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Bisimulation-based stabilization of probabilistic Boolean control networks with state feedback control*

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Abstract: This study is concerned with probabilistic Boolean control networks (PBCNs) with state feedback control. A novel definition of bisimilar PBCNs is proposed to lower computational complexity. To understand more on bisimulation relations between PBCNs, we resort to a powerful matrix manipulation called semi-tensor product (STP). Because stabilization of networks is of critical importance, the propagation of stabilization with probability one between bisimilar PBCNs is then considered and proved to be attainable. Additionally, the transient periods (the maximum number of steps to implement stabilization) of two PBCNs are certified to be identical if these two networks are paired with a bisimulation relation. The results are then extended to the probabilistic Boolean networks.

Key words: Probabilistic Boolean control network; Bisimulation; Stabilization with probability one; State feedback control

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1 Introduction

First introduced by Kauffman (1969), Boolean networks (BNs) have become a highly active focus of study in the past few decades. In particular, BNs are used mostly in the bio-medical domain. Specifically, studying the behavior of genes has always been a prevalent trend and BNs have been proved to be an ideal tool to model genetic regulatory networks. For this reason, biologists use BNs to characterize the dynamics. Simulating neural networks (Wang et al., 1990) and lac operon's behaviors (Veliz-Cuba and Stigler, 2011) by BNs are important cases in point. Actually, BNs are logical systems consisting of a cluster of nodes and a set of logic functions that help connect scattered nodes into a network. In general, only two states (0 represents "off" and 1 represents "on") are assigned to each node and the status updated at any point in time is governed by some logic functions.

When BNs' behaviors are also dependent on some control inputs besides the nodes contained in this network, the classical BN is extended to a model called the Boolean control network (BCN). The emergence of BCNs is partly owing to the

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demand of artificially controlling BNs, for example, steering BNs out of undesirable states to desirable ones uses external controls. More extensive research on BCNs has not sprung up until the semi-tensor product (STP) of matrices was put forward (Cheng et al., 2010b; Lu et al., 2018a). Taking advantage of STP, the logical form of BNs (or BCNs) can be converted to an algebraic representation, which enables algebraic operations to be applied to the investigation of BNs (or BCNs). STP facilitates the exploration of a number of essential and significant properties enjoyed by BNs and BCNs, such as fixed points, cycles, and basin of attractors of BNs (or BCNs) (Cheng et al., 2010b), as well as some control problems, i.e., disturbance decoupling (Liu Y et al., 2017), controllability (Lu et al., 2016; Zhu QX et al., 2019), observability (Cheng and Qi, 2009; Fornasini and Valcher, 2012), optimal control (Zhu QX et al., 2018), output tracking control (Li YY et al., 2019), network synchronization (Li FF and Yu, 2016), and stabilization (Lu et al., 2018b; Li BW et al., 2019; Sun et al., 2020).

Many results on BN (or BCN) control problems have been recently obtained. Nonetheless, BNs (or BCNs) are settled once the logic functions are given. This may make BNs (or BCNs) be unadaptable to certain situations. In reality, the genetic regulation process has the property of uncertainty. Moreover, researchers often have to collect data with unavoidable random errors. Inspired by this, probabilistic BNs (PBNs) as well as probabilistic BCNs (PBCNs) (Shmulevich et al., 2002) were proposed to take the stochastic phenomena into account and cut down the experimental errors. As a matter of fact, PBNs (or PBCNs) can be viewed as an extension of BNs (or BCNs) because of the original characteristics they inherit from BNs (or BCNs). PBNs and PBCNs have been well studied and widely employed in the field of control engineering (Zhu SY et al., 2019), gene regulation (Liang JH and Han, 2012), and modeling of some phenomena of disease (Ma et al., 2008). Also, related control problems have captured the attention of researchers, such as stabilization (Huang et al., 2020), optimal control (Ching et al., 2009), and controllability (Tong et al., 2018a).

Lately, PBCN has become a research hotspot. Various categories of controllers have been used to regulate PBCNs. At the very start, free control sequences were put in use with input-state controllers, which rely on the input network. Then, another kind of controller determined by the current states of BNs, named the state feedback controller, was proposed. Since then, state feedback controllers have been in the highest flight of control topics. On the strength of state feedback controllers, multiple forms of control systems have then been devised. Some late-model controls include, but are not limited to, output feedback control, pinning control, and optimal control. Some important studies applying the controllers mentioned above are summarized in Table 1.

Stabilization of PBCNs has received particular research attention, since stabilization of a network, to a large extent, is equivalent to the convergence of discrete iterations (Huang et al., 2018; Xiong et al., 2019). In addition, stabilizing to achieve desirable goals is of vital importance in practice, e.g., modeling some diseases, where seeking out therapeutic methods and helping patients maintain fitness afterwards are ultimate objectives. Accordingly, studying the stabilization is worthwhile. Up to now, some studies have already covered the stabilization problems with respect to multifarious networks. We classify some influential studies on stabilization according to the networks in Table 2.

