

Differential evolution based computation intelligence solver for elliptic partial differential equations

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Abstract: A differential evolution based methodology is introduced for the solution of elliptic partial differential equations (PDEs) with Dirichlet and/or Neumann boundary conditions. The solutions evolve over bounded domains throughout the interior nodes by minimization of nodal deviations among the population. The elliptic PDEs are replaced by the corresponding system of finite difference approximation, yielding an expression for nodal residues. The global residue is declared as the root-mean-square value of the nodal residues and taken as the cost function. The standard differential evolution is then used for the solution of elliptic PDEs by conversion to a minimization problem of the global residue. A set of benchmark problems consisting of both linear and nonlinear elliptic PDEs has been considered for validation, proving the effectiveness of the proposed algorithm. To demonstrate its robustness, sensitivity analysis has been carried out for various differential evolution operators and parameters. Comparison of the differential evolution based computed nodal values with the corresponding data obtained using the exact analytical expressions shows the accuracy and convergence of the proposed methodology.

Key words: Differential evolution; Boundary value problems; Partial differential equation; Finite difference scheme; Numerical computing

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1 Introduction

Partial differential equations (PDEs) can be found in a wide variety of areas of practical interest, e.g., blood flow in human veins (Spiller et al.,

2017), air flow past vehicles (Carlson et al., 2017), dynamic behavior of structures (Machado et al., 2016), magnetic resonance imaging (Chang and Tsai, 2014), thermal inhibition of tumors (Oden et al., 2016), spatial dynamics with thermo-hydraulic behavior modeling (Carotenuto et al., 2016), nanotechnology (Raja et al., 2016b), fluid dynamics (Raja et al., 2016a), astronomy (Ahmad et al., 2016), thermodynamics (Ahmad et al., 2017), and bioinformatics

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(Raja et al., 2017b).

In the past, research effort focused on finding the exact solution of PDEs (Gao et al., 2017; Yin et al., 2018; Hua et al., 2019). Attempts have also been made for theoretical analysis of the solutions and their applications (Wang et al., 2016, 2017; Dai et al., 2017a, 2017b; Ding et al., 2017; Zhang et al., 2017). Since exact and closed-form analytical solutions for PDEs can be found for only a class of elliptic PDEs, efforts have been made towards finding their approximate numerical solutions. The later ones have been found to match their numerical counterparts to a high degree of accuracy. Due to the wide application of PDEs, fast and reliable solvers are of prime importance in science and engineering. Various available numerical techniques offer ease of use, coupled with the advent of high-speed digital computers, making them the method of choice for the solution of PDEs. However, most of these techniques rely on serial algorithms, which do not fully exploit the computing power of parallel hardware.

The elliptic PDEs with specified boundary conditions have a unique solution $o(h^2)$. Consequently, when multiple proposed solutions progress through iterations, the one closest to that unique solution is expected to have the dominant rate of convergence due to its lowest value of the cost function. In contrast, non-unique scenarios may lead to poor convergence rate due to flipping among multiple alternate solutions.

A literature review reveals a rich diversity of numerical techniques that have been developed over time for the solution of elliptic PDEs. A solution of a Laplace equation posed as a Dirichlet problem using five-point approximation was first proposed by Courant et al. (1928). Subsequently, Gerschgorin (1930) formulated error bounds for elliptic PDEs and these efforts were extended further by various researchers (Motzkin and Wasow, 1952; Bramble et al., 1969). For positive second-order elliptic PDEs, various iterative schemes were developed, including Jacobi, Gauss-Seidel, Frankel-Yound, and Peaceman-Rachford (Peaceman and Rachford, 1955).

In the current study, we take only second-order elliptic equations which have homogeneous and non-homogeneous forms. Homogeneous elliptic PDEs are usually called the Laplace equation, and occur in many fields, such as electrostatics, gravitation, steady state flow of inviscid fluids, and steady state

heat conduction (Shao and Faltinsen, 2014; Gilbarg and Trudinger, 2015; Vétois, 2016; Webster, 2016). Similarly, the Poisson equation appears in electrostatics, elasticity theory, and elsewhere (Selvadurai, 2000; Zhao et al., 2013; Li et al., 2014; Marichal et al., 2014; Taleei and Dehghan, 2014).

2 Materials and methods

In this study, elliptic PEDs are solved by differential evolution and results are compared with the corresponding exact solutions for evaluation of the technique.

