Frontiers of Information Technology & Electronic Engineering
www.jzus.zju.edu.cn; engineering.cae.cn; www.springerlink.com
ISSN 2095-9184 (print); ISSN 2095-9230 (online)
E-mail: jzus@zju.edu.cn

New results on impulsive type inertial bidirectional associative memory neural networks

Chaouki AOUITI[1], Mahjouba Ben REZEG[1], Yang CAO[‡2,3]

[1]*Department of Mathematics, the University of Carthage, Zarzouna 7021, Bizerta, Tunisia*
[2]*School of Cyber Science and Engineering, Southeast University, Nanjing 211189, China*
[3]*Department of Mechanical Engineering, the University of Hong Kong, Pokfulam, Hong Kong, China*
E-mail: chaouki.aouiti@fsb.rnu.tn; mahjouba.benrezek@fsb.rnu.tn; caoyeacy@gmail.com

Received Apr. 7, 2019; Revision accepted May 13, 2019; Crosschecked Jan. 27, 2020

Abstract: This paper is concerned with inertial bidirectional associative memory neural networks with mixed delays and impulsive effects. New and practical conditions are given to study the existence, uniqueness, and global exponential stability of anti-periodic solutions for the suggested system. We use differential inequality techniques to prove our main results. Finally, we give an illustrative example to demonstrate the effectiveness of our new results.

Key words: Inertial neural networks; Anti-periodic solutions; Global exponential stability; Impulsive effect; Time-varying delay; Bidirectional associative memory
https://doi.org/10.1631/FITEE.1900181 **CLC number:** TP183; O175

1 Introduction

In 1982, Kosko furnished answers to an interesting question concerning data-association storage and recall in a dynamic system with nonlinear feedback and two layers (Kosko, 1988). He managed to find a network of neurons that realizes this, known as the bidirectional associative memory neural networks (BAMNNs). This kind of neural network (NN) has been proved to have wide applications in various fields such as medical image edge detection, medical event detection in electronic health records, diagnosis prediction in health care, pattern recognition, and robotics. These applications heavily depend on the dynamic behaviors of bidirectional associative memory (BAM), and the analysis of these dynamic behaviors is a prerequisite for practical design of this kind of NN, because the success of these applications relies on understanding of the underlying dynamic behavior of the model. For this reason, there have been extensive results on the problem of dynamic analysis of BAM (Balasubramaniam et al., 2011; Li HF et al., 2016; Xu and Li, 2016; Aouiti and Assali, 2019). Because neurons cannot communicate instantly, it is important to consider NNs with time delays (M'Hamdi et al., 2016; Aouiti et al., 2017, 2018; Alimi et al., 2018; Aouiti, 2018; Aouiti and Miaadi, 2018, 2019).

Researchers have also investigated NNs by adding an inertial term. This model was first introduced by Wheeler and Schieve (1997). Recently, inertial NNs with a delay have been widely investigated by many researchers because of their role in generating complicated bifurcation behavior and chaos. Ke and Miao (2017) considered a class of inertial BAMNNs (IBAMNNs) and a time delay with constant coefficients. They demonstrated the existence and exponential stability of the suggested NNs, using a Lyapunov function, the Halanay inequality, and the fundamental solution of the coefficient matrix. Xu and Zhang (2015) modified the

‡ Corresponding author
ORCID: Yang CAO, https://orcid.org/0000-0002-6940-0868

system studied in Ke and Miao (2011). They used variable coefficients for the strong points of the connection and external inputs. Using the Lyapunov method and inequality techniques, they showed the uniqueness, existence, and exponential stability of anti-periodic solutions of IBAMNNs. Ke and Miao (2017) discussed the exponential stability of anti-periodic solutions for inertial NNs with time delays, presented hypotheses that help show the existence and exponential stability of anti-periodic solutions for this type of NN, and used the Lyapunov method, uniform convergence, and so on (Ke and Miao, 2011, 2013a, 2017; He et al., 2012; Qi et al., 2015; Xu and Zhang, 2015; Zhang and Quan, 2015; Tu et al., 2016; Liao et al., 2017; Li YK and Xiang, 2019).

The mathematical modeling of various physical processes gives rise to anti-periodic solutions (Batchelor et al., 1995). Okochi (1990) studied the first anti-periodic solutions for nonlinear evolution equations. The investigation of anti-periodic solutions is an important subject because of their applications in engineering, physics, and control theory. In NN theory, much attention has been paid to the study of anti-periodic oscillations of different types of NNs (Li YK et al., 2015; Xu and Zhang, 2015; Li HF et al., 2016; Long, 2016; Xu and Li, 2016; Ke and Miao, 2017; Zhou QY and Shao, 2018; Li YK and Xiang, 2019).

The conditions of numerous current studies on optimal control, science, mechanics, medication, gadgets, and financial matters are susceptible to immediate problems and experience unexpected changes. The term of these progressions is exceptionally short and irrelevant to the duration of the process considered, and can be thought of as momentary changes or impulses. In NN theory, systems with short-term perturbations are naturally described by impulsive differential equations; we refer, for example, to Liu BW (2007), Li YK (2008), Liu B et al. (2008), Zhou JW and Li (2009), Stamova et al. (2014), Li XD and Wu (2016), Li XD and Song (2017), Aouiti and Assali (2019), Aouiti and Dridi (2019a, 2019b), and Li XD et al. (2019).

However, few researchers have been interested in the dynamics of anti-periodic solutions for inertial NNs, in particular IBAMNNs. Also, as far as we know, there is no article that has studied the existence and exponential stability of IBAMNNs with impulsive effect. In this study, we establish new results concerning the existence, uniqueness, and global exponential stability of anti-periodic IBAMNN solutions with mixed delays and impulsive effects defined by the following equations:

$$\begin{cases} \dfrac{\mathrm{d}^2 x_i(t)}{\mathrm{d}t^2} = -\alpha_i \dfrac{x_i(t)}{\mathrm{d}t} - a_i x_i(t) + \displaystyle\sum_{j=1}^{m} c_{ji}(t) \\ \qquad \cdot f_j(y_j(t)) + \displaystyle\sum_{j=1}^{m} d_{ji}(t) f_j\Big(y_j(t-\tau_{ji}(t))\Big) \\ \qquad + \displaystyle\sum_{j=1}^{m} h_{ji}(t) \int_0^{+\infty} K_{ji}(u) \\ \qquad \cdot f_j\Big(y_j(t-u)\Big)\mathrm{d}u + I_i(t), \\ \dfrac{\mathrm{d}^2 y_j(t)}{\mathrm{d}t^2} = -\beta_j \dfrac{y_j(t)}{\mathrm{d}t} - b_j y_j(t) + \displaystyle\sum_{i=1}^{n} p_{ij}(t) \\ \qquad \cdot g_i(x_i(t)) + \displaystyle\sum_{i=1}^{n} q_{ij}(t) g_i\Big(x_i(t-\sigma_{ij}(t))\Big) \\ \qquad + \displaystyle\sum_{i=1}^{n} o_{ij}(t) \int_0^{+\infty} N_{ij}(u) \\ \qquad \cdot g_i\Big(x_i(t-u)\Big)\mathrm{d}u + J_j(t). \end{cases} \tag{1}$$

Herein $n \geq 2$; $t \geq 0$; $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$; the second derivative is called an inertial term of system (1); α_i and β_j are positive constants; $x_i(\cdot)$ and $y_j(\cdot)$ are the external inputs of the i^{th} neuron in the X layer and the external inputs of the j^{th} neuron in the Y layer, respectively; $a_i > 0$ and $b_j > 0$ denote the rate at which the i^{th} neuron and the j^{th} neuron will reset their potential to the resting state in isolation when disconnected from the networks and external inputs, respectively; functions $c_{ji}(\cdot)$, $d_{ji}(\cdot)$, $h_{ji}(\cdot)$, $p_{ij}(\cdot)$, $q_{ij}(\cdot)$, and $o_{ij}(\cdot)$ denote the connection strengths of the NN; $K_{ji}(\cdot)$ and $N_{ij}(\cdot)$ are the delay kernels; $f_j(\cdot)$ and $g_i(\cdot)$ are the activation functions of the j^{th} and i^{th} neurons, respectively; $\tau_{ji}(\cdot)$ and $\sigma_{ij}(\cdot)$ are the external inputs on the i^{th} neuron in the X layer and the j^{th} neuron in the Y layer, respectively; $I_i(\cdot)$ and $J_j(\cdot)$ are the external biases of the X layer and Y layer, respectively.

The initial conditions of system (1) are given by

$$\begin{cases} x_i(s) = \varphi_{xi}(s), \ \dfrac{\mathrm{d}x_i(s)}{\mathrm{d}t} = \psi_{xi}(s), \\ y_j(s) = \varphi_{yj}(s), \ \dfrac{\mathrm{d}y_j(s)}{\mathrm{d}t} = \psi_{yj}(s), \end{cases} \tag{2}$$

where $s \in (-\infty, 0]$ and $\varphi_{xi}(\cdot)$, $\psi_{xi}(\cdot)$, $\varphi_{yj}(\cdot)$, and $\psi_{yj}(\cdot)$ are bounded and continuous functions.

Let $\big(x(t), y(t)\big)^{\mathrm{T}}$ be a solution of system (1) with initial values (2). Let

$$\begin{cases} u_i(t) = \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} + x_i(t), \\ v_j(t) = \dfrac{\mathrm{d}y_j(t)}{\mathrm{d}t} + y_j(t), \end{cases}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. We can rewrite system (1) as

$$\begin{cases} \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -\, x_i(t) + u_i(t), \\ \dfrac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -\, \big(a_i - \alpha_i + 1\big)x_i(t) - \big(\alpha_i - 1\big)u_i(t) \\ \qquad + \displaystyle\sum_{j=1}^{m} c_{ji}(t) f_j(y_j(t)) \\ \qquad + \displaystyle\sum_{j=1}^{m} d_{ji}(t) f_j\Big(y_j(t - \tau_{ji}(t))\Big) \\ \qquad + \displaystyle\sum_{j=1}^{m} h_{ji}(t) \int_0^{+\infty} K_{ji}(u) \\ \qquad \cdot f_j\Big(y_j(t-u)\Big)\mathrm{d}u + I_i(t), \\ \dfrac{\mathrm{d}y_j(t)}{\mathrm{d}t} = -\, y_j(t) + v_j(t), \\ \dfrac{\mathrm{d}v_j(t)}{\mathrm{d}t} = -\, \big(b_j - \beta_j + 1\big)y_j(t) - \big(\beta_j - 1\big)v_j(t) \\ \qquad + \displaystyle\sum_{i=1}^{n} p_{ij}(t) g_i(x_i(t)) \\ \qquad + \displaystyle\sum_{i=1}^{n} q_{ij}(t) g_i\Big(x_i(t - \sigma_{ij}(t))\Big) \\ \qquad + \displaystyle\sum_{i=1}^{n} o_{ij}(t) \int_0^{+\infty} N_{ij}(u) \\ \qquad \cdot g_i\Big(x_i(t-u)\Big)\mathrm{d}u + J_j(t). \end{cases} \tag{3}$$

By adding the impulsive effects, we consider the following IBAMNN:

$$\begin{cases} \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -\, x_i(t) + u_i(t), \\ \dfrac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -\, \big(a_i - \alpha_i + 1\big)x_i(t) - \big(\alpha_i - 1\big)u_i(t) \\ \qquad + \displaystyle\sum_{j=1}^{m} c_{ji}(t) f_j(y_j(t)) \\ \qquad + \displaystyle\sum_{j=1}^{m} d_{ji}(t) f_j\Big(y_j(t - \tau_{ji}(t))\Big) \\ \qquad + \displaystyle\sum_{j=1}^{m} h_{ji}(t) \int_0^{+\infty} K_{ji}(u) \\ \qquad \cdot f_j\Big(y_j(t-u)\Big)\mathrm{d}u + I_i(t), \\ \dfrac{\mathrm{d}y_j(t)}{\mathrm{d}t} = -\, y_j(t) + v_j(t), \\ \dfrac{\mathrm{d}v_j(t)}{\mathrm{d}t} = -\, \big(b_j - \beta_j + 1\big)y_j(t) - \big(\beta_j - 1\big)v_j(t) \\ \qquad + \displaystyle\sum_{i=1}^{n} p_{ij}(t) g_i(x_i(t)) \\ \qquad + \displaystyle\sum_{i=1}^{n} q_{ij}(t) g_i\Big(x_i(t - \sigma_{ij}(t))\Big) \\ \qquad + \displaystyle\sum_{i=1}^{n} o_{ij}(t) \int_0^{+\infty} N_{ij}(u) \\ \qquad \cdot g_i\Big(x_i(t-u)\Big)\mathrm{d}u + J_j(t), \\ x_i(t_k^+) = \vartheta_{ik}^x(x_i(t_k)) = (1 + \widetilde{\vartheta}_{ik}^x)x_i(t_k), \\ u_i(t_k^+) = \vartheta_{ik}^u(u_i(t_k)) = (1 + \widetilde{\vartheta}_{ik}^u)u_i(t_k), \\ y_j(t_k^+) = \vartheta_{jk}^y(y_j(t_k) = (1 + \widetilde{\vartheta}_{jk}^y)y_j(t_k), \\ v_j(t_k^+) = \vartheta_{jk}^v(v_j(t_k)) = (1 + \widetilde{\vartheta}_{jk}^v)v_j(t_k). \end{cases} \tag{4}$$

The impulsive times t_k satisfy: $t_0 < t_1 < \ldots < t_k < \ldots$, $\lim_{t\to+\infty} t_k = +\infty$.