Table 1 A short overview of the literature using diverse controllers

Class of controllers	Subclass of controllers	Literature
Free control sequence	Optimal control	Ching et al. (2009) and Fornasini and Valcher (2014)
	Free sequence	Cheng and Qi (2009), Cheng et al. (2010a), and Laschov and Margaliot (2012)
Feedback control	State feedback control	Li R et al. (2014a), Li HT and Wang (2016), Liu RJ et al. (2016), and Liang JL et al. (2017)
	Output feedback control	Bof et al. (2015)
Pinning control		Chen et al. (2016) and Huang et al. (2020)
Input-state control		Cheng (2009) and Cheng and Qi (2009)
Event-triggered control		Tong et al. (2018a) and Zhu SY et al. (2018)

Table 2 A brief classification of studies on stabilization in terms of various kinds of networks

Network	Studies
BN	Cheng et al. (2011) proposed a necessary and sufficient condition for the stability of BNs. Li FF (2016) gave a necessary and sufficient condition for the stability of BNs, and invented a method of designing the corresponding pinning controllers
BCN	Bof et al. (2015) found a way to examine the existence of output feedback controllers satisfying the stabilization of BCNs. Guo et al. (2015) established a necessary and sufficient condition to set stabilization and carried out a procedure of finding all the stabilizers
PBN	Li R et al. (2014a) devised a state feedback law to accomplish the stabilization of PBNs
PBCN	Tong et al. (2018b) proposed a necessary and sufficient condition for context-sensitive PBCNs with static output feedback to set stabilization and a constructive technique to find such identified controllers

BN: Boolean network; BCN: Boolean control network; PBN: probabilistic Boolean network; PBCN: probabilistic Boolean control network

However, most previous studies neglect computational complexity, and thus to a certain degree, these preceding results are unsatisfactory. Li FF and Xie (2019) involved the process of choosing part of the pinning nodes to put in controls, which unfortunately includes various ways. To choose the minimum number of nodes, the method of exhaustion may be applied, which can definitely increase the computational complexity. Moreover, Li R et al. (2014b) required the locating of the target fixed point, which may demand a huge and expensive effort when PBCNs (or PBNs) reach a certain scale. Likewise, Li YY et al. (2018) required to examine all the possible initial states to identify their transient periods, which presumably requested considerable computing power and time. Naturally, researchers may want to match the states and trajectories of a comparatively large network to those of a relatively small one, so that we can go into the characteristics of the larger network by studying a smaller one instead. Therefore, the concept of bisimulation arises (Li R et al., 2018), and a framework of analyzing bisimulation relation between BCNs was set up. More importantly, it verified the possibility of inferring certain control properties, i.e., controllability and stabilization, of a complicated network by investigating a potentially simpler network via bisimulation relations. However, Li R et al. (2018) focused only on BCNs and the theorems are not necessarily appropriate for PBCNs.

Motivated by the above discussion, we dedicate our study to PBCNs with a state feedback controller, which is the most commonly used controller in industry, and then we intend to develop an approach to examine bisimulation relations of PBCNs. Doing our utmost to diminish computational complexity is the primary goal. Therefore, in parallel to Li R et al. (2018), we try to find a way to simulate such a PBCN with another one of the same kind so that the scale problem may not be a hindrance. To the best of our knowledge, this issue has not been touched upon by other researchers. Using the bisimulation relation, propagation of stabilization then enters into our consideration, where a small-scale PBCN may simulate certain characteristics of the original one. Similar to Li R et al. (2018), we reckon that a complex PBCN can become stabilizable to a state set if and only if the bisimilar small-scale PBCN is stabilizable to a fixed point, which calls for less labor in analysis. Thus, based on the bisimilar simpler PBCN, we can determine whether the large-scale PBCN is stabilizable to a certain state set and whether these two PBCNs can simultaneously achieve stabilization. In a nutshell, the issues we are going to cope with in this study include:

1. How can we generalize the bisimulation relation between BCNs to PBCNs? In short, it is unknown whether we can make a decent definition of the bisimulation relation between PBCNs.

2. Can we prove the presence of necessary and sufficient conditions for a relation between two PBCNs is a bisimulation relation? After determining the definition, we are supposed to present a condition to test whether a given relation is a bisimulation relation between PBCNs, which is full of uncertainty.

3. Is it feasible to deduce the set stabilization of a larger PBCN with probability one by analyzing a smaller BPCN through the bisimulation relation? To be more specific, our preliminary conjecture is that if a smaller PBCN is stabilizable to a certain point

with probability one, then through a bisimulation relation, a more complex PBCN can be stabilized to a state set, which is determined by the stationary point of the former network.

4. Do bisimilar PBCNs share the same transient period? Transient period is the smallest number of steps for a network to reach a steady point or set. If two networks that are matched by a bisimulation relation exhibit the same transient period, we can get to know the larger PBCN's transient period by studying the smaller one.

To tackle the four problems above, we use STP to describe PBCNs in their algebraic forms. To simplify the system, state feedback controllers are also expressed in the algebraic representations such that PBCNs can be shown in forms similar to those of BCNs, which makes the analysis easier. Additionally, at the end of this study, we further extend the definition of the bisimulation relation and the necessary and sufficient conditions to PBNs.

