in both civilian and military domains, where there are generally three basic parts to be designed, i.e., the underlying graph cooperating the interaction between agents, the cooperative algorithm among agents that is concerned with an exact assignment, and the local feedback controller that is responsible for the stability of the overall system.

In recent ten years, there have been an increasing number of studies towards the distributed formation for multi-agent systems (Oh *et al.*, 2015). Roughly, there are mainly three methods dealing with the problem. The first method presents the formation laws with angle-based only approaches

‡ Corresponding author
* Project supported by the National Natural Science Foundation of China (No. 61375072) (50%) and the Natural Science Foundation of Zhejiang Province, China (No. LQ16F030005) (50%)
ORCID: Yin-qiu WANG, http://orcid.org/0000-0002-1410-9619

(Basiri *et al.*, 2010; Bishop, 2011; Zhao *et al.*, 2014). The second method investigates the formation problem using gradient control laws from the relative distance measures perspective (Yu *et al.*, 2007; 2009; Guo *et al.*, 2010). The third method is consensus-based laws related to the Laplacian matrix with offsets to solve the formation problem. Specifically, by employing the consensus method, we can assign the desired formation arbitrarily, and deal with the formation for an arbitrarily large number of agents. If the desired formation or the number of agents has been changed, for each agent the control algorithm should not be updated drastically. Furthermore, the related tools to study consensus, such as matrix theory and algebraic graph, are very mature. Hence, consensus is a better approach for designing formation laws than other approaches, and there exists a great amount of literature in this field (Ren *et al.*, 2007; Xiao *et al.*, 2009; Dimarogonas and Johansson, 2010; Qin *et al.*, 2011; Qin and Gao, 2012; Li *et al.*, 2014; Liu and Jiang, 2014; Shi *et al.*, 2014). Qin *et al.* (2011) investigated two kinds of consensus problems for second-order agents under directed and arbitrarily switching topologies. Qin and Gao (2012) proposed a discrete-time second-order consensus algorithm for networks of agents with nonuniform and time-varying communication delays. Shi *et al.* (2014) considered a graph optimization problem for tracking consensus for first-order multi-agent systems. Xiao *et al.* (2009), Dimarogonas and Johansson (2010), Li *et al.* (2014), and Liu and Jiang (2014) proposed the distributed nonlinear formation laws for multi-agent systems via consensus-based results.

Due to the economic or other reasons, it is natural to design an optimal linear formation algorithm under a given cost function. Dealing with the optimal formation problem we aim to obtain the best performance index, such as the least energy expenditure or the shortest formation time. Moreover, in many situations, the individual goal and collective goal both have critical impact on the performance of the overall multi-agent systems. Therefore, we study optimal linear formation algorithms for multi-agent systems with single-integrator dynamics in three aspects from the linear quadratic regulator (LQR) perspective, and focus on the situation in which both the collective objective of all agents and the individual objective of each agent are considered. The cost functions should contain the interaction-related energy cost item and individual-related energy cost item.

Detailedly, we investigate three cases of LQR formation for multi-agent systems:

1. We first consider the no initial couplings case for agents, in which the optimal communication topology and the optimal feedback matrix should be designed to minimize the cost function. The parameter matrices are obtained from the solution to an algebraic Riccati equation (ARE).

2. If there exist couplings between agents, the optimal formation algorithm is proposed, and the optimal control parameters are obtained from a team of quadratic equations with one unknown, not an ARE.

3. When the communication graph between agents is fixed and only local feedback gain can be designed for each agent, the optimal local feedback gain is solved from a quadratic equation with one unknown. This equation is related to the initial values of all agents and the Laplacian matrix corresponding to the fixed communication graph.

To illustrate the contribution of this paper, we compare our work with the existing literature. Different from Cao and Ren (2010) which considered the optimality aspect of the coordination for multi-agent systems with individual integrator dynamics, the optimal indexes defined in this study not only consider the collective goal of all agents, but also focus on the individual goal of each agent. Movric and Lewis (2014) presented inverse optimality of consensus and pinning control laws for linear multi-agent systems with partial stability theory. Oh *et al.* (2015) considered interconnected systems consisting of identical agents for linear quadratic regulation by networked controllers. However, the above two works did not give the exact parameters of the control algorithms. In this study, we give the solution to the AREs and the exact forms of the proposed controllers. Different from Ghadami and Shafai (2013) which designed only a distributed controller for a continuous-time system composed of a number of identical dynamically coupled agents, in this study, we also consider the optimal cooperation for these agents. Compared with Borrelli and Keviczky (2008) which focused only on distributed controller design of large-scale dynamically isolated systems from the LQR perspective, we not only investigate the optimal formation for initially isolated multi-agent systems, but also study the problem for physically coupled multi-agent systems. Zhang *et al.* (2011) and Ghadami and Shafai (2014) investigated LQR cooperative problems for the general linear system; however, they can maintain only those parameters that are the solutions to a set of AREs. In contrast, we give the exact form of the optimal control parameters. As the most related work, Huang *et al.* (2010) proposed an optimal control law based on LQR for a group of agents to maintain formations while moving towards their destinations. Here, we expand their results to the following cases: (1) optimal formations for initially isolated multi-agent systems; (2) optimal formations for multi-agent systems under fixed and unchangeable topology.

2 Preliminaries

Notations $\mathbb{R}^{m\times n}$ denotes the family of $m\times n$ real matrices. For a given matrix $\boldsymbol{X}\in\mathbb{R}^{m\times n}$, $\boldsymbol{X}^{\mathrm{T}}$ denotes its transpose. The term $\mathrm{diag}(a_1,a_2,\ldots,a_p)$ is a diagonal matrix with diagonal entries a_1 to a_p, $\mathbf{1}_n$ denotes the n-dimensional column vector of all ones, and $\mathbf{0}_n$ and $\boldsymbol{I}_n$ denote the n-dimensional zero matrix and identity matrix, respectively.

Some basic concepts and notations in matrix theory and algebraic graph theory are referred to Horn and Johnson (1985; 1991) and Godsil and Royle (2013).

For a group of N agents, let $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ present the communication topology between all agents, where $\mathcal{V}=\{1,2,\ldots,n\}$ is the set of nodes and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ the set of undirected edges. In an undirected graph, the pairs of nodes are unordered, and an edge $(i,j)\in\mathcal{E}$ denotes that nodes i and j can directly obtain information from each other, that is, $a_{ij}=a_{ji}$. In addition, the set of neighbors of the ith agent, represented by $\mathcal{N}_i=\{j\in\mathcal{V}|(j,i)\in\mathcal{E}\}$, is the set of all agents from which the ith agent can receive the information directly. The weighted adjacency matrix $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{n\times n}$ of graph $\mathcal{G}$ is defined such that $a_{ij}>0$ if and only if $(j,i)\in\mathcal{E}$; otherwise, $a_{ij}=0$. Moreover, we assume that $a_{ii}=0$ for all $i\in\mathcal{V}$. Define the in-degree of the ith agent as $d_i=\sum_{j=1}^{n}a_{ij}$. The Laplacian matrix $\boldsymbol{L}=[l_{ij}]\in\mathbb{R}^{n\times n}$ for the directed graph $\mathcal{G}$ is defined by $\boldsymbol{L}=\mathrm{diag}(d_1,d_2,\ldots,d_n)-\mathcal{A}$. One can see that $\boldsymbol{L}$ has at least one zero eigenvalue with the corresponding eigenvector $\mathbf{1}_n$. If the undirected graph $\mathcal{G}$ is connected, the corresponding Laplacian matrix $\boldsymbol{L}$ is symmetric and has exactly one zero eigenvalue. A matrix $\boldsymbol{A}$ is nonnegative, if all of its entries are nonnegative.

Definition 1 A matrix $\boldsymbol{A}\in\mathbb{R}^{n\times n}$ is called an M-matrix if it has the form $\boldsymbol{A}=s\boldsymbol{I}_n-\boldsymbol{B}$, where $s>0$, $\boldsymbol{B}$ is a nonnegative matrix, and $\rho(\boldsymbol{B})\leq s$ ($\rho(\boldsymbol{B})$ is the spectral radius of $\boldsymbol{B}$). Specifically, $\boldsymbol{A}$ is a nonsingular M-matrix if $\rho(\boldsymbol{B})<s$.

For a nonsingular M-matrix, the following lemma holds:

Lemma 1 (Horn and Johnson, 1991) Let

$$Z_n=\{\boldsymbol{A}=(a_{ij})_{n\times n}\in\mathbb{R}^{n\times n}:a_{ij}\leq 0,\\ i\neq j,\ i,j=1,2,\ldots n\}$$

and $\boldsymbol{A}\in Z_n$. Then the following statements are equivalent:

1. The leading principal minor determinants of $\boldsymbol{A}$ are all positive;
2. The eigenvalues of $\boldsymbol{A}$ have positive real parts;
3. The diagonal entries of $\boldsymbol{A}$ are positive and $\boldsymbol{AD}$ is strictly row diagonally dominant for some positive diagonal matrix $\boldsymbol{D}$;
4. $\boldsymbol{A}^{-1}$ exists and is nonnegative.

Lemma 2 (Alefeld and Schneider, 1982) An M-matrix $\boldsymbol{A}$, which can be written as $\boldsymbol{A}=s(\boldsymbol{I}_n-\boldsymbol{P})$, where $s>0$, $\rho(\boldsymbol{P})\leq 1$, has exactly one M-matrix $\boldsymbol{N}=s^{1/2}(\boldsymbol{I}_n-\boldsymbol{Y}^*)$ as its square root if its characteristic polynomial has at most one simple zero root. $\boldsymbol{Y}^*$ is the limit of the sequence $\boldsymbol{Y}_i$ generated by

$$\boldsymbol{Y}_{i+1}=\frac{1}{2}(\boldsymbol{P}+\boldsymbol{Y}_i^2),\ \boldsymbol{Y}_0=\mathbf{0}. \tag{1}$$

Based on the above definition and lemmas, we have the following lemma:

Lemma 3 The square root of a strictly row diagonally dominant M-matrix $\boldsymbol{A}=[a_{ij}]$ is strictly row diagonally dominant.