Remark 1 The facts that our model can have many applications in the scientific and technical fields (we talked about this in the introduction), and that these applications can be successful by taking into account a good understanding of the dynamic behavior of the model, are motivations to study this model. The investigation of an anti-periodic solution for system (4) does not exist as of this writing. Consequently, the principal reason for this study is to give new conditions to demonstrate the existence, uniqueness, and global exponential stability of anti-periodic solutions for a class of impulsive BAMNNs using differential inequality techniques.

Remark 2 Our principal contributions are as follows:

1. We establish new conditions to prove the existence and uniqueness of anti-periodic solutions for model (4).

2. We prove the global exponential stability of anti-periodic solutions for model (4). Note that our approach for proving the global exponential stability of the anti-periodic solution of system (4) is different from the solutions in other studies (Li YK et al., 2015; Xu and Zhang, 2015; Long, 2016; Xu and Li, 2016; Ke and Miao, 2017; Zhou QY and Shao, 2018; Li YK and Xiang, 2019).

3. We take into account impulsive effects, so our results are more general than the results in Ke and Miao (2013b, 2017), Xu and Zhang (2015), Liao et al. (2017), and Li YK and Xiang (2019).

2 Preliminaries

For convenience, we introduce some notations and define the following class of spaces:

$$\begin{aligned}\overline{c}_{ji} &= \max_{t\in[0,\omega]} |c_{ji}(t)|,\ \overline{d}_{ji} = \max_{t\in[0,\omega]} |d_{ji}(t)|,\\ \overline{h}_{ji} &= \max_{t\in[0,\omega]} |h_{ji}(t)|,\ \overline{p}_{ij} = \max_{t\in[0,\omega]} |p_{ij}(t)|,\\ \overline{q}_{ij} &= \max_{t\in[0,\omega]} |q_{ij}(t)|,\ \overline{o}_{ij} = \max_{t\in[0,\omega]} |o_{ij}(t)|,\\ \overline{I}_{i} &= \max_{t\in[0,\omega]} |I_{i}(t)|,\ \overline{J}_{j} = \max_{t\in[0,\omega]} |J_{j}(t)|.\end{aligned}$$

$|\cdot|$ and $\|\cdot\|$ represent the norm of $\mathbb{R}$ and $\mathbb{R}^n$ $(n>1)$, respectively.

$$\|\boldsymbol{x}\| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}},\ \text{for } \boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}.$$

Definition 1 A solution $\boldsymbol{x}(t)$ of system (4) is said to be ω-anti-periodic, if

$$\begin{aligned}\boldsymbol{x}(t+\omega) &= -\boldsymbol{x}(t),\ t \neq t_k,\\ \boldsymbol{x}(t_k+\omega) &= -\boldsymbol{x}(t_k),\ k=1,2,\ldots,\end{aligned}$$

where ω is a nonnegative small number and it is called the anti-periodic of function $\boldsymbol{x}(t)$.

In addition, for $i=1,2,\ldots,n$ and $j=1,2,\ldots,m$, the following hypotheses are given:

Hypothesis 1 There exist nonnegative constants $L_{fj},\ L_{gi},\ M^f$, and M^g such that for all $u,\ v\in\mathbb{R}$,

$$\begin{aligned}|f_j(u)-f_j(v)| &\le L_{fj}|u-v|,\\ |g_i(u)-g_i(v)| &\le L_{gi}|u-v|,\end{aligned}$$

$$|f_j(u)| \le M^f,\ |g_i(u)| \le M^g.$$

Hypothesis 2 $c_{ji}(\cdot),\ d_{ji}(\cdot),\ p_{ij}(\cdot),\ q_{ij}(\cdot),\ I_i(\cdot),\ J_j(\cdot),\ \tau_{ji}(\cdot),\ \sigma_{ij}(\cdot),\ h_{ji}(\cdot)$, and $o_{ij}(\cdot)$ are continuous functions. They satisfy

$$\begin{aligned}c_{ji}(t+\omega)f_j(u) &= -c_{ji}(t)f_j(-u),\\ d_{ji}(t+\omega)f_j(u) &= -d_{ji}(t)f_j(-u),\\ p_{ij}(t+\omega)g_i(u) &= -p_{ij}(t)g_i(-u),\\ q_{ij}(t+\omega)g_i(u) &= -q_{ij}(t)g_i(-u),\\ I_i(t+\omega) = -I_i(t),\ &J_j(t+\omega) = -J_j(t),\\ \tau_{ji}(t+\omega) = \tau_{ji}(t),\ &\sigma_{ij}(t+\omega) = \sigma_{ij}(t),\end{aligned}$$

$$\begin{aligned}&h_{ji}(t+\omega)\int_0^{+\infty} K_{ji}(u) f_j(y_i(t-u))\mathrm{d}u\\ &\quad = -h_{ji}(t)\int_0^{+\infty} K_{ji}(u) f_j(-y_i(t-u))\mathrm{d}u,\\ &o_{ij}(t+\omega)\int_0^{+\infty} N_{ij}(u) g_i(x_i(t-u))\mathrm{d}u\\ &\quad = -o_{ij}(t)\int_0^{+\infty} N_{ij}(u) g_i(-x_i(t-u))\mathrm{d}u.\end{aligned}$$

Hypothesis 3 The kernels $K_{ji},\ N_{ij}:[0,+\infty)\longrightarrow[0,+\infty)$ satisfy

$$\int_0^{+\infty} K_{ji}(s)\mathrm{d}s = 1,\ \int_0^{+\infty} N_{ij}(s)\mathrm{d}s = 1.$$

Hypothesis 4 There exists a positive integer q, such that for all $k\in\mathbb{N}$,

$$\begin{aligned}t_{k+q} &= t_k + q,\\ \tilde{\vartheta}^x_{i(k+q)} = \tilde{\vartheta}^x_{ik},\ &\tilde{\vartheta}^u_{i(k+q)} = \tilde{\vartheta}^u_{ik},\\ \tilde{\vartheta}^y_{j(k+q)} = \tilde{\vartheta}^y_{jk},\ &\tilde{\vartheta}^v_{j(k+q)} = \tilde{\vartheta}^v_{jk},\end{aligned}$$

and

$$\begin{cases}|1+\tilde{\vartheta}^x_{ik}| \le 1,\ |1+\tilde{\vartheta}^u_{ik}| \le 1,\\ |1+\tilde{\vartheta}^y_{jk}| \le 1,\ |1+\tilde{\vartheta}^v_{jk}| \le 1.\end{cases} \tag{5}$$

Now, we give a few lemmas that serve us later in the proofs of the main theorems.

Lemma 1 Suppose that Hypotheses 1–4 hold. If $\boldsymbol{Z}(t) = (x_i(t), u_i(t), y_j(t), v_j(t))^{\mathrm{T}}$ is a solution of system (4) and

$$\begin{aligned}\alpha_i > 1,\ &\beta_j > 1,\\ \frac{|a_i-\alpha_i+1|}{\alpha_i-1} < 1,\ &\frac{|b_j-\beta_j+1|}{\beta_j-1} < 1,\end{aligned}$$

for all $i=1,2,\ldots,n$ and $j=1,2,\ldots,m$, then for all $t\ge 0$ and $k\in\mathbb{N}$ we have

$$\begin{cases} |x_i(t)| < \widetilde{\gamma}_1,\ |u_i(t)| < \widetilde{\gamma}_2, \\ |y_j(t)| < \widetilde{\gamma}_3,\ |v_j(t)| < \widetilde{\gamma}_4, \\ |x_i(t_k^+)| < \widetilde{\gamma}_1,\ |u_i(t_k^+)| < \widetilde{\gamma}_2, \\ |y_j(t_k^+)| < \widetilde{\gamma}_3,\ |v_j(t_k^+)| < \widetilde{\gamma}_4, \end{cases} \tag{6}$$

where

$$\begin{aligned} \widetilde{\gamma}_1 > & \frac{1}{\alpha_i - 1}\left[\sum_{j=1}^{m}\left(\overline{c}_{ji} + \overline{d}_{ji} + \overline{h}_{ji}\right)M^f + \overline{I}_i\right] \\ & \cdot \left(1 - \frac{|a_i - \alpha_i + 1|}{\alpha_i - 1}\right)^{-1}, \end{aligned} \tag{7}$$

$$\begin{aligned} \widetilde{\gamma}_2 > & \frac{1}{\alpha_i - 1}\left[\frac{|a_i - \alpha_i + 1|}{(\alpha_i - 1) - |a_i - \alpha_i + 1|} + 1\right] \\ & \cdot \left[\sum_{j=1}^{m}\left(\overline{c}_{ji} + \overline{d}_{ji} + \overline{h}_{ji}\right)M^f + \overline{I}_i\right], \end{aligned} \tag{8}$$

$$\begin{aligned} \widetilde{\gamma}_3 > & \frac{1}{\beta_j - 1}\left[\sum_{i=1}^{n}\left(\overline{p}_{ij} + \overline{q}_{ij} + \overline{o}_{ij}\right)M^g + \overline{J}_j\right] \\ & \cdot \left(1 - \frac{|b_j - \beta_j + 1|}{\beta_j - 1}\right)^{-1}, \end{aligned} \tag{9}$$

$$\begin{aligned} \widetilde{\gamma}_4 > & \frac{1}{\beta_j - 1}\left[\frac{|b_j - \beta_j + 1|}{(\beta_j - 1) - |b_j - \beta_j + 1|} + 1\right] \\ & \cdot \left[\sum_{i=1}^{n}\left(\overline{p}_{ij} + \overline{q}_{ij} + \overline{o}_{ij}\right)M^g + \overline{J}_j\right]. \end{aligned} \tag{10}$$

Proof We note that Hypothesis 1 assures the existence and uniqueness of the solution (noted $\boldsymbol{Z}(t)$) of system (4) in $[0, +\infty)$ for any given initial condition.