2.1 Problem formulation

A standard elliptic boundary value problem is considered as follows:

$$\begin{cases} \nabla^2 u = f(x, y), \\ u(x, y_{\min}) = f_1(x, y), \\ u(x, y_{\max}) = f_2(x, y), \\ u(x_{\min}, y) = f_3(x, y), \\ u(x_{\max}, y) = f_4(x, y), \end{cases} \quad (1)$$

where $x \in [x_{\min}, x_{\max}]$ and $y \in [y_{\min}, y_{\max}]$.

The domain of interest is subdivided into $(N+1)$ equal intervals in both variables of space with node spacing given by h in x and l in y such that

$$\begin{cases} x_j = x_{\min} + jh, \\ y_i = y_{\min} + il, \\ u_{i,j} = y(x_j, y_i), \\ u_{i,j-1} = y(x_{j-1}, y_i), \\ u_{i,j+1} = y(x_{j+1}, y_i), \\ u_{i-1,j} = y(x_j, y_{i-1}), \\ u_{i+1,j} = y(x_j, y_{i+1}). \end{cases} \quad (2)$$

Using the three-point central finite difference scheme, the first- and second-order derivatives on a finite grid become

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2}, \quad (3)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{l^2}. \quad (4)$$

The nodal residual (r_i) can then be defined as

$$r_i = \nabla_i^2 u - f(x_i, y_j). \quad (5)$$

The cost function represented by the global residue (R) is given by

$$R = \sqrt{\sum_{i=1}^n r_i^2}. \quad (6)$$

The original elliptic PDE is now posed as a cost function minimization problem.

2.2 Differential evolution

Storn and Price (1997) first proposed the differential evolution algorithm. The salient features of differential equations (DEs) include solvability of non-differentiable, nonlinear multi-objective functions and/or constrained optimization problems with inherently parallel methodology, ease of implementation, robustness, and good convergence capability without getting trapped in local minima. In this study, a novel method of differential evolution is used to solve elliptic PDEs.

DEs, similar to other evolutionary algorithms, have various phases including encoding, cost evaluation, initialization, selection, and reproduction (Guo et al., 2016; Das et al., 2017; Fateh et al., 2017; Raja et al., 2017a, 2018). Our work is based on a fitness function that approximates second-order derivatives. Consequently, the error decrement follows the $o(h^2)$ process.

1. Encoding

Optimization algorithms are widely used for optimization of bit strings and vectors. However, the problem definition asks for the solution of two-dimensional (2D) equations; therefore, a potential solution should take the form of a 2D matrix or array. Following this, the chromosomes or candidate solutions in this method are 2D matrices, where rows and columns represent the two variables in space and the value at each cell represents the solution.

2. Initialization

The initial population for the algorithm is constructed using a three-dimensional (3D) array, where the first two indices represent the rows and columns and the third specifies the chromosome number. Initially, for each such node specified by (i, j, k) , a uniform random number is generated in the range (min, max), where “min” and “max” represent the minimum and maximum values from the specified boundary values, respectively.

3. Fitness evaluation

The fitness evaluation based on the overall nodal residual error is described in the problem formulation. Individuals with small values of overall residuals have better/higher fitness values.

4. Trial vector generation

In DEs, trial vectors are used in crossover with corresponding parents to yield offspring. For each member of the parent population, a trial vector is calculated using the following relationship (Gilbarg and Trudinger, 2015; Webster, 2016):

$$\mathbf{T}_i = \mathbf{P}_j + \beta(\mathbf{P}_k - \mathbf{P}_l), \quad (7)$$

where \mathbf{T}_i is the trial vector corresponding to the i^{th} parent, \mathbf{P}_j is the target vector, and \mathbf{P}_k and \mathbf{P}_l are two randomly selected parent vectors. The parameter β is a scale factor that controls the amplification of the DE process. Clearly, all indices (i, j, k , and l) must be different from each other. A more sophisticated version of the trial vector generation procedure used in this work is

$$\mathbf{T}_i = \mathbf{P} + \beta \sum_{j=1}^{n_v} (\mathbf{P}_{kj} - \mathbf{P}_{jj}), \quad (8)$$

where $n_v = 2$, \mathbf{P}_{kj} and \mathbf{P}_{jj} are two random vectors in the j^{th} difference vector, and \mathbf{P} is the best individual in the population.

5. Crossover operation

In this step, crossover of trial vectors with their corresponding parents is carried out for the generation of offspring. In each such crossover step, one offspring is created. For crossover operation, the operators used in this work include exponential crossover and binomial crossover.