Proof Note the square root of $\boldsymbol{A}$ is $\boldsymbol{A}^{1/2}$. From Definition 1 and Lemma 2, $\boldsymbol{A}^{1/2}=s^{1/2}(\boldsymbol{I}_n-\boldsymbol{Y}^*)$. Because $\boldsymbol{A}$ is strictly row diagonally dominant, noting the ith row sum of $\boldsymbol{P}_i$ as $r_i(\boldsymbol{P})$, we have

$$r_i(\boldsymbol{P})=\sum_{j=1}^{n}p_{ij}<1,$$

and $p_{ij} < 1, i, j = 1, 2, \ldots, n$. From Eq. (1), it can be obtained that $\boldsymbol{Y}_1 = \boldsymbol{P}/2$ and $r_i(\boldsymbol{Y}_1) < 1$. For nonnegative matrix $\boldsymbol{P}$, we have

$$\begin{aligned} r_i(\boldsymbol{P}^2) &= \sum_{j=1}^{n}\sum_{k=1}^{n} p_{ik}p_{kj} = \sum_{k=1}^{n} p_{ik} \sum_{j=1}^{n} p_{kj} \\ &= \sum_{k=1}^{n} p_{ik} r_k(\boldsymbol{P}) < r_i(\boldsymbol{P}) < 1. \end{aligned}$$

It follows $r_i(\boldsymbol{Y}_2) = r_i[(\boldsymbol{P} + \boldsymbol{P}^2/4)/2] < r_i(\boldsymbol{Y}_1) < 1$. In the same way, we can obtain $r_i(\boldsymbol{Y}_{k+1}) < r_i(\boldsymbol{Y}_k), i = 1, 2, \ldots, n$. Then $r_i(\boldsymbol{Y}^*) < 1$, which implies that $\boldsymbol{I}_n - \boldsymbol{Y}^*$ is a strictly row diagonally dominant matrix. As a result, $\boldsymbol{A}^{1/2} = s^{1/2}(\boldsymbol{I}_n - \boldsymbol{Y}^*)$ is strictly row diagonally dominant and $\boldsymbol{A}^{1/2}\boldsymbol{1}_n > 0$.

3 Main results

In this section, we investigate the optimal formation algorithms for first-order linear multi-agent systems from LQR. The dynamics of the ith agent is given by

$$\dot{\boldsymbol{x}}_i = \boldsymbol{u}_i, \tag{2}$$

where $\boldsymbol{x}_i(t) \in \mathbb{R}^m$ and $\boldsymbol{u}_i(t) \in \mathbb{R}^m$ are, respectively, the state vector and the control input vector of the ith agent. In practice, a great number of plants can be viewed as a single integrator. For example, the control inputs of some types of unmanned vehicles are linear velocities, so the dynamics of unmanned vehicles is of first order if the translation is viewed as the output. For quite a lot of types of spacecrafts, such as quadrotors, angular velocities are control inputs; then the first-order integration of angular velocities is the attitude of these spacecrafts. Therefore, we choose the dynamics of agents as single integrators.

The corresponding optimization problem is to find $\boldsymbol{u}_i(t)$ to obtain the minimal quadratic cost function J. Each agent should focus on the collective objective formation and pay attention to its own individual objective. Both objectives are considered in this study.

3.1 LQR formation for initially isolated multi-agent systems

In this subsection, we assume that all agents are isolated initially, which is the simplest case of formation problems for multi-agent systems. In this case, the communication topology between all agents and the local feedback gain for each agent should be designed. Hence, this case is the basis of some more complicated cases, such as the physically coupled case, fixed and unchangeable couplings cases, which will be discussed later.

Similar to the cost function used in LQR control for general linear systems, the cost function combining the interaction-related item and individual control objective item is given by

$$\begin{aligned} J = \int_0^\infty \Bigg(& \sum_{i=1}^{n}\sum_{j=1}^{i-1} h_{ij}(\boldsymbol{x}_i - \boldsymbol{x}_j - \boldsymbol{\Delta}_{ij})^{\mathrm{T}}(\boldsymbol{x}_i - \boldsymbol{x}_j - \boldsymbol{\Delta}_{ij}) \\ & + \sum_{i=1}^{n} e_i(\boldsymbol{x}_i - \boldsymbol{\delta}_i)^{\mathrm{T}}(\boldsymbol{x}_i - \boldsymbol{\delta}_i) + \sum_{i=1}^{n} r_i \boldsymbol{u}_i^{\mathrm{T}}\boldsymbol{u}_i \Bigg) \mathrm{d}t, \end{aligned} \tag{3}$$

where $\boldsymbol{\delta}_i$ is the constant objective of the ith agent, $\boldsymbol{\Delta}_{ij} = \boldsymbol{\delta}_i - \boldsymbol{\delta}_j$ is the desired formation offset between the ith agent and the jth agent, and $h_{ij} \geq 0$, $e_i > 0$, $r_i > 0$. Letting $\tilde{\boldsymbol{x}}_i = \boldsymbol{x}_i - \boldsymbol{\delta}_i$, the cost function is rewritten in matrix form as

$$\begin{aligned} J = \int_0^\infty & [\tilde{\boldsymbol{x}}^{\mathrm{T}}(\boldsymbol{Q} \otimes \boldsymbol{I}_m)\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{x}}^{\mathrm{T}}(\boldsymbol{E} \otimes \boldsymbol{I}_m)\tilde{\boldsymbol{x}} \\ & + \boldsymbol{u}^{\mathrm{T}}(\boldsymbol{R} \otimes \boldsymbol{I}_m)\boldsymbol{u}]\mathrm{d}t, \end{aligned} \tag{4}$$

where $\tilde{\boldsymbol{x}}$ and $\boldsymbol{u}$ are the column stack vectors of $\tilde{\boldsymbol{x}}_i$ and $\boldsymbol{u}_i$, respectively. Similar to Cao and Ren (2010), the matrix $\boldsymbol{Q} = [q_{ij}]$ is a symmetric Laplacian matrix if choosing $q_{ij} = q_{ji} = -h_{ij}$ for $i \neq j$, and $q_{ii} = \sum_{j=1, j\neq i}^{i-1} h_{ij}$. $\boldsymbol{E}$ and $\boldsymbol{R}$ are positive definite diagonal matrices with e_i and r_i being the ith diagonal entries respectively. It is clear that the item $\int_0^\infty \tilde{\boldsymbol{x}}^{\mathrm{T}}(\boldsymbol{Q} \otimes \boldsymbol{I}_m)\tilde{\boldsymbol{x}}\mathrm{d}t$ represents the formation energy of multi-agent systems, while the item $\int_0^\infty \tilde{\boldsymbol{x}}^{\mathrm{T}}(\boldsymbol{E} \otimes \boldsymbol{I}_m)\tilde{\boldsymbol{x}}\mathrm{d}t$ represents the energy of each agent to reach its individual objective.

From Anderson and Moore (2007), if choosing $\boldsymbol{u} = -(\boldsymbol{R} \otimes \boldsymbol{I}_m)^{-1}\boldsymbol{P}\tilde{\boldsymbol{x}}$, we can obtain the minimum J, and $\boldsymbol{P} \in \mathbb{R}^{(mn)\times(mn)}$ is the unique positive semidefinite matrix satisfying the continuous-time algebraic Riccati equation (ARE):

$$\begin{aligned} & (\boldsymbol{I}_n \otimes \boldsymbol{A})^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{I}_n \otimes \boldsymbol{A}) \\ & - \boldsymbol{P}(\boldsymbol{R} \otimes \boldsymbol{I}_m)^{-1}\boldsymbol{P} + (\boldsymbol{Q} + \boldsymbol{E}) \otimes \boldsymbol{I}_m = \boldsymbol{0}_{mn}. \end{aligned}$$

In this case, $\boldsymbol{A} = \boldsymbol{0}_m$ holds because all agents can be considered single integrators. Then the above ARE becomes

$$-\boldsymbol{P}(\boldsymbol{R} \otimes \boldsymbol{I})^{-1}\boldsymbol{P} = -(\boldsymbol{Q} + \boldsymbol{E}) \otimes \boldsymbol{I}_m,$$

which implies that

$$[(\boldsymbol{R}\otimes\boldsymbol{I}_m)^{-1}\boldsymbol{P}]^2=[\boldsymbol{R}^{-1}(\boldsymbol{Q}+\boldsymbol{E})]\otimes\boldsymbol{I}_m. \quad (5)$$

It is necessary to reduce the dimension of $\boldsymbol{P}$ to lower the complexity of the algorithm. Denote a positive definite matrix $\bar{\boldsymbol{P}}\in\mathbb{R}^{n\times n}$ to satisfy $-\bar{\boldsymbol{P}}\boldsymbol{R}^{-1}\bar{\boldsymbol{P}}=-(\boldsymbol{Q}+\boldsymbol{E})$, which also means

$$(\boldsymbol{R}^{-1}\bar{\boldsymbol{P}})^2=\boldsymbol{R}^{-1}(\boldsymbol{Q}+\boldsymbol{E}).$$

Post-multiplying the above equation by $\otimes\boldsymbol{I}_m$, we obtain

$$[(\boldsymbol{R}^{-1}\bar{\boldsymbol{P}})^2]\otimes\boldsymbol{I}_m=[\boldsymbol{R}^{-1}(\boldsymbol{Q}+\boldsymbol{E})]\otimes\boldsymbol{I}_m. \quad (6)$$