Assume by way of contradiction that inequality (6) is not verified. From Hypothesis 4, we obtain

$$\begin{aligned} |x_i(t_k^+)| &= |(1 + \widetilde{\vartheta}_{ik}^x)x_i(t_k)| \le |x_i(t_k)|, \\ |u_i(t_k^+)| &= |(1 + \widetilde{\vartheta}_{ik}^u)u_i(t_k)| \le |u_i(t_k)|, \\ |y_j(t_k^+)| &= |(1 + \widetilde{\vartheta}_{jk}^y)y_j(t_k)| \le |y_j(t_k)|, \\ |v_j(t_k^+)| &= |(1 + \widetilde{\vartheta}_{jk}^v)v_j(t_k)| \le |v_j(t_k)|. \end{aligned}$$

If

$$\begin{aligned} & |x_i(t_k^+)| > \widetilde{\gamma}_1,\ |u_i(t_k^+)| > \widetilde{\gamma}_2, \\ & |y_j(t_k^+)| > \widetilde{\gamma}_3,\ |v_j(t_k^+)| > \widetilde{\gamma}_4, \end{aligned}$$

then

$$\begin{aligned} & |x_i(t_k)| > \widetilde{\gamma}_1,\ |u_i(t_k)| > \widetilde{\gamma}_2, \\ & |y_j(t_k)| > \widetilde{\gamma}_3,\ |v_j(t_k)| > \widetilde{\gamma}_4. \end{aligned}$$

Thus, there must exist $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, m\}$, and $\daleth \in (t_k, t_{k+1}]$ such that for all $t \in (-\infty, \daleth)$, $s = 1, 2, \ldots, n$, and $l = 1, 2, \ldots, m$,

$$\begin{cases} |x_i(\daleth)| = \widetilde{\gamma}_1,\ |u_i(\daleth)| = \widetilde{\gamma}_2, \\ |y_j(\daleth)| = \widetilde{\gamma}_3,\ |v_j(\daleth)| = \widetilde{\gamma}_4, \end{cases} \tag{11}$$

and

$$\begin{cases} |x_s(t)| < \widetilde{\gamma}_1,\ |u_s(t)| < \widetilde{\gamma}_2, \\ |y_l(t)| < \widetilde{\gamma}_3,\ |v_l(t)| < \widetilde{\gamma}_4. \end{cases} \tag{12}$$

By directly computing the upper left derivative of $|x_i(t)|$, $|u_i(t)|$, $|y_j(t)|$, and $|v_j(t)|$ with Hypothesis 1, inequalities (7)–(10), and Eqs. (11) and (12), we have

$$\begin{aligned} 0 \le & D^+(|u_i(\daleth)|) \\ \le & |a_i - \alpha_i + 1||x_i(\daleth)| - (\alpha_i - 1)|u_i(\daleth)| \\ & + \Bigg| \sum_{j=1}^{m} c_{ji}(\daleth)f_j(y_j(\daleth)) + \sum_{j=1}^{m} d_{ji}(\daleth) \\ & \cdot f_j\Big(y_j(\daleth - \tau_{ji}(\daleth))\Big) + \sum_{j=1}^{m} h_{ji}(\daleth)\int_0^{+\infty} K_{ji}(u) \\ & \cdot f_j\Big(y_j(\daleth - u)\Big)\mathrm{d}u + I_i(\daleth)\Bigg| \\ \le & -(\alpha_i - 1)|u_i(\daleth)| + |a_i - \alpha_i + 1||x_i(\daleth)| \\ & + \Bigg| \sum_{j=1}^{m} c_{ji}(\daleth)f_j(y_j(\daleth)) + \sum_{j=1}^{m} d_{ji}(\daleth) \\ & \cdot f_j\Big(y_j(\daleth - \tau_{ji}(\daleth))\Big) + \sum_{j=1}^{m} h_{ji}(\daleth)\int_0^{+\infty} K_{ji}(u) \\ & \cdot f_j\Big(y_j(\daleth - u)\Big)\mathrm{d}u + I_i(\daleth)\Bigg| \\ \le & -(\alpha_i - 1)\widetilde{\gamma}_2 + |a_i - \alpha_i + 1|\widetilde{\gamma}_1 + \sum_{j=1}^{m}\overline{c}_{ji}M^f \\ & + \sum_{j=1}^{m}\overline{d}_{ji}M^f + \sum_{j=1}^{m}\overline{h}_{ji}M^f + \overline{I}_i. \end{aligned} \tag{13}$$

Thus, we have

$$\begin{aligned} \widetilde{\gamma}_2 \le & \frac{1}{\alpha_i - 1}|a_i - \alpha_i + 1|\widetilde{\gamma}_1 + \frac{1}{\alpha_i - 1}\Bigg(\sum_{j=1}^{m}\overline{c}_{ji}M^f \\ & + \sum_{j=1}^{m}\overline{d}_{ji}M^f + \sum_{j=1}^{m}\overline{h}_{ji}M^f + \overline{I}_i\Bigg). \end{aligned} \tag{14}$$

On the other hand, from inequality (14) we obtain

$$\begin{aligned}0 \leq & D^+(|x_i(\daleth)|) \leq -|x_i(t)| + |u_i(t)| \\ \leq & -\widetilde{\gamma}_1 + \widetilde{\gamma}_2 \\ \leq & \left(\frac{|a_i-\alpha_i+1|}{\alpha_i-1} - 1\right)\widetilde{\gamma}_1 + \frac{1}{\alpha_i-1}\left(\sum_{j=1}^{m}\overline{c}_{ji}M^f\right. \\ & \left. + \sum_{j=1}^{m}\overline{d}_{ji}M^f + \sum_{j=1}^{m}\overline{h}_{ji}M^f + \overline{I}_i\right). \end{aligned} \tag{15}$$

Then

$$\begin{aligned}\widetilde{\gamma}_1 \leq & \frac{1}{\alpha_i - 1 - |a_i-\alpha_i+1|}\left(\sum_{j=1}^{m}\overline{c}_{ji}M^f\right. \\ & \left. + \sum_{j=1}^{m}\overline{d}_{ji}M^f + \sum_{j=1}^{m}\overline{h}_{ji}M^f + \overline{I}_i\right). \end{aligned} \tag{16}$$

Using inequalities (13) and (16) we obtain

$$\begin{aligned}0 \leq & D^+(|u_i(\daleth)|) \\ \leq & -(\alpha_i-1)\widetilde{\gamma}_2 + \frac{|a_i-\alpha_i+1|}{(\alpha_i-1)-|a_i-\alpha_i+1|} \\ & \cdot\left(\sum_{j=1}^{m}\overline{c}_{ji}M^f + \sum_{j=1}^{m}\overline{d}_{ji}M^f + \sum_{j=1}^{m}\overline{h}_{ji}M^f\right. \\ & \left. + \overline{I}_i\right) + \sum_{j=1}^{m}\overline{c}_{ji}M^f + \sum_{j=1}^{m}\overline{d}_{ji}M^f \\ & + \sum_{j=1}^{m}\overline{h}_{ji}M^f + \overline{I}_i < 0. \end{aligned} \tag{17}$$

Similarly, by inequalities (15) and (16) we obtain

$$\begin{aligned}0 \leq & D^+(|x_i(\daleth)|) \\ \leq & -\left(1-\frac{|a_i-\alpha_i+1|}{\alpha_i-1}\right)\widetilde{\gamma}_1 + \frac{1}{\alpha_i-1}\left(\sum_{j=1}^{m}\overline{c}_{ji}M^f\right. \\ & \left. + \sum_{j=1}^{m}\overline{d}_{ji}M^f + \sum_{j=1}^{m}\overline{h}_{ji}M^f + \overline{I}_i\right) < 0. \end{aligned} \tag{18}$$

Similarly, we have

$$\begin{aligned}0 \leq & D^+(|v_j(\daleth)|) \\ \leq & -(\beta_j-1)\widetilde{\gamma}_4 + \frac{|a_i-\alpha_i+1|}{(\beta_j-1)-|b_j-\beta_j+1|} \\ & \cdot\left(\sum_{i=1}^{n}\overline{p}_{ij}M^g + \sum_{i=1}^{m}\overline{q}_{ij}M^g + \sum_{i=1}^{n}\overline{o}_{ij}M^g + \overline{J}_j\right) \\ & + \sum_{i=1}^{n}\overline{p}_{ij}M^g + \sum_{i=1}^{m}\overline{q}_{ij}M^g + \sum_{i=1}^{n}\overline{o}_{ij}M^g + \overline{J}_j \\ & < 0, \end{aligned} \tag{19}$$

and

$$\begin{aligned}0 \leq & D^+(|y_j(\daleth)|) \leq -\left(1-\frac{|b_j-\beta_j+1|}{\beta_j-1}\right)\widetilde{\gamma}_3 \\ & + \frac{1}{\beta_j-1}\left(\sum_{i=1}^{n}\overline{p}_{ij}M^g + \sum_{i=1}^{m}\overline{q}_{ij}M^g\right. \\ & \left. + \sum_{i=1}^{n}\overline{o}_{ij}M^g + \overline{J}_j\right) < 0, \end{aligned} \tag{20}$$

which is a contradiction and implies that inequality (6) holds.

Lemma 2 (Aouiti, 2018) Let $\tau \geq 0$ be a given real constant. We suppose that $p(t)$ and $q_i(t)$ $(i=1,2)$ are continuous functions on $[0,+\infty)$ and that $k(s)$ on $[0,+\infty)$ satisfies the following:

1. $\int_0^{+\infty} k(s)\mathrm{d}s \leq k.$
2. $\int_0^{+\infty} k(s)\mathrm{e}^{\mu\tau}\mathrm{d}s < +\infty,\ \mu \geq 0.$

In addition, supposing that there exist nonnegative constants η and M which satisfy

$$\begin{aligned}p(t)-q_1(t)-kq_2(t) \leq \eta,\ \eta>0,\ 0\leq q_1(t)\leq M, \\ 0\leq q_2(t)\leq M,\ \forall t\geq 0,\end{aligned}$$

then we have

$$\begin{aligned}\lambda^* = \inf_{t\geq 0}\Big\{ & \lambda>0, \lambda - p(t) + q_1(t)\mathrm{e}^{\lambda\tau} \\ & + q_2(t)\int_0^{+\infty} k(s)\mathrm{e}^{\lambda s}\,\mathrm{d}s = 0\Big\} > 0.\end{aligned}$$

We also give the following assumption:

Hypothesis 5 Assume that there exist nonnegative constants p_s, $s\in N$ $(N=\{1,2,\ldots,2(n+m)\})$, such that for $t\in[0,+\infty)$, $i=1,2,\ldots,n$, and $j=1,2,\ldots,m$,

$$p_i - p_{n+i} > 0,$$

$$\begin{aligned}p_{n+i}\big|\alpha_i-1\big| + p_i\big|a_i-\alpha_i+1\big| + \sum_{j=1}^{m}p_{2n+j}|c_{ji}(t)|L_{jf} \\ + \sum_{j=1}^{m}p_{2n+j}|d_{ji}(t)|L_{jf} + \sum_{j=1}^{m}p_{2n+j}|h_{ji}(t)| \\ \cdot\int_0^{\infty}|K_{ji}(s)|L_{jf}\mathrm{d}s > 0,\end{aligned}$$

$$p_{2n+j} - p_{2n+m+j} > 0,$$

$$p_{2n+m+j}\big|\beta_j-1\big|+p_{2n+j}\big|b_j-\beta_j+1\big| + \sum_{i=1}^{n} p_i|p_{ij}(t)|L_{ig}+\sum_{i=1}^{n} p_i|q_{ji}(t)|L_{ig}+\sum_{j=1}^{m} p_i|o_{ij}(t)| \cdot \int_0^{\infty}|N_{ij}(s)|L_{ig}\mathrm{d}s>0.$$

Assume that there exist nonnegative vector functions $\big(V_1(t),\ V_2(t),\ \ldots,V_n(t),\ V_{n+1}(t),\ldots, V_{2n}(t),V_{2n+1}(t),\ldots,V_{2n+m}(t),\ldots,V_{2(n+m)}(t)\big)^{\mathrm{T}} \in C(-\infty,\mathbb{R}^{2(n+m)})$, where $V_s(t)$ is continuous at $t\neq t_k$ $(k\in\mathbb{N}^*)$, $t>0$, $s\in N$, and satisfies the following:

$$D^-V_i(t^-)\leq -V_i(t^-)+V_{n+i}(t^-), \tag{21}$$

$$\begin{aligned}D^-V_{n+i}(t^-)\leq& -\big|\alpha_i-1\big|V_{n+i}(t^-)-\big|a_i-\alpha_i+1\big| \\ &\cdot V_i(t^-)+\sum_{j=1}^{m}|c_{ji}(t)|L_{jf}V_{2n+j}(t^-)+\sum_{j=1}^{m}|d_{ji}(t)| \\ &\cdot L_{jf}\overline{V}_{2n+j}(t^-)+\sum_{j=1}^{m}|h_{ji}(t)|L_{jf}\int_0^{+\infty}|K_{ji}(s)| \\ &\cdot V_{2n+j}(t^- -s)\mathrm{d}s,\end{aligned} \tag{22}$$

$$D^-V_{2n+j}(t^-)\leq -V_{2n+j}(t^-)+V_{2n+m+j}(t^-), \tag{23}$$

$$\begin{aligned}&D^-V_{2n+m+j}(t^-)\leq -\big|\beta_j-1\big|V_{2n+m+j}(t^-) \\ &-\big|b_j-\beta_j+1\big|V_{2n+j}(t^-)+\sum_{i=1}^{n}|p_{ij}(t)|L_{ig}V_i(t^-) \\ &+\sum_{i=1}^{n}|q_{ij}(t)|L_{ig}\overline{V}_i(t^-)+\sum_{j=1}^{m}|o_{ij}(t)|L_{ig}\int_0^{+\infty}|N_{ij}(s)| \\ &\cdot V_i(t^- -s)\mathrm{d}s,\end{aligned} \tag{24}$$

$$\begin{cases} V_i(t_k^+)\leq L^xV_i(t_k), \\ V_{n+i}(t_k^+)\leq L^uV_{n+i}(t_k), \\ V_{2n+j}(t_k^+)\leq L^yV_{2n+j}(t_k), \\ V_{2n+m+j}(t_k^+)\leq L^vV_{2n+m+j}(t_k), \end{cases} \tag{25}$$

where $L^x<1, L^u<1, L^y<1, L^v<1$.