Adjacent crossover points are selected and the starting crossover point is selected randomly. A reasonably high value of the crossover probability has been used in this work, since empirical study has shown its effectiveness in improving the convergence rate. The exponential crossover is used for one-dimensional (1D) chromosome or vectors. However, the solution of elliptic PDEs is represented using a space grid or a matrix type representation. To tackle the 2D crossover problem, a variant of the exponential function is introduced. A random number is selected from the indices in the vertical or horizontal direction, and each subsequent row or column is replaced by that of the trial vector until all the rows or columns are replaced or the random number token exceeds the crossover probability.

6. Replacement operator

The greedy selection operator has been employed here. The cost value of offspring is compared with that of the corresponding parent, and the one with a lower cost value is used in the next generation. Consequently, DEs keep on improving the candidate solution through competitive evolution.

7. Termination

Any one of the following can be used to terminate the DE cycles:

(1) Stagnation

The cost value of the best individual remains unchanged for a user-specified number of generations. Alternatively, the improvement in the cost value over a user-specified number of generations becomes smaller than some threshold value.

(2) Cycle limits

The number of generations reaches some user-specified maximum number of generations.

The best individual represents the optimal solution at the termination of DE cycles. At the start, the population is initialized using a suitable scheme. Subsequently, trial vectors are generated using Eq. (8) and crossover is performed between parents and the corresponding trial vectors. After fitness evaluation of offspring, a replacement operation is carried out using the greedy criteria. This process is repeated until the termination criterion is satisfied.

3 Case studies

We have implemented the methodology of differential evolution on the following five benchmark studies of elliptic PDEs and employed the exact solution for comparison.

Case 1: Laplace equation with Dirichlet boundary conditions:

$$\begin{cases} \nabla^2 u = 0, & 1 < x, y < 9, \\ u(x, 1) = x, & 1 \leq x \leq 9, \\ u(x, 9) = 10 - x, & 1 \leq x \leq 9, \\ u(1, y) = y, & 1 \leq y \leq 9, \\ u(9, y) = 10 - y, & 1 \leq y \leq 9. \end{cases} \quad (9)$$

Exact solution:

$$u(x, y) = \frac{1}{4}(5x + 5y - xy - 5).$$

Case 2: Laplace equation with Neumann bound-

ary conditions:

$$\begin{cases} \nabla^2 u = 0, & 1 < x, y < 9, \\ u(x, 1) = x, & 1 \leq x \leq 9, \\ u(x, 9) = 9, & 1 \leq x \leq 9, \\ u(1, y) = y, & 1 \leq y \leq 9, \\ \frac{\partial u(9, y)}{\partial x} = 0, & 1 \leq y \leq 9. \end{cases} \quad (10)$$

Exact solution:

$$u(x, y) = y.$$

Case 3: Linear Poisson equation with Dirichlet boundary conditions:

$$\begin{cases} \frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 4, & 0 < x < 1, 0 < y < 2, \\ u(x, 0) = x^2, & 0 \leq x \leq 1, \\ u(x, 2) = (x - 2)^2, & 0 \leq x \leq 1, \\ u(0, y) = y^2, & 0 \leq y \leq 2, \\ u(1, y) = (y - 1)^2, & 0 \leq y \leq 2. \end{cases} \quad (11)$$

Exact solution:

$$u(x, y) = (x - y)^2.$$

Case 4: Nonlinear Poisson equation I with Dirichlet boundary conditions:

$$\begin{cases} \frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = xe^y, & 0 < x < 2, 0 < y < 1, \\ u(x, 0) = x^2, & 0 \leq x \leq 2, \\ u(x, 1) = e^x, & 0 \leq x \leq 2, \\ u(0, y) = 0, & 0 \leq y \leq 1, \\ u(2, y) = 2e^y, & 0 \leq y \leq 1. \end{cases} \quad (12)$$

Exact solution:

$$u(x, y) = xe^y.$$

Case 5: Nonlinear Poisson equation II with Dirichlet boundary conditions:

$$\begin{cases} \frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = -\cos(x + y) \\ \quad + \cos(x - y), & 0 < x < \pi, 0 < y < \pi/2, \\ u(x, 0) = \cos x, & 0 \leq x \leq \pi, \\ u(x, \pi/2) = 0, & 0 \leq x \leq \pi, \\ u(0, y) = \cos y, & 0 \leq y \leq \pi/2, \\ u(\pi, y) = -\cos y, & 0 \leq y \leq \pi/2. \end{cases} \quad (13)$$