The novelties of this study include:

1. As far as we know, the concept of bisimulation relation between PBCNs is first proposed.

2. Using Boolean product, necessary and sufficient conditions involving the skeleton matrices of PBCNs and a certain logical matrix are given to test bisimulation relations between two PBCNs.

3. Using bisimulation relations, the stabilization of PBCNs with probability one is then propagated.

4. We prove that if two PBCNs are matched by a bisimulation relation, then they share the same transient period.

5. The results are generalized to PBNs.

However, it has been proved in Li R et al. (2018) that checking a bisimulation relation is an NP-hard problem that cannot be solved in polynomial time. Unfortunately, we are yet to find a more effective way to examine a bisimulation relation in polynomial time. Hence, the results obtained in this study may not necessarily contribute to the reduction of computational complexity, but once the efficiency is improved, these results can be significant. Now we will just leave this matter as future work.

Throughout this paper, $\boldsymbol{I}_n$ represents an n-dimensional identity matrix and $\boldsymbol{\delta}_n^i$ is its i^{th} column. $\varDelta_n$ denotes a set containing the vectors $\boldsymbol{\delta}_n^1, \boldsymbol{\delta}_n^2, \ldots, \boldsymbol{\delta}_n^n$. $\mathcal{R}_{m\times n}$ stands for the set of all $m\times n$ real matrices. $\mathcal{L}_{m\times n}$ signifies the set of all $m\times n$ logical matrices in which each column belongs to $\varDelta_m$. $\mathcal{L}$ denotes the set of logical matrices of arbitrary dimensions. We denote $\boldsymbol{\delta}_n[i_1, i_2, \ldots, i_k]$ as a matrix with its j^{th} column being $\boldsymbol{\delta}_n^{i_j}$. For a matrix $\boldsymbol{M}$, its (i,j)-entry is denoted by $(\boldsymbol{M})_{ij}$. A matrix with its entries being either 0 or 1 is called a Boolean matrix. Assume that $\boldsymbol{A}$ is an $m\times l$ Boolean matrix and $\boldsymbol{B}$ is an $l\times n$ Boolean matrix, the Boolean product of $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted by $\boldsymbol{A}\odot\boldsymbol{B}$, is an $m\times n$ Boolean matrix with its (i,j)-entry being $\bigvee\limits_{t=1}^{l}[(\boldsymbol{A}_{it})\wedge(\boldsymbol{B})_{tj}]$. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are $m\times l$ Boolean matrices, the Boolean sum of $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted by $\boldsymbol{A}\vee\boldsymbol{B}$, is then an $m\times l$ Boolean matrix with the (i,j)-entry being $(\boldsymbol{A})_{ij}\vee(\boldsymbol{B})_{ij}$. Given a pair of sets S_1 and S_2, a subset R of $S_1\times S_2$ is called a relation on $S_1\times S_2$. With regard to the relation $R\subseteq S_1\times S_2$, we denote R^{-1} as the inverse relation of R, where $R^{-1}=\{(s_2,s_1):(s_1,s_2)\in R\}$. We define a power-reducing matrix as $\boldsymbol{M}_{\mathrm{r}}=\boldsymbol{\delta}_4[1,4]$. For $\boldsymbol{x}\in\varDelta_2$, the equation $\boldsymbol{x}\ltimes\boldsymbol{x}=\boldsymbol{M}_{\mathrm{r}}\ltimes\boldsymbol{x}$ is a tautology. A swap matrix $\boldsymbol{W}_{[m,n]}$ is an $mn\times mn$ matrix and is defined as follows. Its rows and columns are labeled by the double index (ij). The columns are arranged by the ordered multi-index, i.e., $[(11), (21), \ldots, (m1), (12), (22), \ldots, (m2), \ldots, (1n), (2n), \ldots, (mn)]$, and the rows are arranged in the order like $[(11), (12), \ldots, (1n), (21), (22), \ldots, (2n), \ldots, (m1), (m2), \ldots, (mn)]$. Then, the element at position $[(I,J),(i,j)]$ is

$$w_{(I,J),(i,j)}=\begin{cases}1, & I=i \text{ and } J=j,\\ 0, & \text{otherwise}.\end{cases}$$

$\boldsymbol{\varPhi}_n=\prod\limits_{i=1}^{n}\boldsymbol{I}_{2^{i-1}}\otimes[(\boldsymbol{I}_2\otimes\boldsymbol{W}_{[2,2^{n-1}]})\ltimes\boldsymbol{M}_r]$ is called the group power-reducing matrix. If $\boldsymbol{X}=\boldsymbol{x}_1\ltimes\boldsymbol{x}_2\ltimes\ldots\ltimes\boldsymbol{x}_n$, where $n\geq 1$ and $\boldsymbol{x}_i\in\varDelta_2$ $(i=1,2,\ldots,n)$, then $\boldsymbol{X}\ltimes\boldsymbol{X}=\boldsymbol{\varPhi}_n\ltimes\boldsymbol{X}$.

2 Preliminaries

PBCNs with state feedback control are studied in this section. We first give the formal description of PBCNs.