Thus, for all $t\geq 0$ and $s\in N$, there exists a nonnegative constant M such that

$$\begin{aligned}V_s(t)\leq& M\sum_{l=1}^{2(n+m)}\overline{V}_l(0)\mathrm{e}^{-\lambda^* t}, \\ \lambda^* =& \min\{\lambda_s^*|s\in N\},\end{aligned} \tag{26}$$

where

$$\lambda_i^*=\inf_{t\geq 0}\left\{\lambda(t)>0,\lambda(t)-1+\frac{p_{n+i}}{p_i}=0\right\}>0, \tag{27}$$

$$\begin{aligned}\lambda_{n+i}^*=&\inf_{t\geq 0}\Bigg\{\lambda(t)>0,\lambda(t)-\big|\alpha_i-1\big| \\ &-\frac{p_i}{p_{n+i}}\big|a_i-\alpha_i+1\big|+\sum_{j=1}^{m}\frac{p_{2n+j}}{p_{n+i}}|c_{ji}(t)|L_{jf} \\ &+\sum_{j=1}^{m}\frac{p_{2n+j}}{p_{n+i}}|d_{ji}(t)|L_{jf}\mathrm{e}^{\lambda(t)\tau}+\sum_{j=1}^{m}\frac{p_{2n+j}}{p_{n+i}} \\ &\cdot|h_{ji}(t)|\int_0^{\infty}|K_{ji}(s)|L_{jf}\mathrm{e}^{\lambda(t)s}\mathrm{d}s=0\Bigg\} \\ &>0,\end{aligned} \tag{28}$$

$$\begin{aligned}\widehat{\lambda}_{2nj}^*=&\inf_{t\geq 0,\ 1\leq j\leq m}\Bigg\{\lambda(t)>0,\lambda(t)-1 \\ &+\frac{p_{2n+m+j}}{p_{2n+j}}=0\Bigg\} \\ &>0,\end{aligned} \tag{29}$$

$$\begin{aligned}\lambda_{2n+m+j}^*=&\inf_{t\geq 0,1\leq j\leq m}\Bigg\{\lambda(t)>0,\lambda(t)-\big|\beta_j-1\big| \\ &-\frac{p_{2n+j}}{p_{2n+m+j}}\big|b_j-\beta_j+1\big|+\sum_{i=1}^{n}\frac{p_i}{p_{2n+m+j}} \\ &\cdot|p_{ij}(t)|L_{ig}+\sum_{i=1}^{n}\frac{p_i}{p_{2n+m+j}}|q_{ji}(t)| \\ &\cdot L_{ig}\mathrm{e}^{\lambda(t)\tau}+\sum_{j=1}^{m}\frac{p_i}{p_{2n+m+j}}|o_{ij}(t)| \\ &\cdot\int_0^{\infty}|N_{ij}(s)|L_{ig}\mathrm{e}^{\lambda(t)s}\mathrm{d}s=0\Bigg\} \\ &>0.\end{aligned} \tag{30}$$

Proof Using an analysis similar to that in Lemma 2, we show that there exists a unique λ_s^* $(\lambda_s^*>0)$.

Choose a nonnegative constant $\theta>0$ that satisfies $\min\limits_{s\in N}\{p_s\}\theta>1$. Let $\varPhi_s(t)=V_s(t)/p_s,\ s\in N$. We have

$$\varPsi(t)=\theta\sum_{l\in N}\overline{V}_l(0)\mathrm{e}^{-\lambda^* t}. \tag{31}$$

Then for all $t\in(-\infty,0]$ and $\gamma>1$, we have

$$\gamma\varPsi(t)=\gamma\theta\sum_{l\in N}\overline{V}_l(0)\mathrm{e}^{-\lambda^* t}>\varPhi_s(t). \tag{32}$$

As a consequence,

$$\varPhi_s(t)<\gamma\varPsi(t),\ t\in[0,\infty),\ s\in N. \tag{33}$$

By contradiction, suppose that there exist $s \in N$ and $\bar{t} > 0$ such that

$$\begin{aligned} \Phi_s(\bar{t}^+) \geq & \gamma\Psi(\bar{t}),\ \Phi_{s^*}(t) < \gamma\Psi(t), \\ & \text{for } t \in [0,\bar{t}), s^* \in N. \end{aligned} \tag{34}$$

Based on what has been proved above, two cases can be distinguished:

(I) $\bar{t} \neq t_k$, $t_k \in \mathbb{N}^*$. So, $V_s(t)$ is continuous at $\bar{t}$. By inequality (34), we have

$$\frac{1}{p_s}V_s(\bar{t}) = \gamma\Psi(\bar{t}),\ \frac{1}{p_s}D^-V_s(\bar{t}) > \gamma\Psi'(\bar{t}). \tag{35}$$

From Hypothesis 5, inequality (34), and the definition of λ^*, we have

$$\begin{aligned} & \frac{1}{p_i}D^-V_i(\bar{t}) - \gamma\Psi'(\bar{t}) \leq \frac{1}{p_i}\left(-V_i(t^-) + V_{n+i}(t^-)\right) \\ & + \lambda^*\gamma\Psi(\bar{t}) \leq \gamma\Psi(\bar{t})\left(-1 + \frac{p_{n+i}}{p_i} + \lambda^*\right) < 0, \end{aligned} \tag{36}$$

$$\begin{aligned} & \frac{1}{p_{n+i}}D^-V_{n+i}(\bar{t}) - \gamma\Psi'(\bar{t}) \leq \frac{1}{p_i}\Bigg(- \left|\alpha_i - 1\right| \\ & \cdot V_{n+i}(t^-) - \left|a_i - \alpha_i + 1\right|V_i(t^-) + \sum_{j=1}^{m}|c_{ji}(t)| \\ & \cdot L_{jf}V_{2n+j}(t^-) + \sum_{j=1}^{m}|d_{ji}(t)|L_{jf}\overline{V}_{2n+j}(t^-) \\ & + \sum_{j=1}^{m}|h_{ji}(t)|L_{jf}\int_0^{+\infty}|K_{ji}(s)|_{2n+j}(t^- - s)\mathrm{d}s\Bigg) \\ & + \lambda^*\gamma\Psi(\bar{t}) \\ \leq & \gamma\Psi(\bar{t})\Bigg(- \left|\alpha_i - 1\right| - \frac{p_i}{p_{n+i}}\left|a_i - \alpha_i + 1\right| \\ & + \sum_{j=1}^{m}\frac{p_{2n+j}}{p_{n+i}}|c_{ji}(t)|L_{jf} + \sum_{j=1}^{m}\frac{p_{2n+j}}{p_{n+i}}|d_{ji}(t)| \\ & \cdot L_{jf}\mathrm{e}^{\lambda(t)\tau} + \sum_{j=1}^{m}\frac{p_{2n+j}}{p_{n+i}}|h_{ji}(t)|\int_0^{\infty}|K_{ji}(s)| \\ & \cdot L_{jf}\mathrm{e}^{\lambda(t)s}\mathrm{d}s + \lambda^*\Bigg) \\ < & 0, \end{aligned} \tag{37}$$

$$\begin{aligned} & \frac{1}{p_{2n+j}}D^-V_{2n+j}(\bar{t}) - \gamma\Psi'(\bar{t}) \leq \frac{1}{p_{2n+j}}(-V_{2n+j}(t^-) \\ & + V_{2n+m+j}(t^-)) + \lambda^*\gamma\Psi(\bar{t}) \\ & \leq \gamma\Psi(\bar{t})\Bigg(- 1 + \frac{p_{2n+m+j}}{p_{2n+j}} + \lambda^*\Bigg) < 0, \end{aligned} \tag{38}$$

and

$$\begin{aligned} & \frac{1}{p_{2n+m+j}}D^-V_{2n+m+j}(\bar{t}) - \gamma\Psi'(\bar{t}) \leq \frac{1}{p_{2n+m+j}} \\ & \cdot\Bigg(- \left|\beta_j - 1\right|V_{2n+m+j}(t^-) - \left|b_j - \beta_j + 1\right|V_{2n+j}(t^-) \\ & + \sum_{i=1}^{n}|p_{ij}(t)|L_{ig}V_i(t^-) + \sum_{i=1}^{n}|q_{ij}(t)|L_{ig}\overline{V}_i(t^-) \\ & + \sum_{j=1}^{m}|o_{ij}(t)|L_{ig}\int_0^{+\infty}|N_{ij}(s)|V_i(t^- - s)\mathrm{d}s\Bigg) \\ & + \lambda^*\gamma\Psi(\bar{t}) \leq \gamma\Psi(\bar{t})\Bigg(- \left|\beta_j - 1\right| - \frac{p_{2n+j}}{p_{2n+m+j}} \\ & \cdot \left|b_j - \beta_j + 1\right| + \sum_{i=1}^{n}\frac{p_i}{p_{2n+m+j}}|p_{ij}(t)|L_{ig} \\ & + \sum_{i=1}^{n}\frac{p_i}{p_{2n+m+j}}|q_{ji}(t)|L_{ig}\mathrm{e}^{\lambda(t)\tau} + \sum_{j=1}^{m}\frac{p_i}{p_{2n+m+j}} \\ & \cdot|o_{ij}(t)|\int_0^{\infty}|N_{ij}(s)|L_{ig}\mathrm{e}^{\lambda(t)s}\mathrm{d}s + \lambda^*\Bigg) < 0, \end{aligned} \tag{39}$$

which contradict Eq. (35).

(II) $\exists$ $k_0 \in \mathbb{N}^*$ and $\bar{t} = t_k$, inequality (34) involves

$$\frac{1}{p_s}V_s(\bar{t}) \leq \gamma\Psi(\bar{t}) \leq \frac{1}{p_s}V_s(\bar{t}^+). \tag{40}$$

Given $V_s(\bar{t}^-)/p_s \neq V_s(\bar{t}^+)/p_s$, we have $V_s(\bar{t}^-)/p_s < \gamma\Psi(\bar{t})$ or $\gamma\Psi(\bar{t}) < V_s(\bar{t}^+)/p_s$. We assume that $\gamma\Psi(\bar{t}) < V_s(\bar{t}^+)/p_s$. From inequalities (25) and (40) we have

$$\gamma\Psi(\bar{t}) < \frac{1}{p_i}V_i(\bar{t}^+) \leq \gamma L^x\Psi(\bar{t}), \tag{41}$$

$$\gamma\Psi(\bar{t}) < \frac{1}{p_{n+i}}V_{n+i}(\bar{t}^+) \leq \gamma L^u\Psi(\bar{t}), \tag{42}$$

and

$$\gamma\Psi(\bar{t}) < \frac{1}{p_{2n+j}}V_{2n+j}(\bar{t}^+) \leq \gamma L^y\Psi(\bar{t}), \tag{43}$$

$$\gamma\Psi(\bar{t}) < \frac{1}{p_{2n+m+j}}V_{2n+m+j}(\bar{t}^+) \leq \gamma L^v\Psi(\bar{t}). \tag{44}$$

Simplifying inequalities (41)–(44) we obtain $L^x > 1$, $L^u > 1$, $L^y > 1$, and $L^v > 1$, which contradict $L^x < 1$, $L^u < 1$, $L^y < 1$, and $L^v < 1$.

From (I) and (II), inequality (33) holds. Letting $\gamma \to 1^+$ in inequality (33), we have

$$\Phi_s(t) \leq \gamma\Psi(t),\ t \in [0,\infty),\ s \in N. \tag{45}$$

Consequently, $V_s(t)/p_s \le \Psi(t)$ for all $t \in [0,\infty)$, $s \in N$. Let $\tilde{L} = \max_{s\in N}\{p_s\theta\}$. Then for $t \ge 0$ and $s \in N$, we have

$$V_s(t) \le \tilde{L}\sum_{l\in N}\overline{V}_l(0)\mathrm{e}^{-\lambda^* t}. \tag{46}$$

3 Main results

In this section we present new conditions that demonstrate the uniqueness, existence, and global exponential stability of anti-periodic solutions for system (4).

Theorem 1 Assume that Hypotheses 1–5 hold. Let $\boldsymbol{Z}^*(t) = (x^*(t), u^*(t), y^*(t), v^*(t))$ be a solution of system (4) with initial value $\boldsymbol{\phi}^*(t) = (\varphi_x^*(t), \varphi_u^*(t), \varphi_y^*(t), \varphi_v^*(t))^{\mathrm{T}}$. Then $\boldsymbol{Z}^*(t)$ is globally exponentially stable.