Exact solution:

$$u(x, y) = \cos x \cos y.$$

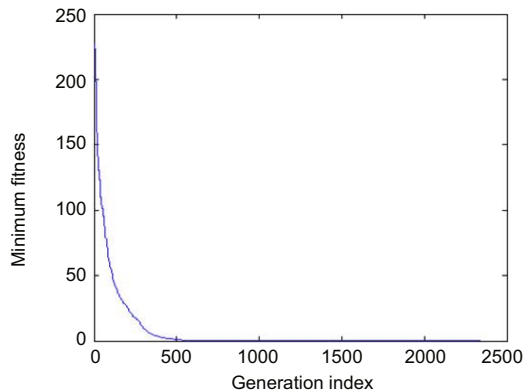


Fig. 9 Convergence curve for case 5 (nonlinear Poisson's equation II with Dirichlet boundary conditions)

Generally, the analysis of computational complexity is given in terms of fitness calls, when analyzing an evolutionary algorithm. However, as the chromosome size increases to a large number, as in our case, the candidate solution consists of 81 nodes, and fitness evaluations no longer remain a good measure of computational cost. Also, the context of this study revolves around the comparison of DE with other stochastic algorithms; it seems suitable for presenting the analysis in terms of computational time. It can be seen from all cases that the time consumed by DE for a solution to evolve is significantly less for the linear cases while more for nonlinear cases. Another important aspect is the increase in time consumption as the grid is refined from coarser to finer. It is also observed that the evolutionary progress of the DE algorithm with respect to generations of the five problems is in good agreement with the nature of the evolutionary algorithms, initially taking large steps and then refining the steps to avoid overshooting from minima.

2. Robustness of DE

As mentioned earlier, we have considered linear and nonlinear case studies of Laplace and Poisson's equations with different boundary conditions, and found that the proposed DE algorithm converges each time. It neither is trapped in any local minimum, nor gives any divergent result. Another important point about DE is that, we need not write separate codes for linear or nonlinear cases, making it a generic technique. Altogether, these features contribute towards the robustness of DE methodology for the application to an elliptic PDE solver.

3. Sensitivity to initialization

Let (min, max) define the expected range in

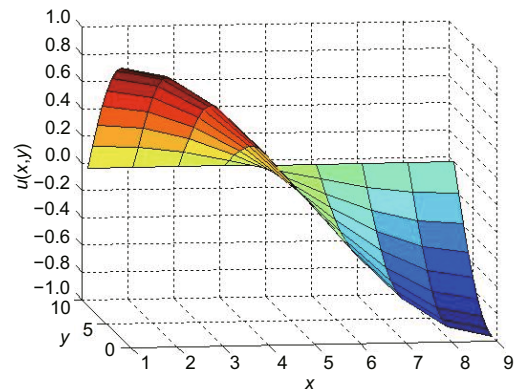


Fig. 10 Plot of the proposed solution for case 5 (nonlinear Poisson's equation II with Dirichlet boundary conditions)

which the solution lies. Four different sets of initialization ranges are defined to test the convergence of the differential evolution algorithm. The test cases are defined as: (1) lower bound less than min and upper bound greater than max; (2) lower bound less than min; (3) upper bound greater than max; (4) lower bound greater than min and upper bound less than max.

The time taken by the differential evolution algorithm for all these cases is almost the same, i.e., 51.6950, 50.5329, 52.7346, and 53.0481 s, which shows that the initialization ranges converge to the same fitness and also highlights the fact that the effect of the initialization method dies out quickly. This shows that the DE algorithm is less sensitive to initialization.

4. Convergence of interior and exterior nodes

The analysis of the convergence of interior and exterior nodes shows that the interior nodes that are farther from the predefined boundary nodes have more turbulence in the convergence process. However, the nodes that are closer to the predefined boundary nodes are less turbulent in the refinement stage of convergence. Also, the absolute error at nodes near the predefined boundary is less than that at the interior nodes, i.e., 2.0×10^{-6} vs. 1.1×10^{-4} .

5 Conclusions

In this study, a novel differential evolution based methodology has been developed for the solution of elliptic PDEs. The results clearly showed the effectiveness of the proposed DE methodology for various benchmark problems. CPU time required

and the amount of historical data required for convergence clearly showed that DE is an effective computing framework for solving PDEs. Also, DE can easily be applied to both linear and nonlinear problems. The random initialization scheme is easy to use and requires minimum information of the nature of the problem. A smaller step size yields more accurate answers, but it entails higher computational cost. Higher convergence rates, wider scope of applicability, inherently parallel nature, and robustness are among the attractive features of the proposed DE solver for elliptic PDEs.

Compliance with ethics guidelines

Muhammad Faisal FATEH, Aneela ZAMEER, Sikander M. MIRZA, Nasir M. MIRZA, Muhammad Saeed ASLAM, and Muhammad Asif Zahoor RAJA declare that they have no conflict of interest.

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