Considering the property of the Kronecker product, ARE (6) is equivalent to ARE (5) if choosing $\boldsymbol{P}=\bar{\boldsymbol{P}}\otimes\boldsymbol{I}_m$. The only solution to Eq. (6) is $\bar{\boldsymbol{P}}=[\boldsymbol{R}(\boldsymbol{Q}+\boldsymbol{E})]^{1/2}$. Then the control input is

$$\boldsymbol{u}=-[(\boldsymbol{R}^{-1}\bar{\boldsymbol{P}})\otimes\boldsymbol{I}_m]\tilde{\boldsymbol{x}}=-[(\boldsymbol{R}^{-1}(\boldsymbol{Q}+\boldsymbol{E}))^{1/2}\otimes\boldsymbol{I}_m]\tilde{\boldsymbol{x}}. \quad (7)$$

In another way, the distributed control law for each agent should employ only the local information and the relative information between the agent and its neighbors. It means that the control input must be subject to the following form:

$$\boldsymbol{u}_i=-k_i(\boldsymbol{x}_i-\boldsymbol{\delta}_i)-\sum_{i=1}^{n}a_{ij}(\boldsymbol{x}_i-\boldsymbol{x}_j-\boldsymbol{\Delta}_{ij}), \quad (8)$$

where k_i is the local feedback gain to let the ith agent achieve its objective, and a_{ij} is the entry of the adjacency matrix of the topology that needs to be designed. The matrix form of $\boldsymbol{u}_i$ is

$$\boldsymbol{u}=-(\boldsymbol{L}\otimes\boldsymbol{I}_m)\tilde{\boldsymbol{x}}-(\boldsymbol{K}\otimes\boldsymbol{I}_m)\tilde{\boldsymbol{x}}, \quad (9)$$

where $\boldsymbol{K}=\mathrm{diag}(k_1,k_2,\ldots,k_n)$. Comparing the form of Eq. (7) with the form of Eq. (9), it is critical to show that $(\boldsymbol{R}^{-1}\boldsymbol{Q}+\boldsymbol{R}^{-1}\boldsymbol{E})^{1/2}$ can be separated into the sum of a Laplacian matrix and a diagonal matrix whose diagonal entries are positive.

Theorem 1 In the LQR formation for single-integrator multi-agent systems with cost function (3), the optimal algorithm is

$$\boldsymbol{u}=-[(\boldsymbol{R}^{-1}(\boldsymbol{Q}+\boldsymbol{E}))^{1/2}\otimes\boldsymbol{I}_m]\tilde{\boldsymbol{x}}=-[(\boldsymbol{L}+\boldsymbol{K})\otimes\boldsymbol{I}_m]\tilde{\boldsymbol{x}}, \quad (10)$$

where

$$\begin{aligned}\boldsymbol{L}=&(\boldsymbol{R}^{-1}\boldsymbol{Q}+\boldsymbol{R}^{-1}\boldsymbol{E})^{1/2}\\&-\mathrm{diag}[(\boldsymbol{R}^{-1}\boldsymbol{Q}+\boldsymbol{R}^{-1}\boldsymbol{E})^{1/2}\boldsymbol{1}_n]\end{aligned} \quad (11)$$

is a Laplacian matrix, and

$$\boldsymbol{K}=\mathrm{diag}[(\boldsymbol{R}^{-1}\boldsymbol{Q}+\boldsymbol{R}^{-1}\boldsymbol{E})^{1/2}\boldsymbol{1}_n] \quad (12)$$

is a positive definite diagonal matrix.

Proof $\boldsymbol{Q}$ is a symmetric Laplacian matrix, so it is also a singular M-matrix. Then it is obvious that matrix $\boldsymbol{R}^{-1}\boldsymbol{Q}$ is a singular M-matrix. As a result, matrix $\boldsymbol{R}^{-1}\boldsymbol{Q}+\boldsymbol{R}^{-1}\boldsymbol{E}$ is a nonsingular M-matrix, and its characteristic polynomial has no zero root. From Lemma 2, $(\boldsymbol{R}^{-1}\boldsymbol{Q}+\boldsymbol{R}^{-1}\boldsymbol{E})^{1/2}$ is a nonsingular M-matrix, and it can be rewritten as $\alpha\boldsymbol{I}-\boldsymbol{C}$, where $\alpha>0$, $\boldsymbol{C}=[c_{ij}]\in\mathbb{R}^{n\times n}$, $\rho(\boldsymbol{C})<\alpha$, and all entries of $\boldsymbol{C}$ are nonnegative. If $\boldsymbol{L}$ is defined as Eq. (11) and $\boldsymbol{K}$ is defined as Eq. (12), we have

$$\alpha\boldsymbol{I}-\boldsymbol{C}=\boldsymbol{L}+\mathrm{diag}[(\alpha\boldsymbol{I}-\boldsymbol{C})\boldsymbol{1}_n]$$

and

$$\mathrm{diag}[(\alpha\boldsymbol{I}-\boldsymbol{C})\boldsymbol{1}_n]=\begin{cases}\alpha-\sum\limits_{i=1}^{n}c_{ij}, & i=j,\\ 0, & i\neq j.\end{cases}$$

Let

$$\begin{aligned}\boldsymbol{C}'&=\boldsymbol{C}+\mathrm{diag}[(\alpha\boldsymbol{I}-\boldsymbol{C})\boldsymbol{1}_n]\\&=\begin{cases}\alpha-\sum\limits_{i=1}^{n}c_{ij}+c_{ii}, & i=j,\\ c_{ij}, & i\neq j.\end{cases}\end{aligned}$$

Therefore,

$$\boldsymbol{L}=\alpha\boldsymbol{I}-\boldsymbol{C}'=\begin{cases}\sum\limits_{i=1,i\neq j}^{n}c_{ij}, & i=j,\\ -c_{ij}, & i\neq j.\end{cases}$$

Because all entries of $\boldsymbol{C}$ are nonnegative, the above is the standard definition of a Laplacian matrix. Thus, $\boldsymbol{L}$ is a Laplacian matrix.

Choosing the local feedback matrix

$$\boldsymbol{K}=\mathrm{diag}[(\boldsymbol{R}^{-1}\boldsymbol{Q}+\boldsymbol{R}^{-1}\boldsymbol{E})^{1/2}\boldsymbol{1}_n],$$

we have that the diagonal entries of $\boldsymbol{K}$ are positive, since $(\boldsymbol{R}^{-1}\boldsymbol{Q}+\boldsymbol{R}^{-1}\boldsymbol{E})^{1/2}$ is strictly row diagonally dominant according to Lemma 3.

Remark 1 Note that the matrix $(\boldsymbol{R}^{-1}\boldsymbol{Q}+\boldsymbol{R}^{-1}\boldsymbol{E})^{1/2}-\mathrm{diag}[(\boldsymbol{R}^{-1}\boldsymbol{Q}+\boldsymbol{R}^{-1}\boldsymbol{E})^{1/2}\boldsymbol{1}_n]$ is not necessarily symmetric in general, so the designed communication topology may be directed. Moreover, the choice of the symmetric Laplacian matrix $\boldsymbol{Q}$ depends on the desired energy cost of formation between two agents that may not be connected physically. The designed Laplacian matrix $\boldsymbol{L}$ is usually different from the corresponding matrix $\boldsymbol{Q}$. Essentially, the communication topology associated with $\boldsymbol{L}$ is a complete graph; i.e., each pair of distinct nodes is connected by an edge. This is because it may be impossible to rely on only local information to obtain the global optimal objective. Similar discussion was given in Cao and Ren (2010). In this subsection, we focus on the global optimization problem, so that each agent should have the full knowledge of all other agents. This problem should not be worked out by using local information. In addition, in this subsection we just design a control algorithm to guarantee that the cost function of the multi-agent systems is optimal, and do not investigate how to solve a distributed control problem. Furthermore, if the number of agents is very small, the all-to-all communication structure among agents is not very complex, and we can also obtain the optimal index for the networked system using the proposed control algorithm.

Remark 2 Different from most literature in which all agents have the same control parameters, the distributed optimal formation algorithm (8) has different local feedback gains k_i for different agents to make the cost function J minimum.

3.2 LQR formation for physically coupled multi-agent systems

In the above subsection we have studied the optimal formation problem for multi-agent systems with no initial couplings. However, in many cases, initially there may exist physical couplings between the agents. For example, in a multi-vehicle coordinated control problem initially some vehicles already have some communication exchanges with other vehicles for other missions. These actions can be viewed as physical couplings. In this subsection, we study the optimal formation for multi-agent systems with physically interconnected couplings. The communication and local feedback matrix need to be designed. The dynamics of the ith agent is given by

$$\dot{\boldsymbol{x}}_i=\sum_{i=1}^{n}a_{ij}(\boldsymbol{x}_j-\boldsymbol{x}_i-\boldsymbol{\Delta}_{ji})+\boldsymbol{u}_i,$$

and its matrix form is

$$\dot{\tilde{\boldsymbol{x}}}=-(\boldsymbol{L}_1\otimes\boldsymbol{I}_m)\tilde{\boldsymbol{x}}+\boldsymbol{u}, \tag{13}$$

where $\tilde{\boldsymbol{x}}$ is defined the same as in the above subsection, a_{ij} is the existing undirected coupling weight between the ith agent and the jth agent, and $\boldsymbol{L}_1$ is the Laplacian matrix corresponding to the couplings. Because it is assumed that the communication topology between the agents is undirected, all agents in the topology have equal importance, and thus all agents have the same optimal weights r_1 and r_2. The cost function is defined by

$$\begin{aligned}J=&\int_0^\infty\left(\sum_{i=1}^{n}\sum_{j=1}^{i-1}a_{ij}(\boldsymbol{x}_i-\boldsymbol{x}_j-\boldsymbol{\Delta}_{ij})^{\mathrm{T}}(\boldsymbol{x}_i-\boldsymbol{x}_j-\boldsymbol{\Delta}_{ij})\right.\\&\left.+\sum_{i=1}^{n}r_1(\boldsymbol{x}_i-\boldsymbol{\delta}_i)^{\mathrm{T}}(\boldsymbol{x}_i-\boldsymbol{\delta}_i)+\sum_{i=1}^{n}r_2\boldsymbol{u}_i^{\mathrm{T}}\boldsymbol{u}_i\right)\mathrm{d}t\\=&\int_0^\infty\left[\tilde{\boldsymbol{x}}^{\mathrm{T}}(\boldsymbol{L}_1\otimes\boldsymbol{I}_m)\tilde{\boldsymbol{x}}+r_1\tilde{\boldsymbol{x}}^{\mathrm{T}}\tilde{\boldsymbol{x}}+r_2\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}\right]\mathrm{d}t.\end{aligned} \tag{14}$$