Definition 1 A PBCN can be presented as

$$\boldsymbol{X}(t+1)=f_i\big(\boldsymbol{U}(t),\boldsymbol{X}(t)\big), \tag{1}$$

where $\boldsymbol{X}(t)$ is the state variable at time t taking values in $\{0,1\}^n$, $t=0,1,2,\ldots$, and $i=1,2,\ldots,r$. $\boldsymbol{U}(t)=h\big(\boldsymbol{X}(t)\big)\in\{0,1\}^m$ denotes the input at time

t with $h:\{0,1\}^n \rightarrow \{0,1\}^m$ being a fixed logic function. $f_i:\{0,1\}^{m+n} \rightarrow \{0,1\}^n$ is a logic function chosen from $f_1, f_2, ..., f_r$ at time t with probability $p_i > 0$. The relation $\sum_{i=1}^{r} p_i = 1$ must be satisfied.

However, it is more useful to describe Eq. (1) in an algebraic form using STP. Consequently, we will introduce the definition and some properties of STP first.

Definition 2 Let $\boldsymbol{A} \in \mathcal{R}_{p\times q}$, and $\boldsymbol{B} \in \mathcal{R}_{m\times n}$. STP of $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined as (Cheng et al., 2010b)

$$\boldsymbol{A} \ltimes \boldsymbol{B} = \left(\boldsymbol{A} \otimes \boldsymbol{I}_{\frac{l}{q}}\right)\left(\boldsymbol{B} \otimes \boldsymbol{I}_{\frac{l}{m}}\right),$$

where l denotes the least common multiple of q and m, and $\otimes$ stands for the Kronecker product of matrices.

Note that if $q = m$, then $\boldsymbol{A} \ltimes \boldsymbol{B} = \boldsymbol{AB}$. Therefore, STP can be regarded as an extension of the conventional matrix product. Using STP, a logical system can be described in an algebraic representation. Let the canonical vectors $\boldsymbol{\delta}_2^1$ and $\boldsymbol{\delta}_2^2$ denote the Boolean values 1 and 0, respectively. Then, a logic function of $\{1,0\}^n \rightarrow \{1,0\}^m$ can be equivalently expressed as a mapping from Δ_{2^n} to Δ_{2^m}, since the mapping from $(\Delta_2)^n$ to Δ_{2^n} converting $(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)$ to $\boldsymbol{x}_1 \ltimes \boldsymbol{x}_2 \ltimes \ldots \ltimes \boldsymbol{x}_n$ $(\boldsymbol{x}_i \in \Delta_2)$ is bijective.

Then, we will lead up to a technique to transform a logical system into an algebraic expression.

Lemma 1 Consider the logic function $f:\{0,1\}^n \rightarrow \{0,1\}^m$. Let $f(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n) = (\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_m)$, where $\boldsymbol{X}_i \in \{0,1\}$ $(i = 1, 2, \ldots, n)$ and $\boldsymbol{Y}_j \in \{0,1\}$ $(j = 1, 2, \ldots, m)$. Then, there exists a unique logical matrix $\boldsymbol{L} \in \mathcal{L}_{2^m \times 2^n}$ such that (Cheng et al., 2010b)

$$\boldsymbol{y}_1 \times \boldsymbol{y}_2 \ltimes \ldots \ltimes \boldsymbol{y}_m = \boldsymbol{L} \ltimes \boldsymbol{x}_1 \ltimes \boldsymbol{x}_2 \ltimes \ldots \ltimes \boldsymbol{x}_n,$$

where $\boldsymbol{L}$ is called the structure matrix of f and $\boldsymbol{x}_i \in \Delta_2$ and $\boldsymbol{y}_j \in \Delta_2$ are the corresponding vector representations of $\boldsymbol{X}_i$ and $\boldsymbol{Y}_j$, respectively.

Now, consider Σ and $\widetilde{\Sigma}$ as two different PBCNs in the form of Eq. (1). Then, these two PBCNs can be modeled by Lemma 1 in their algebraic forms as

$$\Sigma: \ \boldsymbol{x}(t+1) = \boldsymbol{L}_i \ltimes \boldsymbol{u}(t) \ltimes \boldsymbol{x}(t), \tag{2}$$

$$\widetilde{\Sigma}: \ \boldsymbol{z}(t+1) = \widetilde{\boldsymbol{L}}_j \ltimes \boldsymbol{v}(t) \ltimes \boldsymbol{z}(t), \tag{3}$$

where $\boldsymbol{x}(t) \in \Delta_N$, $\boldsymbol{u}(t) \in \Delta_M$, $\boldsymbol{z}(t) \in \Delta_{\widetilde{N}}$, and $\boldsymbol{v}(t) \in \Delta_{\widetilde{M}}$. $\boldsymbol{L}_i \in \mathcal{L}_{N\times MN}$ and $\widetilde{\boldsymbol{L}}_j \in \mathcal{L}_{\widetilde{N}\times\widetilde{M}\widetilde{N}}$ $(i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, k)$ are structure matrices in accordance with the logic functions picked at time t; i.e., $\boldsymbol{L}_i$ $(\widetilde{\boldsymbol{L}}_j)$ can be deemed structure matrices chosen from a pool of candidate structure matrices in terms of a given probability distribution.