Proof We denote $\boldsymbol{Z}(t) = (x(t), u(t), y(t), v(t))^{\mathrm{T}}$ as an arbitrary solution of system (4) with the initial value $\boldsymbol{\phi}(t) = (\varphi_x(t), \varphi_u(t), \varphi_y(t), \varphi_v(t))^{\mathrm{T}}$.

Let

$$\begin{aligned}
V_i(t) &= |x_i(t) - x_i^*(t)|,\\
V_{n+j}(t) &= |u_{n+j}(t) - u_{n+j}^*(t)|,\\
V_{2n+j}(t) &= |y_{2n+j}(t) - y_{2n+j}^*(t)|,\\
V_{2n+m+j}(t) &= |v_{2n+m+j}(t) - v_{2n+m+j}^*(t)|,
\end{aligned}$$

for $t \in \mathbb{R}_+$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Then, we have

$$D^- V_i(t^-) \le -V_i(t^-) + V_{n+i}(t^-), \tag{47}$$

$$\begin{aligned}
&D^- V_{n+i}(t^-) \le -\big|\alpha_i - 1\big|V_{n+i}(t^-) - \big|a_i - \alpha_i + 1\big|\\
&\cdot V_i(t^-) + \sum_{j=1}^{m}|c_{ji}(t)|L_{jf}V_{2n+j}(t^-) + \sum_{j=1}^{m}|d_{ji}(t)|\\
&\cdot L_{jf}\overline{V}_{2n+j}(t^-) + \sum_{j=1}^{m}|h_{ji}(t)|L_{jf}\int_0^{+\infty}|K_{ji}(s)|\\
&\cdot V_{2n+j}(t^- - s)\mathrm{d}s,
\end{aligned} \tag{48}$$

$$D^- V_{2n+j}(t^-) \le -V_{2n+j}(t^-) + V_{2n+m+j}(t^-), \tag{49}$$

$$\begin{aligned}
&D^- V_{2n+m+j}(t^-) \le -\big|\beta_j - 1\big|V_{2n+m+j}(t^-)\\
&- \big|b_j - \beta_j + 1\big|V_{2n+j}(t^-) + \sum_{i=1}^{n}|p_{ij}(t)|L_{ig}V_i(t^-)\\
&+ \sum_{i=1}^{n}|q_{ij}(t)|L_{ig}\overline{V}_i(t^-) + \sum_{j=1}^{m}|o_{ij}(t)|L_{ig}\int_0^{+\infty}|N_{ij}(s)|\\
&\cdot V_i(t^- - s)\mathrm{d}s,
\end{aligned} \tag{50}$$

$$V_i(t_k^+) \le L^x V_i(t_k),\ i = 1, 2, \ldots, n, \tag{51}$$

$$V_{n+i}(t_k^+) \le L^u V_{n+i}(t_k),\ i = 1, 2, \ldots, n, \tag{52}$$

$$V_{2n+j}(t_k^+) \le L^y V_{2n+j}(t_k),\ j = 1, 2, \ldots, m, \tag{53}$$

$$V_{2n+m+j}(t_k^+) \le L^v V_{2n+m+j}(t_k),\ j = 1, 2, \ldots, m. \tag{54}$$

By inequalities (47)–(54) and Hypotheses 1–5, there exists a nonnegative constant M that verifies

$$V_s(t) \le M\sum_{l=1}^{2(n+m)}\overline{V}_l(0)\mathrm{e}^{\lambda^* t}, \tag{55}$$

where λ^* is defined in Eq. (26).

Theorem 2 Assume that Hypotheses 1–5 are satisfied. Then system (4) has a unique T-anti-periodic solution that is also globally exponentially stable.

Proof We denote

$$\boldsymbol{x}(t) = \Big(x_1(t), x_2(t), \ldots, x_n(t)\Big)^{\mathrm{T}},$$

$$\boldsymbol{u}(t) = \Big(u_1(t), u_2(t), \ldots, u_n(t)\Big)^{\mathrm{T}},$$

$$\boldsymbol{y}(t) = \Big(y_1(t), y_2(t), \ldots, y_n(t)\Big)^{\mathrm{T}},$$

$$\boldsymbol{v}(t) = \Big(v_1(t), v_2(t), \ldots, v_n(t)\Big)^{\mathrm{T}}.$$

Let $\big(\boldsymbol{x}(t),\ \boldsymbol{u}(t),\ \boldsymbol{y}(t),\ \boldsymbol{v}(t)\big)^{\mathrm{T}}$ be a solution of system (4) with initial conditions $x_i(s) = \varphi_{x_i}(s)$, $u_i(s) = \varphi_{u_i}(s)$, $y_j(s) = \varphi_{y_j}(s), v_j(s) = \varphi_{v_j}(s)$, $|\varphi_{x_i}(s)| < \widetilde{\gamma}_1, |\varphi_{u_i}(s)| < \widetilde{\gamma}_2, |\varphi_{y_j}(s)| < \widetilde{\gamma}_3, |\varphi_{v_j}(s)| < \widetilde{\gamma}_4$, for $s \in (-\infty, 0]$, $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$. It follows from system (4) and Hypothesis 2 that for all $p \in \mathbb{N}$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[(-1)^{p+1}x_i(t + (p+1)T)]\\
&=(-1)^{p+1}\frac{\mathrm{d}}{\mathrm{d}t}[x_i(t + (p+1)T)]\\
&=(-1)^{p+1}[-x_i(t + (p+1)T) + u_i(t + (p+1)T)]\\
&=-(-1)^{p+1}x_i(t + (p+1)T)\\
&\quad+(-1)^{p+1}u_i(t + (p+1)T),\ t \ne t_k,
\end{aligned} \tag{56}$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[(-1)^{p+1}x_i(t_k + (p+1)T)^+]\\
&=(-1)^{p+1}\vartheta^x_{i(k+(p+1)q)}(x_i(t_k + (p+1)T))\\
&=\vartheta^x_{ik}((-1)^{p+1}(x_i(t_k + (p+1)T))),\ k = 1, 2, \ldots,
\end{aligned} \tag{57}$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[(-1)^{p+1}u_i(t+(p+1)T)]\\
=&(-1)^{p+1}\Big[-\big(a_i-\alpha_i+1\big)x_i(t+(p+1)T)\\
&-\big(\alpha_i-1\big)u_i(t+(p+1)T)+\sum_{j=1}^{m}c_{ji}(t+(p+1)T)\\
&\cdot f_j(y_j(t+(p+1)T))+\sum_{j=1}^{m}d_{ji}(t+(p+1)T)\\
&\cdot f_j(y_j(t+(p+1)T)-\tau_{ji}(t+(p+1)T))\\
&+\sum_{j=1}^{m}h_{ji}(t+(p+1)T)\int_0^{+\infty}K_{ji}(u)\\
&\cdot f_j(y_j(t+(p+1)T)-u)\mathrm{d}u+I_i(t+(p+1)T)\Big]\\
=&-\big(a_i-\alpha_i+1\big)(-1)^{p+1}x_i(t+(p+1)T)\\
&-\big(\alpha_i-1\big)(-1)^{p+1}u_i(t+(p+1)T)\\
&+\sum_{j=1}^{m}c_{ji}(t)f_j((-1)^{p+1}y_j(t+(p+1)T))\\
&+\sum_{j=1}^{m}d_{ji}(t)f_j((-1)^{p+1}y_j(t+(p+1)T)-\tau_{ji}(t))\\
&+\sum_{j=1}^{m}h_{ji}(t)\int_0^{+\infty}K_{ji}(u)f_j(y_j(t+(p+1)T)\\
&-u)\mathrm{d}u+I_i(t),
\end{aligned}\tag{58}$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[(-1)^{p+1}u_i(t_k+(p+1)T)^+]=\vartheta_{ik}^{u}((-1)^{p+1}\\
&\quad\cdot(u_i(t_k+(p+1)T))),\ k=1,2,\ldots.
\end{aligned}\tag{59}$$

Similarly, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[(-1)^{p+1}y_j(t+(p+1)T)]=-(-1)^{p+1}\\
&\cdot y_j(t+(p+1)T)+(-1)^{p+1}v_j(t+(p+1)T),
\end{aligned}\tag{60}$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[(-1)^{p+1}y_j(t_k+(p+1)T)^+]=\vartheta_{jk}^{y}((-1)^{p+1}\\
&\cdot(y_j(t_k+(p+1)T))),\ k=1,2,\ldots,
\end{aligned}\tag{61}$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[(-1)^{p+1}v_j(t+(p+1)T)]=-\big(b_j-\beta_j+1\big)\\
&\quad\cdot(-1)^{p+1}y_j(t+(p+1)T)-\big(\beta_j-1\big)(-1)^{p+1}\\
&\quad\cdot v_j(t+(p+1)T)+\sum_{i=1}^{n}p_{ij}(t)g_i((-1)^{p+1}\\
&\quad\cdot x_i(t+(p+1)T))+\sum_{i=1}^{n}q_{ij}(t)g_i((-1)^{p+1}\\
&\quad\cdot x_i(t+(p+1)T-\sigma_{ij}(t)))+\sum_{i=1}^{n}o_{ij}(t)\\
&\quad\cdot\int_0^{+\infty}N_{ij}(u)g_i((-1)^{p+1}x_i(t+(p+1)T\\
&\quad-u))\mathrm{d}u+J_j(t),
\end{aligned}\tag{62}$$

and

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[(-1)^{p+1}v_j(t_k+(p+1)T)^+]=\vartheta_{jk}^{v}((-1)^{p+1}\\
&\quad\cdot(v_j(t_k+(p+1)T))),\ k=1,2,\ldots.
\end{aligned}\tag{63}$$

Let

$$\breve{\boldsymbol{x}}(t)=\begin{pmatrix}(-1)^{p+1}x_1(t+(p+1)T)\\(-1)^{p+1}x_2(t+(p+1)T)\\\vdots\\(-1)^{p+1}x_n(t+(p+1)T)\end{pmatrix},$$

$$\breve{\boldsymbol{u}}(t)=\begin{pmatrix}(-1)^{p+1}u_1(t+(p+1)T)\\(-1)^{p+1}u_2(t+(p+1)T)\\\vdots\\(-1)^{p+1}u_n(t+(p+1)T)\end{pmatrix},$$

$$\breve{\boldsymbol{y}}(t)=\begin{pmatrix}(-1)^{p+1}y_1(t+(p+1)T)\\(-1)^{p+1}y_2(t+(p+1)T)\\\vdots\\(-1)^{p+1}y_m(t+(p+1)T)\end{pmatrix},$$

$$\breve{\boldsymbol{v}}(t)=\begin{pmatrix}(-1)^{p+1}v_1(t+(p+1)T)\\(-1)^{p+1}v_2(t+(p+1)T)\\\vdots\\(-1)^{p+1}v_m(t+(p+1)T)\end{pmatrix}.$$