The control input is subject to the form

$$\boldsymbol{u}_i=-\sum_{i=1}^{n}b_{ij}(\boldsymbol{x}_i-\boldsymbol{x}_j-\boldsymbol{\Delta}_{ij})-k_i(\boldsymbol{x}_i-\boldsymbol{\delta}_i),$$

where k_i is the local feedback gain, and b_{ij} is the edge weight that needs to be designed. Its matrix form is

$$\boldsymbol{u}=-(\boldsymbol{L}_2\otimes\boldsymbol{I}_m)\tilde{\boldsymbol{x}}-(\boldsymbol{K}\otimes\boldsymbol{I}_m)\tilde{\boldsymbol{x}},$$

where $\boldsymbol{K}=\mathrm{diag}(k_1,k_2,\ldots,k_n)$, and

$$\boldsymbol{L}_2=\begin{cases}\sum\limits_{i=1,i\neq j}^{n}b_{ij}, & i=j,\\ -b_{ij}, & i\neq j.\end{cases}$$

It is obvious that $\boldsymbol{L}_2\boldsymbol{1}_n=\boldsymbol{0}_n$ always holds. Note that b_{ij} may be negative, so $\boldsymbol{L}_2$ is not a Laplacian matrix in some cases. To solve the optimal formation from the LQR perspective, $\boldsymbol{u}$ should be designed to obtain the minimal J. Then we have the following theorem:

Theorem 2 In the optimal formation under undirected couplings for the cost function (14) subject to Eq. (13), the optimal control parameter matrices are $\boldsymbol{L}_2=r_2^{-1}\bar{\boldsymbol{P}}-(r_1/r_2)^{1/2}\boldsymbol{I}_n$ and $\boldsymbol{K}=(r_1/r_2)^{1/2}\boldsymbol{I}_n$, where $\bar{\boldsymbol{P}}$ can be constructed as $\bar{\boldsymbol{P}}=\boldsymbol{M}^{\mathrm{T}}\mathrm{diag}(p_1,p_2,\ldots,p_n)\boldsymbol{M}$ with $\boldsymbol{M}$ being an orthogonal matrix such that $\boldsymbol{L}_1=\boldsymbol{M}^{\mathrm{T}}\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)\boldsymbol{M}$; moreover, we have

$$p_i=-\lambda_ir_2+[\lambda_i^2r_2^2+r(\lambda_i+r_1)]^{1/2},\quad i=1,2,\ldots,n.$$

Proof From Anderson and Moore (2007), the optimal solution to Eq. (14) subject to Eq. (13) is

$\boldsymbol{u}=-r_2^{-1}\boldsymbol{P}\tilde{\boldsymbol{x}}$, where $\boldsymbol{P}\in\mathbb{R}^{(mn)\times(mn)}$ is the solution to the following ARE:

$$(-\boldsymbol{L}_1\otimes\boldsymbol{I}_m)^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}(-\boldsymbol{L}_1\otimes\boldsymbol{I}_m)-r_2^{-1}\boldsymbol{P}^2 + (\boldsymbol{L}_1+r_1\boldsymbol{I}_n)\otimes\boldsymbol{I}_m=\boldsymbol{0}_{mn}. \tag{15}$$

Next, to reduce the algorithm complexity, we denote a symmetrical matrix $\bar{\boldsymbol{P}}\in\mathbb{R}^{n\times n}$ to reduce the complexity of Eq. (15). Assume $\bar{\boldsymbol{P}}$ satisfies

$$(-\boldsymbol{L}_1)^{\mathrm{T}}\bar{\boldsymbol{P}}+\bar{\boldsymbol{P}}(-\boldsymbol{L}_1)-r_2^{-1}\bar{\boldsymbol{P}}^2+(\boldsymbol{L}_1+r_1\boldsymbol{I}_n)=\boldsymbol{0}_n. \tag{16}$$

Then post-multiplying Eq. (16) by $\otimes\boldsymbol{I}_m$, we obtain

$$[(-\boldsymbol{L}_1^{\mathrm{T}}\bar{\boldsymbol{P}})\otimes\boldsymbol{I}_m]+\{[\bar{\boldsymbol{P}}(-\boldsymbol{L}_1)]\otimes\boldsymbol{I}_m\} - r_2^{-1}\bar{\boldsymbol{P}}^2\otimes\boldsymbol{I}_m+(\boldsymbol{L}_1+r_1\boldsymbol{I}_n)\otimes\boldsymbol{I}_m=\boldsymbol{0}_{mn}. \tag{17}$$

If $\boldsymbol{P}=\bar{\boldsymbol{P}}\otimes\boldsymbol{I}_m$, according to the property of the Kronecker product, ARE (17) is equivalent to ARE (16). It means that the solution to ARE (17) is given by $\boldsymbol{P}=\bar{\boldsymbol{P}}\otimes\boldsymbol{I}_m$.

Because the couplings are undirected, $\boldsymbol{L}_1$ is symmetric. Thus, there exists an orthogonal matrix $\boldsymbol{M}$ which satisfies $\boldsymbol{L}_1=\boldsymbol{M}^{\mathrm{T}}\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)\boldsymbol{M}$. Constructing $\bar{\boldsymbol{P}}=\boldsymbol{M}^{\mathrm{T}}\mathrm{diag}(p_1,p_2,\ldots,p_n)\boldsymbol{M}$, we obtain

$$\begin{aligned}\boldsymbol{L}_1\bar{\boldsymbol{P}}=&\boldsymbol{M}^{\mathrm{T}}\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)\boldsymbol{M}\\&\cdot\boldsymbol{M}^{\mathrm{T}}\mathrm{diag}(p_1,p_2,\ldots,p_n)\boldsymbol{M}\\=&\boldsymbol{M}^{\mathrm{T}}\mathrm{diag}(p_1,p_2,\ldots,p_n)\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)\boldsymbol{M}\\=&\bar{\boldsymbol{P}}\boldsymbol{L}_1.\end{aligned}$$

Pre- and post-multiplying ARE (16) by $\boldsymbol{M}^{\mathrm{T}}$ and $\boldsymbol{M}$ respectively, we have

$$\begin{aligned}&-2\boldsymbol{M}^{\mathrm{T}}\boldsymbol{L}_1\bar{\boldsymbol{P}}\boldsymbol{M}-r_2^{-1}\boldsymbol{M}^{\mathrm{T}}\bar{\boldsymbol{P}}^2\boldsymbol{M}\\&+\boldsymbol{M}^{\mathrm{T}}(\boldsymbol{L}_1+r_1\boldsymbol{I}_n)\boldsymbol{M}=\boldsymbol{0}_n,\end{aligned}$$

which means that

$$\begin{aligned}&-2\mathrm{diag}(p_1,p_2,\ldots,p_n)\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)\\&-r_2^{-1}\mathrm{diag}(p_1^2,p_2^2,\ldots,p_n^2)\\&+\mathrm{diag}(\lambda_1+r_1,\lambda_2+r_1,\ldots,\lambda_n+r_1)=\boldsymbol{0}_n.\end{aligned}$$

From the entry-wise aspect, the above matrix equation becomes

$$-2\lambda_ip_i-r_2^{-1}p_i^2+(\lambda_i+r_1)=0,\quad i=1,2,\ldots,n.$$

The solution to the above equation is

$$p_i=-\lambda_ir_2\pm[\lambda_i^2r_2^2+r_2(\lambda_i+r_1)]^{1/2}.$$

Because $\bar{\boldsymbol{P}}$ is positive semi-definite, p_i should be $-\lambda_ir+[\lambda_i^2r_2^2+r_2(\lambda_i+r_1)]^{1/2}$, which is positive. It means that $\boldsymbol{P}$ is positive definite. Letting $\bar{\boldsymbol{P}}=r_2(\boldsymbol{L}_2+\boldsymbol{K})$ and $\boldsymbol{K}=\mathrm{diag}(r_2^{-1}\bar{\boldsymbol{P}}\boldsymbol{1}_n)$, $\boldsymbol{L}_2=r_2^{-1}\bar{\boldsymbol{P}}-\mathrm{diag}(r_2^{-1}\bar{\boldsymbol{P}}\boldsymbol{1}_n)$ must be a symmetric matrix because $\bar{\boldsymbol{P}}$ and $\boldsymbol{K}$ are symmetric. Next, if choosing the first column of $\boldsymbol{M}$ as $\boldsymbol{1}_n/\sqrt{n}$, $\boldsymbol{1}_n/\sqrt{n}$ is also an eigenvector of $\bar{\boldsymbol{P}}$ with the corresponding eigenvalue $(r_1r_2)^{1/2}$. Therefore, $\boldsymbol{K}=(r_1/r_2)^{1/2}\boldsymbol{I}_n$ is a positive definite diagonal matrix.

Remark 3 It is obvious that $\boldsymbol{L}_2\boldsymbol{1}_n=\boldsymbol{0}_n$, but $\boldsymbol{L}_2$ may not be a Laplacian matrix, because the nondiagonal entries of $\boldsymbol{L}_2$ may be positive. It means that the designed edge weights may be negative, that is, $b_{ij}<0$. In Altafini (2013), the communication topology with negative edge weights has been discussed. If a multi-agent system network has negative edge weights and nonnegative weights, it is more plausible that some agents collaborate, while others compete. In a sense, the traditional consensus Laplacian schemes are a special case for the consensus scheme with edge weights which may be positive or nonnegative.