Likewise, the control systems of Σ and $\widetilde{\Sigma}$ can be represented as

$$\boldsymbol{u}(t) = \boldsymbol{H} \ltimes \boldsymbol{x}(t), \ \boldsymbol{H} \in \mathcal{L}_{M\times N}, \tag{4}$$

$$\boldsymbol{v}(t) = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}(t), \ \widetilde{\boldsymbol{H}} \in \mathcal{L}_{\widetilde{M}\times\widetilde{N}}, \tag{5}$$

where $\boldsymbol{H}$ and $\widetilde{\boldsymbol{H}}$ will not change with time.

This study aims at figuring out the indispensable condition so that two PBCNs can mutually simulate each other. In other words, the purpose of this study is to find out under what circumstances there would be a bisimulation relation between two PBCNs. Therefore, it is helpful to give the definitions of the (bi)simulation relation first.

Definition 3 Consider PBCNs Σ and $\widetilde{\Sigma}$ as in Eqs. (2) and (3), respectively. A relation $R \subseteq \Delta_N \times \Delta_{\widetilde{N}}$ is a simulation relation of Σ by $\widetilde{\Sigma}$ if the following two conditions are satisfied:

1. For every $\boldsymbol{x} \in \Delta_N$, there exists a $\boldsymbol{z} \in \Delta_{\widetilde{N}}$ such that $(\boldsymbol{x}, \boldsymbol{z}) \in R$.

2. For every $(\boldsymbol{x}, \boldsymbol{z}) \in R$ and every possible choice of $\boldsymbol{L}_i$ with $\boldsymbol{u} = \boldsymbol{H} \ltimes \boldsymbol{x}$ being the present input of Σ, there exists an $\widetilde{\boldsymbol{L}}_j$ such that $(\boldsymbol{L}_i \ltimes \boldsymbol{u} \ltimes \boldsymbol{x}, \widetilde{\boldsymbol{L}}_j \ltimes \boldsymbol{v} \ltimes \boldsymbol{z}) \in R$ with $\boldsymbol{v} = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}$ being the input of $\widetilde{\Sigma}$.

Definition 4 Consider PBCNs Σ and $\widetilde{\Sigma}$ as in Eqs. (2) and (3), respectively. The relation $R \subseteq \Delta_N \times \Delta_{\widetilde{N}}$ is a bisimulation relation between Σ and $\widetilde{\Sigma}$ if R is a simulation relation of Σ by $\widetilde{\Sigma}$ and R^{-1} is a simulation relation of $\widetilde{\Sigma}$ by Σ.

If we construct a mapping $\boldsymbol{C}: \Delta_N \rightarrow \Delta_{\widetilde{N}}$ between the state sets of Σ and $\widetilde{\Sigma}$, and regard $\boldsymbol{C}$ as a logical matrix in $\boldsymbol{L}_{\widetilde{N}\times N}$, then a relation R can be described as

$$R = \{(\boldsymbol{x}, \boldsymbol{z}): \boldsymbol{z} = \boldsymbol{C} \ltimes \boldsymbol{x}, \ \boldsymbol{x} \in \Delta_N, \ \boldsymbol{z} \in \Delta_{\widetilde{N}}\}. \tag{6}$$

According to the definitions above, $\boldsymbol{C}$ must be a surjection if R is a bisimulation relation between Σ and $\widetilde{\Sigma}$. In the next section, a necessary and sufficient condition will be imposed to ensure that R can be a bisimulation relation between Σ and $\widetilde{\Sigma}$.

3 Bisimilar PBCNs with state feedback controller

Consider PBCNs Σ and $\widetilde{\Sigma}$ as in Eqs. (2) and (3) with their state feedback controllers as given in Eqs. (4) and (5), respectively. Then, these two PBCNs can be modeled by Lemma 1 as

$$\begin{aligned}\Sigma:\ \boldsymbol{x}(t+1)=&\boldsymbol{L}_i\ltimes\boldsymbol{u}(t)\ltimes\boldsymbol{x}(t)\\=&\boldsymbol{L}_i\ltimes\boldsymbol{H}\ltimes\boldsymbol{x}(t)\ltimes\boldsymbol{x}(t)\\=&\boldsymbol{L}_i\ltimes\boldsymbol{H}\ltimes\boldsymbol{\Phi}_N\ltimes\boldsymbol{x}(t),\end{aligned}\tag{7}$$

$$\widetilde{\Sigma}:\ \boldsymbol{z}(t+1)=\widetilde{\boldsymbol{L}}_j\ltimes\widetilde{\boldsymbol{H}}\ltimes\boldsymbol{\Phi}_{\widetilde{N}}\ltimes\boldsymbol{z}(t),\tag{8}$$

where $i=1,2,\ldots,r$, $j=1,2,\ldots,k$. Let $\boldsymbol{F}_i=\boldsymbol{L}_i\ltimes\boldsymbol{H}\ltimes\boldsymbol{\Phi}_N$ and $\widetilde{\boldsymbol{F}}_j=\widetilde{\boldsymbol{L}}_j\ltimes\widetilde{\boldsymbol{H}}\ltimes\boldsymbol{\Phi}_{\widetilde{N}}$. Then, we construct two square matrices called the skeleton matrices of Σ and $\widetilde{\Sigma}$:

$$\boldsymbol{Q}_F=\boldsymbol{F}_1\vee\boldsymbol{F}_2\vee\ldots\vee\boldsymbol{F}_r,\tag{9}$$

$$\boldsymbol{Q}_{\widetilde{F}}=\widetilde{\boldsymbol{F}}_1\vee\widetilde{\boldsymbol{F}}_2\vee\ldots\vee\widetilde{\boldsymbol{F}}_k.\tag{10}$$