Thus, for any $k\in\mathbb{N}$, $\big(\breve{\boldsymbol{x}}(t),\ \breve{\boldsymbol{u}}(t),\ \breve{\boldsymbol{y}}(t),\ \breve{\boldsymbol{v}}(t)\big)^{\mathrm{T}}$ is likewise a solution of system (4). If the initial values (2) are bounded, from Theorem 1, we can deduce that there exists a positive constant γ that verifies

$$\begin{aligned}
&|(-1)^{p+1}x_i(t+(p+1)T)-(-1)^{p}x_i(t+pT)|\\
&\quad=|x_i(t+pT)-(-x_i(t+pT+T))|\\
&\quad\le M\mathrm{e}^{-\gamma(t+pT)}\sup_{-\infty\le s\le 0}\sum_{i=1}^{n}|x_i(s+T)+x_i(s)|^2\\
&\quad\le 2M\tilde{\gamma}_1\mathrm{e}^{-\gamma(t+pT)}\\
&\quad\le F\mathrm{e}^{-\gamma(t+pT)},
\end{aligned}\tag{64}$$

$$\begin{aligned}&|(-1)^{p+1}x_i((t_k+(p+1)T)^+)-(-1)^k x_i((t_k+pT)^+)|\\&\quad=|x_i((t_k+pT)^+)+x_i((t_k+pT+T)^+)|\\&\quad=|1+\tilde{\vartheta}_{ik}^x||x_i(t_k+(p+1)T)+x_i(t_k+pT)|\\&\quad\le 2M\tilde{\gamma}_1\mathrm{e}^{-\gamma(t+pT)}\\&\quad\le F\mathrm{e}^{-\gamma(t_k+pT)},\end{aligned}\tag{65}$$

$$\begin{aligned}&|(-1)^{p+1}u_i(t+(p+1)T)-(-1)^p u_i(t+pT)|\\&\quad=|u_i(t+pT)-(-u_i(t+pT+T))|\\&\quad\le M\mathrm{e}^{-\gamma(t+pT)}\sup_{-\infty\le s\le 0}\sum_{i=1}^{n}|u_i(s+T)+u_i(s)|^2\\&\quad\le 2M\tilde{\gamma}_2\mathrm{e}^{-\gamma(t+pT)}\\&\quad\le F\mathrm{e}^{-\gamma(t+pT)},\end{aligned}\tag{66}$$

$$\begin{aligned}&|(-1)^{p+1}u_i((t_k+(p+1)T)^+)-(-1)^k u_i((t_k+pT)^+)|\\&\quad=|u_i((t_k+pT)^+)+u_i((t_k+pT+T)^+)|\\&\quad=|1+\tilde{\vartheta}_{ik}^u||u_i(t_k+pT+T)+u_i(t_k+pT)|\\&\quad\le 2M\tilde{\gamma}_2\mathrm{e}^{-\gamma(t_k+pT)}\\&\quad\le F\mathrm{e}^{-\gamma(t_k+pT)},\end{aligned}\tag{67}$$

$$\begin{aligned}&|(-1)^{p+1}y_j(t+(p+1)T)-(-1)^p y_j(t+pT)|\\&\quad=|y_j(t+pT)-(-y_j(t+pT+T))|\\&\quad\le M\mathrm{e}^{-\gamma(t+pT)}\sup_{-\infty\le s\le 0}\sum_{j=1}^{m}|y_j(s+T)+y_j(s)|^2\\&\quad\le 2M\tilde{\gamma}_3\mathrm{e}^{-\gamma(t+pT)}\\&\quad\le F\mathrm{e}^{-\gamma(t+pT)},\end{aligned}\tag{68}$$

$$\begin{aligned}&|(-1)^{p+1}y_j((t_k+(p+1)T)^+)-(-1)^k y_j((t_k+pT)^+)|\\&\quad=|y_j((t_k+pT)^+)+y_j((t_k+pT+T)^+)|\\&\quad=|1+\tilde{\vartheta}_{jk}^y||y_j(t_k+pT+T)+y_j(t_k+pT)|\\&\quad\le 2M\tilde{\gamma}_3\mathrm{e}^{-\gamma(t_k+pT)}\\&\quad\le F\mathrm{e}^{-\gamma(t_k+pT)},\end{aligned}\tag{69}$$

$$\begin{aligned}&|(-1)^{p+1}v_j(t+(p+1)T)-(-1)^p v_j(t+pT)|\\&\quad=|v_j(t+pT)-(-v_j(t+pT+T))|\\&\quad\le M\mathrm{e}^{-\beta(t+pT)}\sup_{-\infty\le s\le 0}\sum_{j=1}^{m}|v_j(s+T)+v_j(s)|^2\\&\quad\le 2M\tilde{\gamma}_4\mathrm{e}^{-\gamma(t+pT)}\\&\quad\le F\mathrm{e}^{-\gamma(t+pT)},\end{aligned}\tag{70}$$

$$\begin{aligned}&|(-1)^{p+1}v_j((t_k+(p+1)T)^+)-(-1)^k v_j((t_k+pT)^+)|\\&\quad=|v_j((t_k+pT)^+)+v_j((t_k+pT+T)^+)|\\&\quad=|\tilde{\vartheta}_{jk}^v||v_j(t_k+pT+T)+v_j(t_k+pT)|\\&\quad\le 2M\tilde{\gamma}_4\mathrm{e}^{-\gamma(t_k+pT)},\end{aligned}\tag{71}$$

where $t+kT>0,\ i=1,2,\ldots,n,\ j=1,2,\ldots,m.$ For any $k\in\mathbb{N}$ and $t\neq t_k$, we have

$$\begin{cases}(-1)^{p+1}x_i(t+(p+1)T)=x_i(t)+\sum\limits_{z=0}^{p}[(-1)^{z+1}\\\quad\cdot x_i(t+(z+1)T)-(-1)^z x_i(t+zT)],\\(-1)^{p+1}u_i(t+(p+1)T)=u_i(t)+\sum\limits_{z=0}^{p}[(-1)^{z+1}\\\quad\cdot u_i(t+(z+1)T)-(-1)^z u_i(t+zT)],\\(-1)^{p+1}y_j(t+(p+1)T)=y_j(t)+\sum\limits_{z=0}^{p}[(-1)^{z+1}\\\quad\cdot y_j(t+(z+1)T)-(-1)^z y_j(t+zT)],\\(-1)^{p+1}v_j(t+(p+1)T)=v_j(t)+\sum\limits_{z=0}^{p}[(-1)^{z+1}\\\quad\cdot v_j(t+(z+1)T)-(-1)^z v_j(t+zT)].\end{cases}$$

Then, we have

$$\begin{aligned}&(-1)^{p+1}x_i(t+(p+1)T)\\&\qquad\le|x_i(t)|+\sum_{z=0}^{p}|(-1)^{z+1}x_i(t+(z+1)T)\\&\qquad\quad-(-1)^z x_i(t+zT)|,\\&(-1)^{p+1}x_i((t_k+(p+1)T)^+)\\&\qquad=|\vartheta_{ik}^x((-1)^{p+1}x_i(t_k+(p+1)T))|\\&\qquad\le|1+\tilde{\vartheta}_{ik}^x(-1)^{p+1}x_i(t_k+(p+1)T)|\\&\qquad\le|(-1)^{p+1}x_i(t_k+(p+1)T)|,\\&(-1)^{p+1}u_i(t+(p+1)T)\\&\qquad\le|u_i(t)|+\sum_{z=0}^{p}|(-1)^{z+1}u_i(t+(z+1)T)\\&\qquad\quad-(-1)^z u_i(t+zT)|,\\&(-1)^{p+1}u_i((t_k+(p+1)T)^+)\\&\qquad=|\vartheta_{ik}^u((-1)^{p+1}u_i(t_k+(p+1)T))|\\&\qquad\le|1+\tilde{\vartheta}_{ik}^u(-1)^{p+1}u_i(t_k+(p+1)T)|\\&\qquad\le|(-1)^{p+1}u_i(t_k+(p+1)T)|,\\&(-1)^{p+1}y_j(t+(p+1)T)\\&\qquad\le|y_j(t)|+\sum_{z=0}^{p}|(-1)^{z+1}y_j(t+(z+1)T)\\&\qquad\quad-(-1)^p y_j(t+zT)|,\end{aligned}$$

$$
\begin{aligned}
&(-1)^{p+1}y_j((t_k+(p+1)T)^+)\\
&\quad=|\vartheta^y_{jk}((-1)^{p+1}y_j(t_k+(p+1)T))|\\
&\quad\le|1+\tilde{\vartheta}^y_{jk}(-1)^{p+1}y_j(t_k+(p+1)T)|\\
&\quad\le|(-1)^{p+1}y_j(t_k+(p+1)T)|,\\
&(-1)^{p+1}v_j(t+(p+1)T)\\
&\quad\le|v_j(t)|+\sum_{z=0}^{p}|(-1)^{z+1}v_j(t+(z+1)T)\\
&\qquad-(-1)^z v_j(t+zT)|,\\
&(-1)^{p+1}v_j((t_k+(p+1)T)^+)\\
&\quad=|\vartheta^v_{jk}((-1)^{p+1}v_j(t_k+(p+1)T))|\\
&\quad\le|1+\tilde{\vartheta}^v_{jk}(-1)^{p+1}v_j(t_k+(p+1)T)|\\
&\quad\le|(-1)^{p+1}v_j(t_k+(p+1)T)|,
\end{aligned}\tag{72}
$$

where $i=1,2,\ldots,n,\ j=1,2,\ldots,m$. In view of Lemma 1, the solutions of system (4) are bounded. By inequalities (64)–(72), we can deduce that $\Big\{(-1)^{k+1}x_i(t+(k+1)T), (-1)^{k+1}u_i(t+(k+1)T), (-1)^{k+1}y_j(t+(k+1)T), (-1)^{k+1}v_j(t+(k+1)T)\Big\}$ is a fundamental sequence on any compact set of $\mathbb{R}_+$. Evidently, $(-1)^k\boldsymbol{x}(t+kT)$, $(-1)^k\boldsymbol{u}(t+kT)$, $(-1)^k\boldsymbol{y}(t+kT)$, and $(-1)^k\boldsymbol{v}(t+kT)$ uniformly converge to piecewise continuous functions $\boldsymbol{x}^*(t)$, $\boldsymbol{u}^*(t)$, $\boldsymbol{y}^*(t)$, and $\boldsymbol{v}^*(t)$ on any compact set of $\mathbb{R}_+$, respectively.

Thus, we can show that $\boldsymbol{Z}^*(t)$ is a T-anti-periodic solution of system (4). As

$$
\begin{aligned}
\boldsymbol{Z}^*(t+T)&=\lim_{k\to\infty}(-1)^k\boldsymbol{Z}(t+T+kT)\\
&=-\lim_{k+1\to\infty}(-1)^{k+1}\boldsymbol{Z}(t+(k+1)T)\\
&=-\boldsymbol{Z}^*(t),
\end{aligned}
$$

and

$$
\begin{aligned}
&\boldsymbol{Z}^*((t_k+T)^+)=\lim_{k\to\infty}(-1)^k\boldsymbol{Z}((t_k+T+kT)^+)\\
&=-\lim_{k+1\to\infty}(-1)^{k+1}\boldsymbol{Z}((t_k+(k+1)T)^+)\\
&=-\boldsymbol{Z}^*(t_k),
\end{aligned}
$$

$\boldsymbol{Z}^*(t)=\big(\boldsymbol{x}^*(t),\boldsymbol{u}^*(t),\boldsymbol{y}^*(t),\boldsymbol{v}^*(t)\big)^{\mathrm{T}}$ is T-anti-periodic.

Next, we prove that $\boldsymbol{Z}^*(t)=\big(\boldsymbol{x}^*(t),\ \boldsymbol{u}^*(t),\ \boldsymbol{y}^*(t),\ \boldsymbol{v}^*(t)\big)$ is a solution of system (4). Because the right-hand side of system (4) is piecewise continuous, we can conclude that $((-1)^{k+1}x_i(t+(k+1)T))'$, $((-1)^{k+1}u_i(t+(k+1)T))'$, $((-1)^{k+1}y_j(t+(k+1)T))'$, and $((-1)^{k+1}v_j(t+(k+1)T))'$ uniformly converge to piecewise continuous functions on any compact subset of $\mathbb{R}_+$, respectively. Therefore, letting $p\to+\infty$ on Eqs. (56)–(63), we obtain

$$
\begin{cases}
\dfrac{\mathrm{d}x_i^*(t)}{\mathrm{d}t}=-x_i^*(t)+u_i^*(t),\ t\neq t_k,\\
\dfrac{\mathrm{d}u_i^*(t)}{\mathrm{d}t}=-\big(a_i-\alpha_i+1\big)x_i^*(t)\\
\quad-\big(\alpha_i-1\big)u_i^*(t)+\displaystyle\sum_{j=1}^{m}c_{ji}(t)f_j(y_j^*(t))\\
\quad+\displaystyle\sum_{j=1}^{m}d_{ji}(t)f_j(y_j^*(t-\tau_{ji}(t)))\\
\quad+\displaystyle\sum_{j=1}^{m}h_{ji}(t)\int_0^{+\infty}K_{ji}(u)\\
\quad\cdot f_j(y_j^*(t-u))\mathrm{d}u+I_i(t),\ t\neq t_k,\\
\dfrac{\mathrm{d}y_j^*(t)}{\mathrm{d}t}=-y_j^*(t)+v_j^*(t),\ t\neq t_k,\\
\dfrac{\mathrm{d}v_j^*(t)}{\mathrm{d}t}=-\big(b_j-\beta_j+1\big)y_j^*(t)\\
\quad-\big(\beta_j-1\big)v_j^*(t)+\displaystyle\sum_{i=1}^{n}p_{ij}(t)g_i(x_i^*(t))\\
\quad+\displaystyle\sum_{i=1}^{n}q_{ij}(t)g_i(x_i^*(t-\sigma_{ij}(t)))\\
\quad+\displaystyle\sum_{i=1}^{n}o_{ij}(t)\int_0^{+\infty}N_{ij}(u)\\
\quad\cdot g_i(x_i^*(t-u))\mathrm{d}u+J_j(t),\ t\neq t_k,\\
\Delta x_i^*(t=t_k)=x_i^*(t_k)-x_i^*(t_k^-)\\
\quad=\vartheta^x_{ik}(x_i^*(t_k)),\ k=1,2,\ldots,\\
\Delta u_i^*(t=t_k)=u_i^*(t_k)-u_i^*(t_k^-)\\
\quad=\vartheta^u_{ik}(u_i^*(t_k)),\ k=1,2,\ldots,\\
\Delta y_j^*(t=t_k)=y_j^*(t_k)-y_j^*(t_k^-)\\
\quad=\vartheta^y_{jk}(y_j^*(t_k)),\ k=1,2,\ldots,\\
\Delta v_j^*(t=t_k)=v_j^*(t_k)-v_j^*(t_k^-)\\
\quad=\vartheta^v_{jk}(v_j^*(t_k)),\ k=1,2,\ldots.
\end{cases}\tag{73}
$$

Thus, $\boldsymbol{Z}^*(t)=\big(\boldsymbol{x}^*(t),\ \boldsymbol{u}^*(t),\ \boldsymbol{y}^*(t),\ \boldsymbol{v}^*(t)\big)^{\mathrm{T}}$ is the T-anti-periodic solution of system (4). Finally, results in Theorem 1 affirm that $\boldsymbol{Z}^*(t)=\big(\boldsymbol{x}^*(t),\ \boldsymbol{u}^*(t),\ \boldsymbol{y}^*(t),\ \boldsymbol{v}^*(t)\big)^{\mathrm{T}}$ is globally exponentially stable.