Remark 4 The formation performance for multi-agent systems essentially depends on matrix $\boldsymbol{L}_1+\boldsymbol{L}_2$. Considering that $\boldsymbol{L}_2=r_2^{-1}\bar{\boldsymbol{P}}-(r_1/r_2)^{1/2}\boldsymbol{I}_n$ and $\bar{\boldsymbol{P}}$ have the same eigenvectors as $\boldsymbol{L}_1$, the eigenvalue of $\boldsymbol{L}_1+\boldsymbol{L}_2$ is $[\lambda_i^2+(\lambda_i+r_1)/r_2]^{1/2}-(r_1/r_2)^{1/2}$, which is nonnegative, and thus $\boldsymbol{L}_1+\boldsymbol{L}_2$ is positive semi-definite. All eigenvalues of the closed-loop system matrix $-(\boldsymbol{L}_1+\boldsymbol{L}_2+\boldsymbol{K})$ locate on the open left-half plane.

For a general case in which the couplings are directed and the gains for different agents are not equal, the cost function becomes

$$\begin{aligned}J=\int_0^\infty&[\tilde{\boldsymbol{x}}^{\mathrm{T}}(\frac{\boldsymbol{L}_1+\boldsymbol{L}_1^{\mathrm{T}}}{2}\otimes\boldsymbol{I}_m)\tilde{\boldsymbol{x}}+\tilde{\boldsymbol{x}}^{\mathrm{T}}(\boldsymbol{E}\otimes\boldsymbol{I}_m)\tilde{\boldsymbol{x}}\\&+\boldsymbol{u}^{\mathrm{T}}(\boldsymbol{R}\otimes\boldsymbol{I}_m)\boldsymbol{u}]\mathrm{d}t,\end{aligned} \tag{18}$$

where $\boldsymbol{E}$ and $\boldsymbol{R}$ are positive definite diagonal matrices to represent the weight of the individual goal of each agent and the energy of the control inputs of the overall system, respectively. We also use $(\boldsymbol{L}_1+\boldsymbol{L}_1^{\mathrm{T}})/2$ to describe the formation relative errors between agents.

Corollary 1 Let the dynamics of multi-agent systems in matrix form be $\dot{\tilde{\boldsymbol{x}}}=-(\boldsymbol{L}_1\otimes\boldsymbol{I}_m)\tilde{\boldsymbol{x}}+\boldsymbol{u}$, and suppose there exists a positive definite matrix $\bar{\boldsymbol{P}}$ satisfying the following ARE:

$$(-\boldsymbol{L}_1)^{\mathrm{T}}\bar{\boldsymbol{P}}+\bar{\boldsymbol{P}}(-\boldsymbol{L}_1)-\bar{\boldsymbol{P}}\boldsymbol{R}^{-1}\bar{\boldsymbol{P}}+\left(\frac{\boldsymbol{L}_1+\boldsymbol{L}_1^{\mathrm{T}}}{2}+\boldsymbol{E}\right)=\mathbf{0}_n.$$

Then the control law $\boldsymbol{u} = -[(\boldsymbol{R}^{-1}\bar{\boldsymbol{P}})\otimes\boldsymbol{I}_m]\tilde{\boldsymbol{x}}$ is optimal with respect to cost function (18).

The proof is similar to that of Theorem 2, and it is omitted here.

3.3 LQR formation for multi-agent systems under fixed topology

The above two subsections allow the communication topology between agents to be constructed. However, in many conditions it is impossible to design or change the communication topology. It means that the optimal formation problem has to be studied under a fixed and unchangeable topology. In this subsection, we investigate the case in which the topology between single-integrator multi-agent systems $\dot{\boldsymbol{x}}_i = \boldsymbol{u}_i$ is fixed and has been determined in advance. It means that only the local feedback can be designed. For simplicity, we assume all agents employ the same local feedback gain. The dynamics of each agent is subject to

$$\begin{cases}\dot{\boldsymbol{x}}=\boldsymbol{u}_i,\\ \boldsymbol{u}_i=-\sum_{i=1}^{n}a_{ij}(\boldsymbol{x}_i-\boldsymbol{x}_j-\boldsymbol{\Delta}_{ij})-k(\boldsymbol{x}_i-\boldsymbol{\delta}_i).\end{cases}\tag{19}$$

It is assumed that the communication topology between agents is undirected, so all agents in the topology have equal importance. Thus, all agents have the same optimal weights r_1 and r_2. The cost function is

$$\begin{aligned}J=&\int_0^{\infty}\left(\sum_{i=1}^{n}\sum_{j=1}^{i-1}a_{ij}(\boldsymbol{x}_i-\boldsymbol{x}_j-\boldsymbol{\Delta}_{ij})^2\right.\\&\left.+\sum_{i=1}^{n}r_1(\boldsymbol{x}_i-\boldsymbol{\delta}_i)^2+\sum_{i=1}^{n}r_2\boldsymbol{u}_i^{\mathrm{T}}\boldsymbol{u}_i\right)\mathrm{d}t\\=&\int_0^{\infty}\left[\tilde{\boldsymbol{x}}^{\mathrm{T}}(\boldsymbol{L}\otimes\boldsymbol{I}_m)\tilde{\boldsymbol{x}}+r_1\tilde{\boldsymbol{x}}^{\mathrm{T}}\tilde{\boldsymbol{x}}+r_2\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}\right]\mathrm{d}t,\end{aligned}\tag{20}$$

where $\boldsymbol{L}$ is a symmetric Laplacian matrix corresponding to the undirected topology between the agents. The corresponding optimization formation problem is to find $\boldsymbol{u}_i$ for each agent to minimize J subject to Eq. (19).

Theorem 3 In the LQR formation control problem with the cost function J proposed in Eq. (20) subject to Eq. (19), if the communication topology is fixed and undirected, the optimal control parameter k is given by

$$\begin{aligned}k=&-\frac{\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)(\boldsymbol{L}\otimes\boldsymbol{I}_m)\tilde{\boldsymbol{x}}(0)}{\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)\tilde{\boldsymbol{x}}(0)}\\&+\frac{1}{r_2\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)\tilde{\boldsymbol{x}}(0)}\{r_2^2[\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)(\boldsymbol{L}\otimes\boldsymbol{I}_m)\tilde{\boldsymbol{x}}(0)]^2\\&-r_2\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)\tilde{\boldsymbol{x}}(0)\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)[r_2(\boldsymbol{L}\otimes\boldsymbol{I}_m)^2\\&-\boldsymbol{L}\otimes\boldsymbol{I}_m-r_1\boldsymbol{I}_{mn}]\tilde{\boldsymbol{x}}(0)\}^{1/2},\end{aligned}$$

where $r_2 \leq 1/(2d_{\max})$, and $d_{\max} = \max_i d_i$, $i = 1,2,\ldots,n$.

Proof The dynamics of $\tilde{x}_i$ is rewritten in matrix form as

$$\dot{\tilde{\boldsymbol{x}}}=-(\boldsymbol{L}\otimes\boldsymbol{I}_m)\tilde{\boldsymbol{x}}-k\tilde{\boldsymbol{x}},$$

so

$$\tilde{\boldsymbol{x}}(t)=\mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}\tilde{\boldsymbol{x}}(0).$$

Then we have

$$\boldsymbol{u}=-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})\mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}\tilde{\boldsymbol{x}}(0).$$

The cost function J becomes

$$\begin{aligned}J=&\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)\int_0^{\infty}[\mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}(\boldsymbol{L}\otimes\boldsymbol{I}_m\\&+r_1\boldsymbol{I}_{mn})\mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_{mn}+k\boldsymbol{I}_{mn})t}\\&+r_2\mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}(\boldsymbol{L}\otimes\boldsymbol{I}_m\\&+k\boldsymbol{I}_{mn})^2\mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}]\mathrm{d}t\tilde{\boldsymbol{x}}(0).\end{aligned}\tag{21}$$

Because the matrices $\boldsymbol{L}\otimes\boldsymbol{I}_m$, $\boldsymbol{L}\otimes\boldsymbol{I}_m+r_1\boldsymbol{I}_{mn}$, $\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn}$, and $\mathrm{e}^{(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}$ have the same eigenvectors, the multiplications of these matrices are commutative. To solve the integration (21), we employ the approach of integration by parts. In this way, we have

$$\begin{aligned}&\int_0^{\infty}\mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}(\boldsymbol{L}\otimes\boldsymbol{I}_m\\&+r_1\boldsymbol{I}_{mn})\mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}\mathrm{d}t\\=&\,\mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}(\boldsymbol{L}\otimes\boldsymbol{I}_m+r_1\boldsymbol{I}_{mn})[-(\boldsymbol{L}\otimes\boldsymbol{I}_m\\&+k\boldsymbol{I}_{mn})^{-1}\mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}]\Big|_0^{\infty}\\&-\int_0^{\infty}\frac{\mathrm{d}\mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}(\boldsymbol{L}\otimes\boldsymbol{I}_m+r_1\boldsymbol{I}_{mn})}{\mathrm{d}t}\\&\cdot[-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})^{-1}\mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}]\mathrm{d}t\end{aligned}$$

$$= (\boldsymbol{L}\otimes\boldsymbol{I}_m + r_1\boldsymbol{I}_{mn})(\boldsymbol{L}\otimes\boldsymbol{I}_m + k\boldsymbol{I}_{mn})^{-1} - \int_0^\infty \mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}(\boldsymbol{L}\otimes\boldsymbol{I}_m + r_1\boldsymbol{I}_{mn})\mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}\mathrm{d}t,$$

so

$$\int_0^\infty \mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}(\boldsymbol{L}\otimes\boldsymbol{I}_m + r_1\boldsymbol{I}_{mn})\mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}\mathrm{d}t = \frac{1}{2}(\boldsymbol{L}\otimes\boldsymbol{I}_m + r_1\boldsymbol{I}_{mn})(\boldsymbol{L}\otimes\boldsymbol{I}_m + k\boldsymbol{I}_{mn})^{-1}.$$