As for an m-dimensional skeleton matrix, its (i,j)-entry equals 1 if and only if there exists a one-step transition from $\boldsymbol{\delta}_m^j$ to $\boldsymbol{\delta}_m^i$. Hence, the skeleton matrices defined as in Eqs. (9) and (10) indicate all the possible one-step state transitions of Σ and $\widetilde{\Sigma}$.

Theorem 1 Consider PBCNs Σ and $\widetilde{\Sigma}$ as in Eqs. (7) and (8) with their state feedback control systems as given in Eqs. (4) and (5), respectively. Let $\boldsymbol{Q}_F$ and $\boldsymbol{Q}_{\widetilde{F}}$ be the skeleton matrices and $\boldsymbol{C}\in\mathcal{L}_{\widetilde{N}\times N}$. Then the relation R defined in Eq. (6) is a bisimulation relation between Σ and $\widetilde{\Sigma}$ if and only if

$$\boldsymbol{C}\odot\boldsymbol{Q}_F=\boldsymbol{Q}_{\widetilde{F}}\odot\boldsymbol{C}.\tag{11}$$

Proof Necessity ($\Longrightarrow$): Suppose that the relation R is a bisimulation relation between Σ and $\widetilde{\Sigma}$.

First, R is a simulation relation of Σ by $\widetilde{\Sigma}$. According to Definition 3, for any $\boldsymbol{x}(t)\in\Delta_N$, there exists $\boldsymbol{z}(t)\in\Delta_{\widetilde{N}}$ such that $(\boldsymbol{x}(t),\boldsymbol{z}(t))\in R$. Then by Eq. (6), we have $\boldsymbol{z}(t)=\boldsymbol{C}\ltimes\boldsymbol{x}(t)$. Condition (2) in Definition 3 should also be satisfied. That is, for any given state $\boldsymbol{x}(t)$ and arbitrary structure matrix $\boldsymbol{L}_i$, there is a structure matrix $\widetilde{\boldsymbol{L}}_j$ of $\widetilde{\Sigma}$ satisfying

$$\begin{aligned}\boldsymbol{z}(t+1)=&\boldsymbol{C}\boldsymbol{x}(t+1)=\boldsymbol{C}\ltimes\boldsymbol{L}_i\ltimes\boldsymbol{H}\ltimes\boldsymbol{\Phi}_N\ltimes\boldsymbol{x}(t),\\\boldsymbol{z}(t+1)=&\widetilde{\boldsymbol{L}}_j\widetilde{\boldsymbol{H}}\boldsymbol{\Phi}_{\widetilde{N}}\boldsymbol{z}(t)=\widetilde{\boldsymbol{L}}_j\ltimes\widetilde{\boldsymbol{H}}\ltimes\boldsymbol{\Phi}_{\widetilde{N}}\ltimes\boldsymbol{C}\ltimes\boldsymbol{x}(t).\end{aligned}$$

We then have

$$\begin{aligned}&\boldsymbol{C}\ltimes\boldsymbol{L}_i\ltimes\boldsymbol{H}\ltimes\boldsymbol{\Phi}_N=\widetilde{\boldsymbol{L}}_j\ltimes\widetilde{\boldsymbol{H}}\ltimes\boldsymbol{\Phi}_{\widetilde{N}}\ltimes\boldsymbol{C},\\\Longrightarrow&\boldsymbol{C}\ltimes(\boldsymbol{L}_i\ltimes\boldsymbol{H}\ltimes\boldsymbol{\Phi}_N)=(\widetilde{\boldsymbol{L}}_j\ltimes\widetilde{\boldsymbol{H}}\ltimes\boldsymbol{\Phi}_{\widetilde{N}})\ltimes\boldsymbol{C}.\end{aligned}$$

Equivalently,

$$\boldsymbol{C}\boldsymbol{F}_i=\widetilde{\boldsymbol{F}}_j\boldsymbol{C},\tag{12}$$

$$\Longrightarrow\boldsymbol{C}\odot\boldsymbol{F}_i=\widetilde{\boldsymbol{F}}_j\odot\boldsymbol{C}.\tag{13}$$

Note that the multiplied matrices in Eq. (12) have compatible sizes. Hence, the symbol of STP is omitted here. The conversion from Eq. (12) to Eq. (13) is trivial. In view of the fact that $(\boldsymbol{C}\odot\boldsymbol{F}_i)_{pq}=1$ if $(\boldsymbol{C}\boldsymbol{F}_i)_{pq}>0$, Eq. (13) is absolutely satisfied. All in all, for any $\boldsymbol{F}_i$ with $\boldsymbol{x}(t)$ being the current state, we can find an $\widetilde{\boldsymbol{F}}_j$ such that Eq. (13) holds. In consequence, for a given $\boldsymbol{F}_i$ regardless of the present state, there are at most $\min\{k,N\}$ $\widetilde{\boldsymbol{F}}_j$'s so that Eq. (13) can be satisfied. This is because for an $\boldsymbol{F}_i$, the choice of $\widetilde{\boldsymbol{F}}_j$ is dependent on the value of $\boldsymbol{x}(t)$. Because there are N alternative values of $\boldsymbol{x}(t)$, the number of $\widetilde{\boldsymbol{F}}_j$'s that correspond to a given $\boldsymbol{F}_i$ should be no larger than N. When taking the number of $\widetilde{\boldsymbol{F}}_j$'s into account, the minimum values of N and k are taken.