4 Examples and comparisons

In this section, we present an example to show the effectiveness of the results given in the previous section. Let $n = m = 2$. Then $N = 1, 2, \ldots, 8$. We have

$$\begin{cases} \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -x_i(t) + u_i(t), \\ \dfrac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -\big(a_i - \alpha_i + 1\big)x_i(t) - \big(\alpha_i - 1\big)u_i(t) \\ \quad + \displaystyle\sum_{j=1}^{2} c_{ji}(t)f_j(y_j(t)) + \sum_{j=1}^{2} d_{ji}(t)f_j(y_j(t-\tau_{ji}(t))) \\ \quad + \displaystyle\sum_{j=1}^{2} h_{ji}(t)\int_0^{+\infty} K_{ji}(u)f_j(y_j(t-u))\mathrm{d}u + I_i(t), \\ \dfrac{\mathrm{d}y_j(t)}{\mathrm{d}t} = -y_j(t) + v_j(t), \\ \dfrac{\mathrm{d}v_j(t)}{\mathrm{d}t} = -\big(b_j - \beta_j + 1\big)y_j(t) - \big(\beta_j - 1\big)v_j(t) \\ \quad + \displaystyle\sum_{i=1}^{2} p_{ij}(t)g_i(x_i(t)) + \sum_{i=1}^{2} q_{ij}(t)g_i(x_i(t-\sigma_{ij}(t))) \\ \quad + \displaystyle\sum_{i=1}^{2} o_{ij}(t)\int_0^{+\infty} N_{ij}(u)g_i(x_i(t-u))\mathrm{d}u + J_j(t), \\ \Delta x_1(t=t_k) = \tilde{\vartheta}_{1k}^{x} x_1(t_k),\ k = 1, 2, \ldots, \\ \Delta x_2(t=t_k) = \tilde{\vartheta}_{2k}^{x} x_2(t_k),\ k = 1, 2, \ldots, \\ \Delta u_1(t=t_k) = \tilde{\vartheta}_{1k}^{u} u_1(t_k),\ k = 1, 2, \ldots, \\ \Delta u_2(t=t_k) = \tilde{\vartheta}_{2k}^{u} u_2(t_k),\ k = 1, 2, \ldots, \\ \Delta y_1(t=t_k) = \tilde{\vartheta}_{1k}^{y} y_1(t_k),\ k = 1, 2, \ldots, \\ \Delta y_2(t=t_k) = \tilde{\vartheta}_{2k}^{y} y_2(t_k),\ k = 1, 2, \ldots, \\ \Delta v_1(t=t_k) = \tilde{\vartheta}_{1k}^{v} v_1(t_k),\ k = 1, 2, \ldots, \\ \Delta v_2(t=t_k) = \tilde{\vartheta}_{2k}^{v} v_2(t_k),\ k = 1, 2, \ldots, \end{cases} \tag{74}$$

where for all $u \in \mathbb{R}, i = j = 1, 2$, we have

$$\begin{aligned} f_j(u) =& g_i(u) = \frac{1}{2}\big(|u+1| - |u-1|\big), \\ &\Rightarrow L_{fj} = L_{gi} = 1, \\ K_{ji}(u) =& N_{ij}(u) = \mathrm{e}^{-u}. \end{aligned}$$

For $k = 1, 2$, we give

$$\begin{aligned} &\tilde{\vartheta}_{1k}^{x} = \tilde{\vartheta}_{2k}^{x} = -0.4,\ \tilde{\vartheta}_{1k}^{u} = \tilde{\vartheta}_{2k}^{u} = -0.7, \\ &\tilde{\vartheta}_{1k}^{y} = \tilde{\vartheta}_{2k}^{y} = -0.5, \tilde{\vartheta}_{1k}^{v} = \tilde{\vartheta}_{2k}^{v} = -0.4. \end{aligned}$$

Thus, $L^x = 0.4 < 1$, $L^u = 0.7 < 1$, $L^y = 0.5 < 1$, and $L^v = 0.4 < 1$. Let

$$(c_{ji}(t))_{1\le i,j\le 2} = \begin{pmatrix} 0.51|\sin t| & 0.62|\sin t| \\ 0.35|\cos t| & 0.54|\cos t| \end{pmatrix},$$

$$\Rightarrow (\overline{c}_{ji})_{1\le i,j\le 2} = \begin{pmatrix} 0.51 & 0.62 \\ 0.35 & 0.54 \end{pmatrix}.$$

$$(d_{ji}(t))_{1\le i,j\le 2} = \begin{pmatrix} 0.44|\sin t| & 0.52|\sin t| \\ 0.34|\cos t| & 0.70|\cos t| \end{pmatrix},$$

$$\Rightarrow (\overline{d}_{ji})_{1\le i,j\le 2} = \begin{pmatrix} 0.44 & 0.52 \\ 0.34 & 0.70 \end{pmatrix}.$$

$$(h_{ji}(t))_{1\le i,j\le 2} = \begin{pmatrix} 0.75|\sin t| & 0.54|\sin t| \\ 0.15|\cos t| & 0.44|\cos t| \end{pmatrix},$$

$$\Rightarrow (\overline{h}_{ji})_{1\le i,j\le 2} = \begin{pmatrix} 0.75 & 0.54 \\ 0.15 & 0.44 \end{pmatrix}.$$

$$(p_{ij}(t))_{1\le i,j\le 2} = \begin{pmatrix} 0.27|\sin t| & 0.39|\sin t| \\ 0.59|\cos t| & 0.21|\cos t| \end{pmatrix},$$

$$\Rightarrow (\overline{p}_{ij})_{1\le i,j\le 2} = \begin{pmatrix} 0.27 & 0.39 \\ 0.59 & 0.21 \end{pmatrix}.$$

$$(q_{ij}(t))_{1\le i,j\le 2} = \begin{pmatrix} 0.75|\sin t| & 0.63|\sin t| \\ 0.35|\cos t| & 0.55|\cos t| \end{pmatrix},$$

$$\Rightarrow (\overline{q}_{ij})_{1\le i,j\le 2} = \begin{pmatrix} 0.75 & 0.63 \\ 0.35 & 0.55 \end{pmatrix}.$$

$$(o_{ij}(t))_{1\le i,j\le 2} = \begin{pmatrix} 0.25|\sin t| & 0.68|\sin t| \\ 0.14|\cos t| & 0.53|\cos t| \end{pmatrix},$$

$$\Rightarrow (\overline{o}_{ij})_{1\le i,j\le 2} = \begin{pmatrix} 0.25 & 0.68 \\ 0.14 & 0.53 \end{pmatrix}.$$

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ a_1 & a_2 \end{pmatrix} = \begin{pmatrix} 3.1 & 3 \\ 3.1 & 3.2 \end{pmatrix}.$$

$$\begin{pmatrix} \beta_1 & \beta_2 \\ b_1 & b_2 \end{pmatrix} = \begin{pmatrix} 2.85 & 3 \\ 3 & 3 \end{pmatrix}.$$

$$(\tau_{ji}(t))_{1\le i,j\le 2} = \begin{pmatrix} 0.50\sin(2t) & 0.50\sin(2t) \\ 0.63\sin(2t) & 0.36\sin(2t) \end{pmatrix},$$

$$\Rightarrow (\overline{\tau}_{ji})_{1\le i,j\le 2} = \begin{pmatrix} 0.50 & 0.50 \\ 0.63 & 0.36 \end{pmatrix}.$$

$$(\sigma_{ij}(t))_{1\le i,j\le 2} = \begin{pmatrix} 0.72|\sin t| & 0.61|\sin t| \\ 0.53|\sin t| & 0.51|\sin t| \end{pmatrix},$$

$$\Rightarrow (\overline{\sigma}_{ij})_{1\le i,j\le 2} = \begin{pmatrix} 0.72 & 0.61 \\ 0.53 & 0.51 \end{pmatrix}.$$

$$(I_i(t))_{1\le i\le 2} = \begin{pmatrix} 0.45\sin t \\ 0.59\cos t \end{pmatrix} \Rightarrow \overline{I}_i = \begin{pmatrix} 0.45 \\ 0.59 \end{pmatrix}.$$

$$(J_j(t))_{1\le j\le 2} = \begin{pmatrix} 0.32\sin t \\ 0.52\cos t \end{pmatrix} \Rightarrow \overline{J}_j = \begin{pmatrix} 0.32 \\ 0.52 \end{pmatrix}.$$

Thus, we have

$$\frac{|a_1-\alpha_1+1|}{\alpha_1-1}=0.5<1,\ \frac{|a_2-\alpha_2+1|}{\alpha_2-1}=0.45<1,$$

$$\frac{|b_1-\beta_1+1|}{\beta_1-1}=0.6<1,\ \frac{|b_2-\beta_2+1|}{\beta_2-1}=0.5<1.$$

Thus, Hypotheses 1–4 are satisfied.

Let $p_1 = 10.5,\ p_2 = 9.25,\ p_3 = 8.75,\ p_4 = 8,\ p_5 = 6.85,\ p_6 = 6.25, p_7 = 5.25, p_8 = 4.75$.

From the above assumption, Hypothesis 5 is satisfied.

Therefore, all conditions from Theorems 1 and 2 are satisfied, and the IBAMNN with time-varying delay and impulsive effect (74) has a unique π-anti-periodic solution, which is globally exponentially stable. A numerical simulation is given in Figs. 1–4 to check these results.

Aouiti (2018) obtained some new sufficient conditions that ensure the existence and exponential stability of periodic solutions for IBAMNNs with time delay using the Weierstrass criteria, the boundedness of solutions, and the Lyapunov function. A constant coefficient was considered without impulsive effect. Zhang and Quan (2015) investigated the global exponential stability of an equilibrium point for inertial delayed BAMNNs with constant coefficients (for example, the connection strengths c_{ji}, d_{ji}, h_{ji}, p_{ij}, q_{ij}, and o_{ij} are all constants) using inequality techniques. In our model we use variable coefficients for the strong points of the connection and external inputs, because our model is more general than the models in Zhang and Quan (2015) and Aouiti (2018). Liao et al. (2017) investigated the same model as in Zhang and Quan (2015) with a variable coefficient. By combining Mawhin's continuation theorem of coincidence degree theory with the Lyapunov functional method and using inequality techniques, the existence and global exponential stability of periodic solutions for NNs were established. The models in Zhang and Quan (2015) and Liao et al. (2017) are not concerned with the impulsive effect and distributed delays. Besides, our approach for demonstrating the global exponential stability is different from the models in these studies. Xu and Zhang (2015) gave some conditions to demonstrate the existence and global exponential stability of an anti-periodic solution for a BAMNN with the inertial term and delay using the inequality technique and Lyapunov method. The model in Xu and Zhang (2015) is not concerned with impulsive effect and time-varying delays. Our results on the exponential stability for impulsive BAMNNs with mixed delays and the inertial term are essentially new, and the investigation methods used in this study can be used to study the piecewise anti-periodic solutions for some other types of NNs.