Similarly, we have

$$\begin{aligned}&\int_0^\infty \mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}(\boldsymbol{L}\otimes\boldsymbol{I}_m + k\boldsymbol{I}_{mn})^2\mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}\mathrm{d}t\\ &= \mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}(\boldsymbol{L}\otimes\boldsymbol{I}_m + k\boldsymbol{I}_{mn})^2[-(\boldsymbol{L}\otimes\boldsymbol{I}_m + k\boldsymbol{I}_{mn})^{-1}\mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}]\Big|_0^\infty\\ &\quad+\int_0^\infty \frac{\mathrm{d}\mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}}{\mathrm{d}t}(\boldsymbol{L}\otimes\boldsymbol{I}_m + k\boldsymbol{I}_{mn})\mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}\mathrm{d}t\\ &= (\boldsymbol{L}\otimes\boldsymbol{I}_m + k\boldsymbol{I}_{mn}) - \int_0^\infty \mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}(\boldsymbol{L}\otimes\boldsymbol{I}_m + k\boldsymbol{I}_{mn})^2\mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}\mathrm{d}t.\end{aligned}$$

Therefore,

$$\int_0^\infty \mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m+k\boldsymbol{I}_{mn})t}(\boldsymbol{L}\otimes\boldsymbol{I}_m + k\boldsymbol{I}_{mn})^2 \cdot \mathrm{e}^{-(\boldsymbol{L}\otimes\boldsymbol{I}_m)t}\mathrm{e}^{k\boldsymbol{I}_{mn}t}\mathrm{d}t = (\boldsymbol{L}\otimes\boldsymbol{I}_m + k\boldsymbol{I}_{mn})/2$$

holds. Then the cost function becomes

$$J = \tilde{\boldsymbol{x}}^{\mathrm{T}}(0)\left[\frac{1}{2}(\boldsymbol{L}\otimes\boldsymbol{I}_m + r_1\boldsymbol{I}_{mn})(\boldsymbol{L}\otimes\boldsymbol{I}_{mn} + k\boldsymbol{I}_{mn})^{-1} + \frac{r_2}{2}(\boldsymbol{L}\otimes\boldsymbol{I}_m + k\boldsymbol{I}_{mn})\right]\tilde{\boldsymbol{x}}(0).$$

Taking the derivative of J with respect to k gives

$$\frac{\mathrm{d}J}{\mathrm{d}k} = \frac{1}{2}\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)[-(\boldsymbol{L}\otimes\boldsymbol{I}_m + r_1\boldsymbol{I}_{mn})(\boldsymbol{L}\otimes\boldsymbol{I}_{mn} + k\boldsymbol{I}_{mn})^{-2} + r_2\boldsymbol{I}_{mn}]\tilde{\boldsymbol{x}}(0).$$

Setting $\mathrm{d}J/\mathrm{d}k = 0$ gives $\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)[-(\boldsymbol{L}\otimes\boldsymbol{I}_m + r_1\boldsymbol{I}_{mn})(\boldsymbol{L}\otimes\boldsymbol{I}_m + k\boldsymbol{I}_{mn})^{-2} + r_2\boldsymbol{I}_{mn}]\tilde{\boldsymbol{x}}(0) = 0$, which means

$$k^2[r_2\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)\tilde{\boldsymbol{x}}(0)] + 2k[r_2\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)(\boldsymbol{L}\otimes\boldsymbol{I}_m)\tilde{\boldsymbol{x}}(0)] + \tilde{\boldsymbol{x}}^{\mathrm{T}}(0)[r_2(\boldsymbol{L}\otimes\boldsymbol{I}_m)^2 - \boldsymbol{L}\otimes\boldsymbol{I}_m - r_1\boldsymbol{I}_{mn}]\tilde{\boldsymbol{x}}(0) = 0 \tag{22}$$

for any vector $\tilde{\boldsymbol{x}}(0)$.

It is obvious that the above equation is a quadratic equation with respect to k, so the solution is

$$\begin{aligned}k = &-\frac{\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)(\boldsymbol{L}\otimes\boldsymbol{I}_m)\tilde{\boldsymbol{x}}(0)}{\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)\tilde{\boldsymbol{x}}(0)}\\ &\pm\frac{1}{[r\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)\tilde{\boldsymbol{x}}(0)]}\{r^2[\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)(\boldsymbol{L}\otimes\boldsymbol{I}_m)\tilde{\boldsymbol{x}}(0)]^2\\ &-r_2\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)\tilde{\boldsymbol{x}}(0)\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)[r_2(\boldsymbol{L}\otimes\boldsymbol{I}_m)^2 - \boldsymbol{L}\otimes\boldsymbol{I}_m\\ &-r_1\boldsymbol{I}_{mn}]\tilde{\boldsymbol{x}}(0)\}^{1/2},\end{aligned}$$

To guarantee that k is a real number, the following inequality should be satisfied:

$$\begin{aligned}&r_2^2[\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)(\boldsymbol{L}\otimes\boldsymbol{I}_m)\tilde{\boldsymbol{x}}(0)]^2\\ &-r_2\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)\tilde{\boldsymbol{x}}(0)\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)[r_2(\boldsymbol{L}\otimes\boldsymbol{I}_m)^2 - \boldsymbol{L}\otimes\boldsymbol{I}_m\\ &-r_1\boldsymbol{I}_{mn}]\tilde{\boldsymbol{x}}(0)\geq 0.\end{aligned} \tag{23}$$

Because all eigenvalues of $\boldsymbol{L}$ locate in the circle whose radius is $d_{\max}$ with the center at $(d_{\max}, 0)$, $r_2\lambda_i - 1 \leq 0$ holds if $2r_2d_{\max} \leq 1$. Then $r_2\boldsymbol{L}^2 - \boldsymbol{L}$ is negative semi-definite, inequality (23) holds, and k is real. Moreover, as k should be positive, '$\pm$' is chosen as '$+$'. Considering that $r_2\boldsymbol{L}^2 - \boldsymbol{L}$ is positive semi-definite and $[4r_2^2(\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)(\boldsymbol{L}\otimes\boldsymbol{I}_m)\tilde{\boldsymbol{x}}(0))^2 + 4r_2\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)\tilde{\boldsymbol{x}}(0)\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)(r_1\boldsymbol{I}_{mn})\tilde{\boldsymbol{x}}(0)]^{1/2} > |2r_2\tilde{\boldsymbol{x}}^{\mathrm{T}}(0)(\boldsymbol{L}\otimes\boldsymbol{I}_m)\tilde{\boldsymbol{x}}(0)|$, k must be positive.

Remark 5 We obtain constraint (23) from quadratic equation (22) with respect to k. It makes the control parameter k a real constant, and we have simplified it to $r_2 \leq d_{\max}/2$. Moreover, if constraint (23) is not satisfied, k will have the imaginary part, which is impossible to apply in practical systems. The condition $r_2 \leq d_{\max}/2$ will limit the weights of the control actions; that is, if r_2 is larger than $d_{\max}/2$ which is related with the fixed communication topology, or constraint (23) is not satisfied, the optimal control parameter k is not applied in physical systems.

Remark 6 In practice, if inequality (23) holds, k is a real number. As a result, in many cases, the condition $r_2 \leq 1/(2d_{\max})$ is not necessary.

4 Simulation

In this section, we give three examples to illustrate the optimal formation problems from the

LQR perspective discussed in Sections 3.1–3.3, respectively. To observe the performance of the relative formations during convergence more intuitively, we consider the problems in a 2D plane with six agents. The initial values of all agents are chosen as $\boldsymbol{x}_1(0) = [-1,-1]^{\mathrm{T}}$, $\boldsymbol{x}_2(0) = [-1,1]^{\mathrm{T}}$, $\boldsymbol{x}_3(0) = [1,-1]^{\mathrm{T}}$, $\boldsymbol{x}_4(0) = [0.5,-0.5]^{\mathrm{T}}$, $\boldsymbol{x}_5(0) = [2,-1]^{\mathrm{T}}$, and $\boldsymbol{x}_6(0) = [1.5,-1]^{\mathrm{T}}$, and the objectives are given by $\boldsymbol{\delta}_1 = [0,0]^{\mathrm{T}}$, $\boldsymbol{\delta}_2 = [1,0]^{\mathrm{T}}$, $\boldsymbol{\delta}_3 = [3/2,\sqrt{3}/2]^{\mathrm{T}}$, $\boldsymbol{\delta}_4 = [1,\sqrt{3}]^{\mathrm{T}}$, $\boldsymbol{\delta}_5 = [0,\sqrt{3}]^{\mathrm{T}}$, and $\boldsymbol{\delta}_6 = [-1/2,\sqrt{3}/2]^{\mathrm{T}}$ to constitute a regular hexagon.

Example 1 To verify the effectiveness of Theorem 1, we simply choose

$$\boldsymbol{Q} = \begin{bmatrix} 3 & -1 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{E} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Then from Theorem 1, we obtain

$$\boldsymbol{L} = \begin{bmatrix} 0.9984 & -0.4793 & -0.4883 & -0.0233 & -0.0030 & -0.0045 \\ -0.2398 & 0.5455 & -0.2848 & -0.0156 & -0.0021 & -0.0033 \\ -0.1634 & -0.1901 & 0.5615 & -0.1821 & -0.0108 & -0.0151 \\ -0.0059 & -0.0021 & -0.0022 & 0.3980 & -0.1769 & -0.2109 \\ -0.0066 & -0.0026 & -0.0027 & -0.0004 & 0.2217 & -0.2094 \\ -0.1059 & -0.0197 & -0.0206 & -0.0026 & -0.0005 & 0.1493 \end{bmatrix}.$$

And the optimal local feedback matrix is given by

$$\boldsymbol{K} = \begin{bmatrix} 1.4109 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.4117 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.4045 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.3337 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8736 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.6663 \end{bmatrix}.$$

Note that $\boldsymbol{L}$ is a nonsymmetric Laplacian matrix and $\boldsymbol{K}$ a positive diagonal matrix consistent with Theorem 1. The simulation time is set as 7 s. Figs. 1 and 2 show the trajectories of the six agents.