Second, R^{-1} is a simulation relation of $\widetilde{\Sigma}$ by Σ. Given any $\boldsymbol{z}\in\Delta_{\widetilde{N}}$, there is at least one $\boldsymbol{x}_i\in\Delta_N$ $(1\le i\le N)$, so that $(\boldsymbol{z},\boldsymbol{x}_i)\in R^{-1}$. This is because $\boldsymbol{C}$ is a surjection from Δ_N to $\Delta_{\widetilde{N}}$. Consequently, we define a set $\mathcal{S}_i$ $(1\le i\le\widetilde{N})$ as

$$\mathcal{S}_i=\{\boldsymbol{\delta}_N^j:\boldsymbol{C}\ltimes\boldsymbol{\delta}_N^j=\boldsymbol{\delta}_{\widetilde{N}}^i\},\tag{14}$$

with d_i being its cardinality. Therefore, for any $\boldsymbol{z}(t)=\boldsymbol{\delta}_{\widetilde{N}}^i$, there exists $\boldsymbol{x}(t)\in\mathcal{S}_i$. It immediately follows that $(\boldsymbol{z}(t),\boldsymbol{x}(t))\in R^{-1}$.

Similarly, for a given pair $(\boldsymbol{z}(t),\boldsymbol{x}(t))\in R^{-1}$ and any $\widetilde{\boldsymbol{L}}_j$, we can find an $\boldsymbol{L}_i$ satisfying

$$\begin{aligned}\widetilde{\boldsymbol{L}}_j\ltimes\widetilde{\boldsymbol{H}}\ltimes\boldsymbol{\Phi}_{\widetilde{N}}\ltimes\boldsymbol{z}(t)&=\boldsymbol{z}(t+1)\\&=\boldsymbol{C}\ltimes\boldsymbol{x}(t+1)\\&=\boldsymbol{C}\ltimes\boldsymbol{L}_i\ltimes\boldsymbol{H}\ltimes\boldsymbol{\Phi}_N\ltimes\boldsymbol{x}(t).\end{aligned}$$

As defined in Eq. (14), $\boldsymbol{z}(t)=\boldsymbol{C}\ltimes\boldsymbol{x}(t)$ when $\boldsymbol{x}(t)\in\mathcal{S}_i$. Thus, we have

$$\begin{aligned}\widetilde{\boldsymbol{L}}_j\ltimes\widetilde{\boldsymbol{H}}\ltimes\boldsymbol{\Phi}_{\widetilde{N}}\ltimes\boldsymbol{z}(t)&=\widetilde{\boldsymbol{L}}_j\ltimes\widetilde{\boldsymbol{H}}\ltimes\boldsymbol{\Phi}_{\widetilde{N}}\ltimes\boldsymbol{C}\ltimes\boldsymbol{x}(t)\\&=\boldsymbol{C}\ltimes\boldsymbol{L}_i\ltimes\boldsymbol{H}\ltimes\boldsymbol{\Phi}_N\ltimes\boldsymbol{x}(t),\end{aligned}$$

or

$$\begin{aligned}&\boldsymbol{C}\ltimes\boldsymbol{L}_i\ltimes\boldsymbol{H}\ltimes\boldsymbol{\Phi}_N=\widetilde{\boldsymbol{L}}_j\ltimes\widetilde{\boldsymbol{H}}\ltimes\boldsymbol{\Phi}_{\widetilde{N}}\ltimes\boldsymbol{C},\\\Longrightarrow&\boldsymbol{C}\odot\boldsymbol{F}_i=\widetilde{\boldsymbol{F}}_j\odot\boldsymbol{C}.\end{aligned}\tag{15}$$

Accordingly, with regard to any $\widetilde{\boldsymbol{F}}_j$ with $\boldsymbol{z}=\boldsymbol{\delta}_{\widetilde{N}}^i$ being the present state, similar to the previous analysis, there are at most $\min\{r, d_i\}$ $\boldsymbol{F}_i$'s satisfying Eq. (15). Thus, for a random $\widetilde{\boldsymbol{F}}_j$, there exist at most $\min\{r, d_1\} + \min\{r, d_2\} + \ldots + \min\{r, d_{\widetilde{N}}\}$ $\boldsymbol{F}_i$'s by which Eq. (15) can be established.