5 Conclusions

In this study, we consider a class of IBAMNNs with time-varying delays and distributed delays. First, we have transformed the second-order inertial

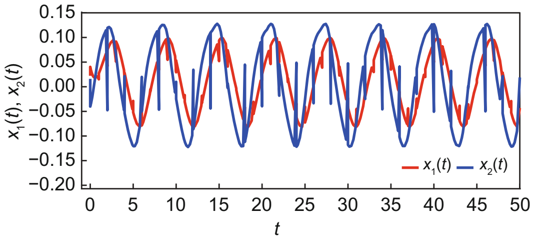

Fig. 1 Transient responses of states $x_1(t)$ and $x_2(t)$

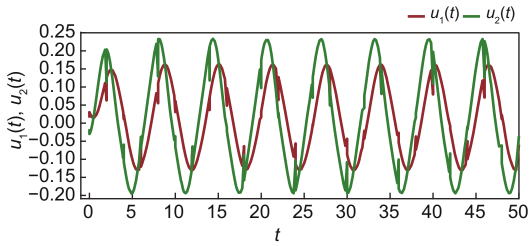

Fig. 2 Transient responses of states $u_1(t)$ and $u_2(t)$

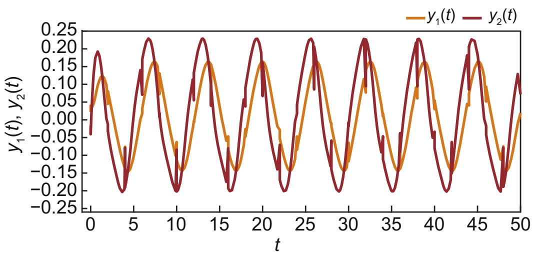

Fig. 3 Transient responses of states $y_1(t)$ and $y_2(t)$

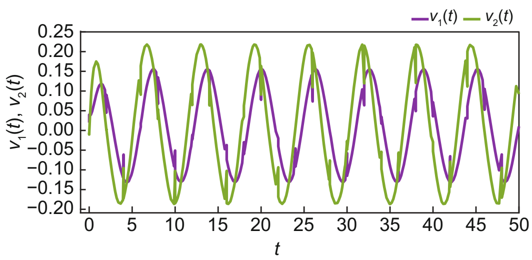

Fig. 4 Transient responses of states $v_1(t)$ and $v_2(t)$

NNs into a first-order differential system and applied the Lyapunov method and differential inequality techniques to present new conditions that ensure the existence and exponential stability of anti-periodic solutions of model (4). Note that our results are new and that the system studied is more general than the system in Xu and Zhang (2015). In addition, we have used a technique different from the one used in Xu and Zhang (2015) to show the exponential stability of the anti-periodic solution, which can be applied to many concrete examples of NNs. At last, we have detailed some open issues. We might want to stretch our results to examine the global stability criteria for impulsive IBAMNNs with unbounded delays (Ke and Miao, 2013c) and the dynamics of Clifford-valued IBAMNNs with impulsive effect (Li YK and Xiang, 2019). These topics will be our main focus in the future.

Contributors

Yang CAO processed the conceptualization. Chaouki AOUITI and Mahjouba Ben REZEG conducted the analysis and validation, and drafted the manuscript. Yang CAO polished the paper.

Compliance with ethics guidelines

Chaouki AOUITI, Mahjouba Ben REZEG, and Yang CAO declare that they have no conflict of interest.

References

Alimi AM, Aouiti C, Chérif F, et al., 2018. Dynamics and oscillations of generalized high-order Hopfield neural networks with mixed delays. *Neurocomputing*, 321:274-295. https://doi.org/10.1016/j.neucom.2018.01.061

Aouiti C, 2018. Oscillation of impulsive neutral delay generalized high-order Hopfield neural networks. *Neur Comput Appl*, 29(9):477-495. https://doi.org/10.1007/s00521-016-2558-3

Aouiti C, Assali EA, 2019. Stability analysis for a class of impulsive bidirectional associative memory (BAM) neural networks with distributed delays and leakage time-varying delays. *Neur Process Lett*, 50(1):851-885. https://doi.org/10.1007/s11063-018-9937-y

Aouiti C, Dridi F, 2019a. New results on impulsive Cohen–Grossberg neural networks. *Neur Process Lett*, 49(3):1459-1483. https://doi.org/10.1007/s11063-018-9880-y

Aouiti C, Dridi F, 2019b. Piecewise asymptotically almost automorphic solutions for impulsive non-autonomous high-order Hopfield neural networks with mixed delays. *Neur Comput Appl*, 31(9):5527-5545. https://doi.org/10.1007/s00521-018-3378-4

Aouiti C, Miaadi F, 2018. Finite-time stabilization of neutral Hopfield neural networks with mixed delays. *Neur Process Lett*, 48(3):1645-1669. https://doi.org/10.1007/s11063-018-9791-y

Aouiti C, Miaadi F, 2019. Pullback attractor for neutral Hopfield neural networks with time delay in the leakage term and mixed time delays. *Neur Comput Appl*, 31(8):4113-4122. https://doi.org/10.1007/s00521-017-3314-z

Aouiti C, Coirault P, Miaadi F, et al., 2017. Finite time boundedness of neutral high-order Hopfield neural networks with time delay in the leakage term and mixed time delays. *Neurocomputing*, 260:378-392. https://doi.org/10.1016/j.neucom.2017.04.048

Aouiti C, Abed Assali E, Cao JD, et al., 2018. Global exponential convergence of neutral-type competitive neural networks with multi-proportional delays, distributed delays and time-varying delay in leakage delays. *Int J Syst Sci*, 49(10):2202-2214. https://doi.org/10.1080/00207721.2018.1496297

Balasubramaniam P, Kalpana M, Rakkiyappan R, 2011. Global asymptotic stability of BAM fuzzy cellular neural networks with time delay in the leakage term, discrete and unbounded distributed delays. *Math Comput Model*, 53(5-6):839-853. https://doi.org/10.1016/j.mcm.2010.10.021

Batchelor M, Baxter R, O'Rourke M, et al., 1995. Exact solution and interfacial tension of the six-vertex model with anti-periodic boundary conditions. *J Phys A*, 28(10): 2759-2770. https://doi.org/10.1088/0305-4470/28/10/009

He X, Li CD, Shu Y, 2012. Bogdanov-Takens bifurcation in a single inertial neuron model with delay. *Neurocomputing*, 89:193-201. https://doi.org/10.1016/j.neucom.2012.02.019

Ke YQ, Miao CF, 2011. Stability analysis of BAM neural networks with inertial term and time delay. *WSEAS Trans Syst*, 10(12):425-438.

Ke YQ, Miao CF, 2013a. Stability analysis of inertial Cohen-Grossberg-type neural networks with time delays. *Neurocomputing*, 117:196-205. https://doi.org/10.1016/j.neucom.2013.01.026

Ke YQ, Miao CF, 2013b. Stability and existence of periodic solutions in inertial BAM neural networks with time delay. *Neur Comput Appl*, 23(3-4):1089-1099. https://doi.org/10.1007/s00521-012-1037-8

Ke YQ, Miao CF, 2013c. Exponential stability of periodic solutions in inertial neural networks with unbounded delay. *Int J Math Comput Phys Electr Comput Eng*, 7(3):477-486.

Ke YQ, Miao CF, 2017. Anti-periodic solutions of inertial neural networks with time delays. *Neur Process Lett*, 45(2):523-538. https://doi.org/10.1007/s11063-016-9540-z

Kosko B, 1988. Bidirectional associative memories. *IEEE Trans Syst Man Cybern*, 18(1):49-60. https://doi.org/10.1109/21.87054

Li HF, Jiang HJ, Hu C, 2016. Existence and global exponential stability of periodic solution of memristor-based BAM neural networks with time-varying delays. *Neur Netw*, 75:97-109. https://doi.org/10.1016/j.neunet.2015.12.006

Li XD, Song SJ, 2017. Stabilization of delay systems: delay-dependent impulsive control. *IEEE Trans Autom Contr*, 62(1):406-411. https://doi.org/10.1109/TAC.2016.2530041

Li XD, Wu JH, 2016. Stability of nonlinear differential systems with state-dependent delayed impulses. *Automatica*, 64:63-69. https://doi.org/10.1016/j.automatica.2015.10.002

Li XD, Ho DWC, Cao JD, 2019. Finite-time stability and settling-time estimation of nonlinear impulsive systems. *Automatica*, 99:361-368. https://doi.org/10.1016/j.automatica.2018.10.024

Li YK, 2008. Positive periodic solutions of nonlinear differential systems with impulses. *Nonl Anal Theory Meth Appl*, 68(8):2389-2405. https://doi.org/10.1016/j.na.2007.01.066

Li YK, Xiang JL, 2019. Existence and global exponential stability of anti-periodic solution for Clifford-valued inertial Cohen-Grossberg neural networks with delays. *Neurocomputing*, 332:259-269. https://doi.org/10.1016/j.neucom.2018.12.064

Li YK, Yang L, Wu WQ, 2015. Anti-periodic solution for impulsive BAM neural networks with time-varying leakage delays on time scales. *Neurocomputing*, 149:536-545. https://doi.org/10.1016/j.neucom.2014.08.020

Liao HY, Zhang ZQ, Ren L, et al., 2017. Global asymptotic stability of periodic solutions for inertial delayed BAM neural networks via novel computing method of degree and inequality techniques. *Chaos Sol Fract*, 104:785-797. https://doi.org/10.1016/j.chaos.2017.09.035

Liu B, Teo KL, Liu XZ, 2008. Robust exponential stabilization for large-scale uncertain impulsive systems with coupling time-delays. *Nonl Anal Theory Meth Appl*, 68(5):1169-1183. https://doi.org/10.1016/j.na.2006.12.025

Liu BW, 2007. Almost periodic solutions for Hopfield neural networks with continuously distributed delays. *Math Comput Simul*, 73(5):327-335. https://doi.org/10.1016/j.matcom.2006.05.027

Long ZW, 2016. New results on anti-periodic solutions for SICNNs with oscillating coefficients in leakage terms. *Neurocomputing*, 171:503-509. https://doi.org/10.1016/j.neucom.2015.06.070

M'Hamdi MS, Aouiti C, Touati A, et al., 2016. Weighted pseudo almost-periodic solutions of shunting inhibitory cellular neural networks with mixed delays. *Acta Math Sci*, 36(6):1662-1682. https://doi.org/10.1016/S0252-9602(16)30098-4

Okochi H, 1990. On the existence of anti-periodic solutions to a nonlinear evolution equation associated with odd subdifferential operators. *J Funct Anal*, 91(2):246-258. https://doi.org/10.1016/0022-1236(90)90143-9

Qi JT, Li CD, Huang TW, 2015. Stability of inertial BAM neural network with time-varying delay via impulsive control. *Neurocomputing*, 161:162-167. https://doi.org/10.1016/j.neucom.2015.02.052

Stamova I, Stamov T, Li XD, 2014. Global exponential stability of a class of impulsive cellular neural networks with supremums. *Int J Adapt Contr Signal Process*, 28(11):1227-1239. https://doi.org/10.1002/acs.2440

Tu ZW, Cao JD, Hayat T, 2016. Global exponential stability in Lagrange sense for inertial neural networks with time-varying delays. *Neurocomputing*, 171:524-531. https://doi.org/10.1016/j.neucom.2015.06.078

Wheeler DW, Schieve WC, 1997. Stability and chaos in an inertial two-neuron system. *Phys D*, 105(4):267-284. https://doi.org/10.1016/S0167-2789(97)00008-0

Xu CJ, Li PL, 2016. Existence and exponentially stability of anti-periodic solutions for neutral BAM neural networks with time-varying delays in the leakage terms. *J Nonl Sci Appl*, 9(3):1285-1305. https://doi.org/10.22436/jnsa.009.03.52

Xu CJ, Zhang QM, 2015. Existence and global exponential stability of anti-periodic solutions for BAM neural networks with inertial term and delay. *Neurocomputing*, 153:108-116. https://doi.org/10.1016/j.neucom.2014.11.047

Zhang ZQ, Quan ZY, 2015. Global exponential stability via inequality technique for inertial BAM neural networks with time delays. *Neurocomputing*, 151:1316-1326. https://doi.org/10.1016/j.neucom.2014.10.072

Zhou JW, Li YK, 2009. Existence and multiplicity of solutions for some Dirichlet problems with impulsive effects. *Nonl Anal Theory Meth Appl*, 71(7-8):2856-2865. https://doi.org/10.1016/j.na.2009.01.140

Zhou QY, Shao JY, 2018. Weighted pseudo-anti-periodic SICNNs with mixed delays. *Neur Comput Appl*, 29(10):865-872. https://doi.org/10.1007/s00521-016-2582-3