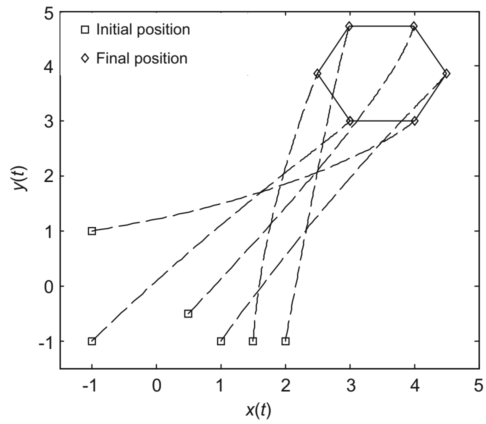

Fig. 1 Formation for the six agents (Example 1)

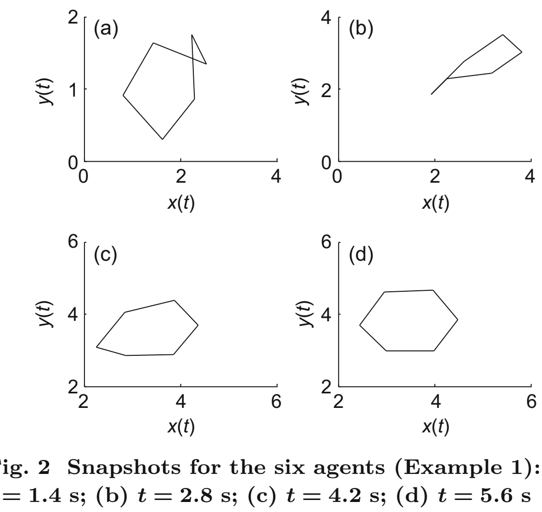

Fig. 2 Snapshots for the six agents (Example 1): (a) $t = 1.4$ s; (b) $t = 2.8$ s; (c) $t = 4.2$ s; (d) $t = 5.6$ s

To show that the proposed method is optimal, we present four trajectories of the cost function (3) in Fig. 3. The solid line is the result under the the proposed algorithm, and the other three lines are the results under the algorithms whose parameter matrices $\boldsymbol{K}$ and $\boldsymbol{L}$ are chosen randomly. It is seen that the proposed algorithm is better than the other three algorithms.

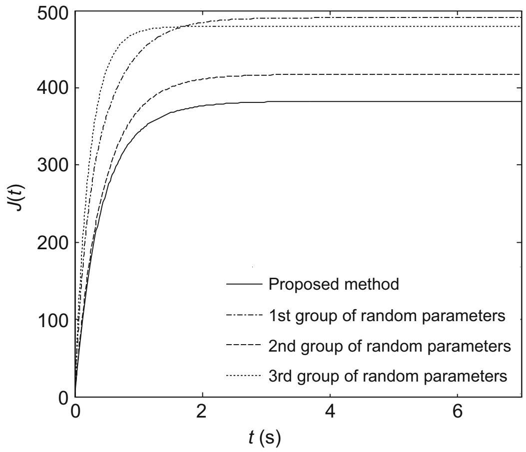

Fig. 3 Trajectories of cost function J (Example 1)

Example 2 To show the effectiveness of Theorem 2, we choose

$$\boldsymbol{L}_1=\begin{bmatrix} 0.3 & -0.1 & -0.1 & 0 & 0 & -0.1 \\ -0.1 & 0.3 & -0.1 & 0 & -0.1 & 0 \\ -0.1 & -0.1 & 0.3 & -0.1 & 0 & 0 \\ 0 & 0 & -0.1 & 0.1 & 0 & 0 \\ 0 & -0.1 & 0 & 0 & 0.1 & 0 \\ -0.1 & 0 & 0 & 0 & 0 & 0.1 \end{bmatrix},$$

$r_1 = 0.6$, and $r_2 = 1/3$, and set the simulation time as 10 s. It then follows from Theorem 2 that the optimal state feedback matrices are given by

$$\boldsymbol{L}_2=\begin{bmatrix} 0.0271 & -0.0084 & -0.0084 & -0.0006 & -0.0006 & -0.0090 \\ -0.0084 & 0.0271 & -0.0084 & -0.0006 & -0.0090 & -0.0006 \\ -0.0084 & -0.0084 & 0.0271 & -0.0090 & -0.0006 & -0.0006 \\ -0.0006 & -0.0006 & -0.0090 & 0.0103 & -0.0000 & -0.0000 \\ -0.0006 & -0.0090 & -0.0006 & -0.0000 & 0.0103 & -0.0000 \\ -0.0090 & -0.0006 & -0.0006 & -0.0000 & -0.0000 & 0.0103 \end{bmatrix}$$

and $\boldsymbol{K} = 1.3416\boldsymbol{I}_n$. The simulation results are demonstrated in Figs. 4 and 5. These six agents not only make a regular hexagon, but also reach the objectives of themselves. Fig. 6 shows the trajectories of the cost function in (14) with the proposed method and three random groups of parameters. At a given t, the $J(t)$ obtained using the proposed method is smaller than those of three other methods with random parameter matrices, which shows that the control algorithm proposed in Theorem 2 is optimal.

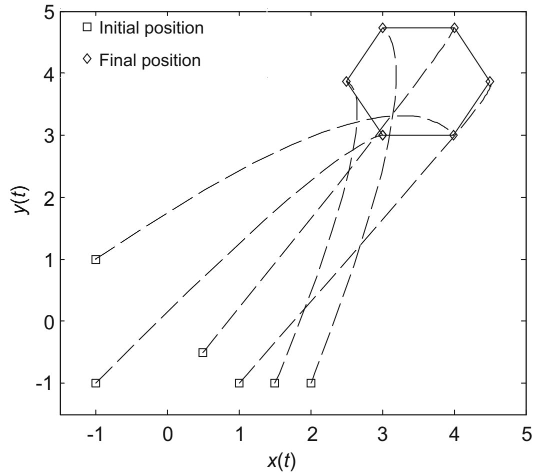

Fig. 4 Formation for the six agents (Example 2)

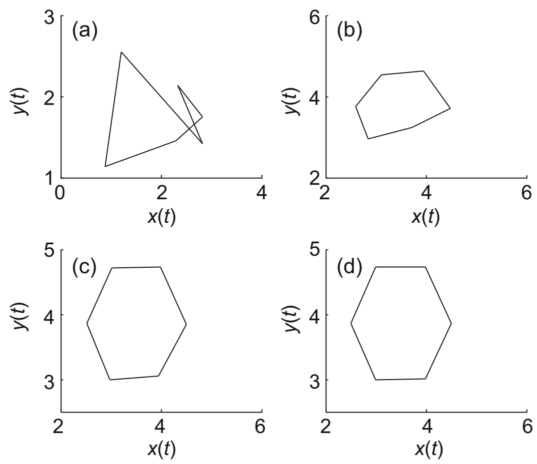

Fig. 5 Snapshots for the six agents (Example 2): (a) $t = 2$ s; (b) $t = 4$ s; (c) $t = 6$ s; (d) $t = 8$ s

The designed $\boldsymbol{L}_2$ is a Laplacian matrix in this example, so the designed communication topology has positive or nonnegative edge weights. However, if choosing different $\boldsymbol{L}_1$, r_1, or r_2, the designed topology with negative edge weights may be better than the topology with nonnegative edge weights according to optimal index (14). In future research more attention should be paid to the study on this phenomenon.

Example 3 In this example, we choose the Laplacian matrix corresponding to the fixed topology between agents as

$$\boldsymbol{L}=\begin{bmatrix} 3 & -1 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We also choose $r_1 = 1$ and $r_2 = 0.15$. According to Theorem 3, it can be computed that the optimal local feedback gain is $k = 2.4976$. The simulation time is 4 s. The trajectories of these six agents are shown in Figs. 7 and 8.

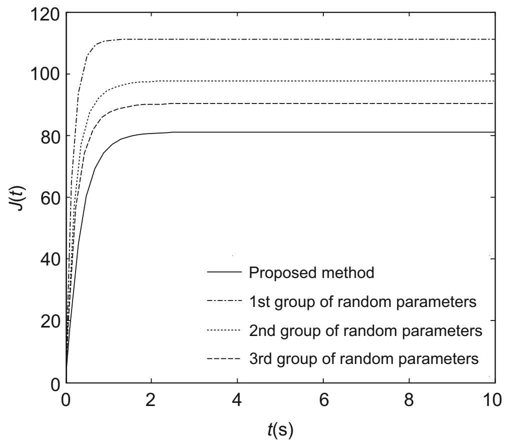

Fig. 6 Trajectories of cost function J (Example 2)

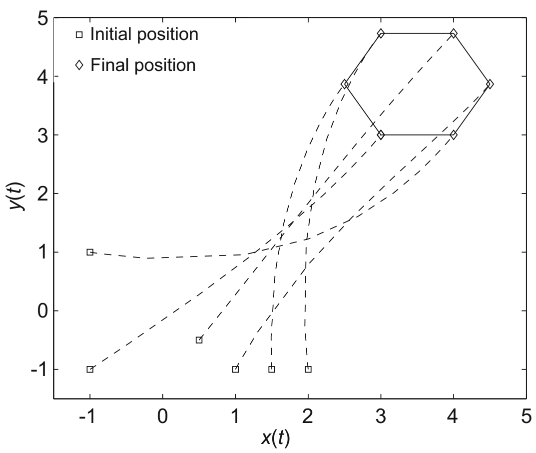

Fig. 7 Formation for the six agents (Example 3)

It is seen that all agents reach their objectives and form a formation. Fig. 9 demonstrates the evolution of the cost function J in (20) as k varies. When $k = 2.4976$, the proposed controller minimizes cost function J, which is consistent with the theoretical result.

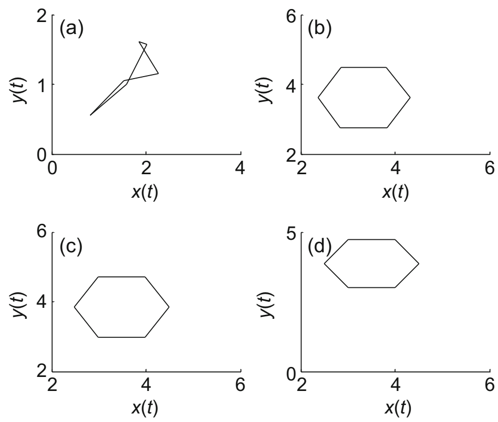

Fig. 8 Snapshots for the six agents (Example 3): (a) $t = 0.8$ s; (b) $t = 1.6$ s; (c) $t = 2.4$ s; (d) $t = 3.2$ s

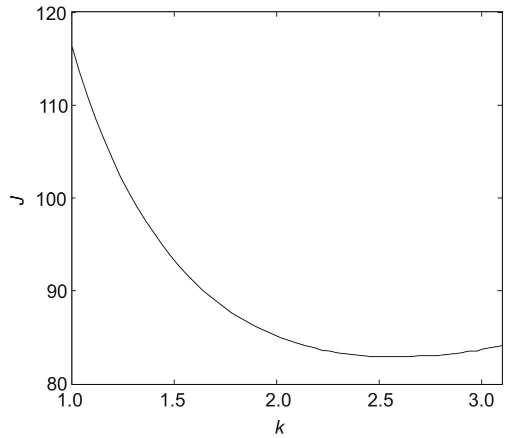

Fig. 9 Trajectory of cost function J as a function of k (Example 3)

5 Conclusions

In this paper, we have investigated three optimal formation problems for first-order multi-agent systems from the LQR perspective. Different from some existing works, these formation problems not only focus on the collective objective for all agents, but also consider each agent's self-objective. Hence, the cost functions should contain the interaction energy cost and the individual energy cost. The optimal formation algorithm for agents with no initial coupling is designed. The corresponding optimal communication topology and local feedback matrix have been proposed from the solution to an ARE. For the physically interconnected multi-agent systems, the optimal formation problem has been discussed. The corresponding topology and local feedback matrix are obtained from a group of quadratic equations with one unknown. Due to the situation in which the communication cannot be changed, optimal formation with fixed and unchangeable topology has been considered. The local feedback gain has been derived by letting the derivative of the cost function be zero.

References

Alefeld, G., Schneider, N., 1982. On square roots of M-matrices. *Linear Algebra Appl.*, **42**:119-132. http://dx.doi.org/10.1016/0024-3795(82)90143-4

Altafini, C., 2013. Consensus problems on networks with antagonistic interactions. *IEEE Trans. Autom. Contr.*, **58**(4):935-946. http://dx.doi.org/10.1109/TAC.2012.2224251

Anderson, B.D.O., Moore, J.B., 2007. Optimal Control: Linear Quadratic Methods. Dover Publications, USA.

Basiri, M., Bishop, A.N., Jensfelt, P., 2010. Distributed control of triangular formations with angle-only constraints. *Syst. Contr. Lett.*, **59**(2):147-154. http://dx.doi.org/10.1016/j.sysconle.2009.12.010

Bishop, A.N., 2011. A very relaxed control law for bearing-only triangular formation control. Proc. 18th IFAC World Congress, p.5991-5998.

Borrelli, F., Keviczky, T., 2008. Distributed LQR design for identical dynamically decoupled systems. *IEEE Trans. Autom. Contr.*, **53**(8):1901-1912. http://dx.doi.org/10.1109/TAC.2008.925826

Cao, Y., Ren, W., 2010. Optimal linear-consensus algorithms: an LQR perspective. *IEEE Trans. Syst. Man Cybern. Part B*, **40**(3):819-830. http://dx.doi.org/10.1109/TSMCB.2009.2030495

Dimarogonas, D.V., Johansson, K.H., 2010. Stability analysis for multi-agent systems using the incidence matrix: quantized communication and formation control. *Automatica*, **46**(4):695-700. http://dx.doi.org/10.1016/j.automatica.2010.01.012

Ghadami, R., Shafai, B., 2013. Decomposition-based distributed control for continuous-time multi-agent systems. *IEEE Trans. Autom. Contr.*, **58**(1):258-264. http://dx.doi.org/10.1109/TAC.2012.2204153

Ghadami, R., Shafai, B., 2014. Distributed observer-based LQR design for multi-agent systems. World Automation Congress, p.520-525. http://dx.doi.org/10.1109/WAC.2014.6936025

Godsil, C., Royle, G.F., 2013. Algebraic Graph Theory. Springer, USA.

Guo, J., Lin, Z., Cao, M., *et al.*, 2010. Adaptive control schemes for mobile robot formations with triangularised structures. *IET Contr. Theory Appl.*, **4**(9):1817-1827. http://dx.doi.org/10.1049/iet-cta.2009.0513

Horn, R.A., Johnson, C.R., 1985. Matrix Analysis. Cambridge University Press, USA.

Horn, R.A., Johnson, C.R., 1991. Topics in Matrix Analysis. Cambridge University Press, USA.

Huang, H., Yu, C., Wu, Q., 2010. Distributed LQR design for multi-agent formations. Proc. 49th IEEE Conf. on Decision and Control, p.4535-4540. http://dx.doi.org/10.1109/CDC.2010.5716988

Li, Z., Duan, Z., Lewis, F.L., 2014. Distributed robust consensus control of multi-agent systems with heterogeneous matching uncertainties. *Automatica*, **50**(3):883-889. http://dx.doi.org/10.1016/j.automatica.2013.12.008

Liu, T., Jiang, Z., 2014. Distributed nonlinear control of mobile autonomous multi-agents. *Automatica*, **50**(4):1075-1086. http://dx.doi.org/10.1016/j.automatica.2014.02.023

Movric, K.H., Lewis, F.L., 2014. Cooperative optimal control for multi-agent systems on directed graph topologies. *IEEE Trans. Autom. Contr.*, **59**(3):769-774. http://dx.doi.org/10.1109/TAC.2013.2275670

Oh, K.K., Park, M.C., Ahn, H.S., 2015. A survey of multi-agent formation control. *Automatica.*, **53**(3):424-440. http://dx.doi.org/10.1016/j.automatica.2014.10.022

Qin, J., Gao, H., 2012. A sufficient condition for convergence of sampled-data consensus for double-integrator dynamics with nonuniform and time-varying communication delays. *IEEE Trans. Autom. Contr.*, **57**(9):2417-2422. http://dx.doi.org/10.1109/TAC.2012.2188425

Qin, J., Gao, H., Zheng, W.X., 2011. Second-order consensus for multi-agent systems with switching topology and communication delay. *Syst. Contr. Lett.*, **60**(6):390-397. http://dx.doi.org/10.1016/j.sysconle.2011.03.004

Ren, W., Beard, R.W., Atkins, E.M., 2007. Information consensus in multivehicle cooperative control. *IEEE Contr. Syst. Mag.*, **27**(2):71-82. http://dx.doi.org/10.1109/MCS.2007.338264

Shi, G., Sou, K.C., Sandberg, H., *et al.*, 2014. A graph-theoretic approach on optimizing informed-node selection in multi-agent tracking control. *Phys. D*, **267**:104-111. http://dx.doi.org/10.1016/j.physd.2013.07.014

Xiao, F., Wang, L., Chen, J., *et al.*, 2009. Finite-time formation control for multi-agent systems. *Automatica*, **45**(11):2605-2611. http://dx.doi.org/10.1016/j.automatica.2009.07.012

Yu, C., Hendrickx, J.M., Fidan, B., *et al.*, 2007. Three and higher dimensional autonomous formations: rigidity, persistence and structural persistence. *Automatica*, **43**(3):387-402.
http://dx.doi.org/10.1016/j.automatica.2006.08.025

Yu, C., Anderson, B.D.O., Dasgupta, S., *et al.*, 2009. Control of minimally persistent formations in the plane. *SIAM J. Contr. Optim.*, **48**(1):206-233.
http://dx.doi.org/10.1137/060678592

Zhang, H., Lewis, F.L., Das, A., 2011. Optimal design for synchronization of cooperative systems: state feedback, observer and output feedback. *IEEE Trans. Autom. Contr.*, **56**(8):1948-1952.
http://dx.doi.org/10.1109/TAC.2011.2139510

Zhao, S., Lin, F., Peng, K., *et al.*, 2014. Distributed control of angle-constrained cyclic formations using bearing-only measurements. *Syst. Contr. Lett.*, **63**:12-24.
http://dx.doi.org/10.1016/j.sysconle.2013.10.003

Dr. Chang-bin YU, first author of this invited paper, received his Bachelor degree in Computer Engineering from Nanyang Technological Unviersity, Singapore, in 2004, and PhD degree in Engineering from the Australian National University, Australia, in 2008. Since then, he has been an academic staff with the Australian National University and subsequently held various appointments with Hangzhou Dianzi University (China), National ICT Australia (Australia), etc. His current research interests include control of autonomous formations, multi-agent systems, mobile sensor networks, human-robot interation, and graph theory. He is Associate Editor of *International Journal of Robust and Nonlinear Control*.