To conclude, if R is a simulation relation of Σ by $\widetilde{\Sigma}$, then for any $\boldsymbol{F}_i$, there are some $\widetilde{\boldsymbol{F}}_j$'s so that Eq. (13) holds. Thus,

$$\begin{aligned}\boldsymbol{C} \odot \boldsymbol{Q}_F &= \boldsymbol{C} \odot (\boldsymbol{F}_1 \vee \boldsymbol{F}_2 \vee \ldots \vee \boldsymbol{F}_r) \\ &= (\boldsymbol{C} \odot \boldsymbol{F}_1) \vee (\boldsymbol{C} \odot \boldsymbol{F}_2) \vee \ldots \vee (\boldsymbol{C} \odot \boldsymbol{F}_r) \\ &= (\widetilde{\boldsymbol{F}}_{k1} \vee \widetilde{\boldsymbol{F}}_{k2} \vee \ldots \vee \widetilde{\boldsymbol{F}}_{kr}) \odot \boldsymbol{C} \\ &\subseteq \boldsymbol{Q}_{\widetilde{F}} \odot \boldsymbol{C}.\end{aligned} \tag{16}$$

Analogously, if R^{-1} is a simulation relation of $\widetilde{\Sigma}$ by Σ, for any $\widetilde{\boldsymbol{F}}_j$, various $\boldsymbol{F}_i$'s can be found so that Eq. (15) is true. As a result,

$$\begin{aligned}\boldsymbol{Q}_{\widetilde{F}} \odot \boldsymbol{C} &= (\widetilde{\boldsymbol{F}}_1 \vee \widetilde{\boldsymbol{F}}_2 \vee \ldots \vee \widetilde{\boldsymbol{F}}_k) \odot \boldsymbol{C} \\ &= \boldsymbol{C} \odot (\boldsymbol{F}_{r1} \vee \boldsymbol{F}_{r2} \vee \ldots \vee \boldsymbol{F}_{rk}) \\ &\subseteq \boldsymbol{C} \odot \boldsymbol{Q}_F.\end{aligned} \tag{17}$$

Combining Eqs. (16) and (17), we conclude that if R is a bisimulation relation between Σ and $\widetilde{\Sigma}$, then $\boldsymbol{C} \odot \boldsymbol{Q}_F = \boldsymbol{Q}_{\widetilde{F}} \odot \boldsymbol{C}$.

Sufficiency ($\Longleftarrow$): Assume that $\boldsymbol{C} \odot \boldsymbol{Q}_F = \widetilde{\boldsymbol{Q}}_{\widetilde{F}} \odot \boldsymbol{C}$. Then, we must show that the relation R is a bisimulation relation between Σ and $\widetilde{\Sigma}$. Observe that Eq. (11) is equivalent to the statement that for any $1 \leq i \leq \widetilde{N}$ and $1 \leq j \leq N$, $(\boldsymbol{C} \odot \boldsymbol{Q}_F)_{ij} = 1$ if and only if $(\boldsymbol{Q}_{\widetilde{F}} \odot \boldsymbol{C})_{ij} = 1$. According to the Boolean matrix product defined, we have

$$(\boldsymbol{C} \odot \boldsymbol{Q}_F)_{ij} = \bigvee_{t=1}^{N} \left[(\boldsymbol{C})_{it} \wedge (\boldsymbol{Q}_F)_{tj}\right],$$

$$(\boldsymbol{Q}_{\widetilde{F}} \odot \boldsymbol{C})_{ij} = \bigvee_{l=1}^{\widetilde{N}} \left[(\boldsymbol{Q}_{\widetilde{F}})_{il} \wedge (\boldsymbol{C})_{lj}\right].$$

Hence, if $(\boldsymbol{C} \odot \boldsymbol{Q}_F)_{ij} = 1$, there exists some t $(1 \leq t \leq N)$ such that

$$(\boldsymbol{C})_{it} = (\boldsymbol{Q}_F)_{tj} = 1.$$

If $(\boldsymbol{Q}_{\widetilde{F}} \odot \boldsymbol{C})_{ij} = 1$, then we can find some l $(1 \leq l \leq \widetilde{N})$ such that

$$(\boldsymbol{C})_{lj} = (\boldsymbol{Q}_{\widetilde{F}})_{il} = 1.$$

First, we show that R is a simulation relation of Σ by $\widetilde{\Sigma}$. Condition (1) in Definition 3 is clearly satisfied. To show that condition (2) in Definition 3 is also satisfied, let $\boldsymbol{x} = \boldsymbol{\delta}_N^j$ for some j $(1 \leq j \leq N)$. For this j, we can find $(\boldsymbol{Q}_F)_{tj} = 1$ for some t $(1 \leq t \leq N)$. Hence, Σ admits a one-step transition from $\boldsymbol{\delta}_N^j$ to $\boldsymbol{\delta}_N^t$, i.e., the conditional probability $P\{\boldsymbol{x}(1) = \boldsymbol{\delta}_N^t \mid \boldsymbol{x}(0) = \boldsymbol{\delta}_N^j, \boldsymbol{u}(0) = \boldsymbol{H} \ltimes \boldsymbol{\delta}_N^j\} > 0$. For $\boldsymbol{x} = \boldsymbol{\delta}_N^t$, we can find $(\boldsymbol{C})_{it} = 1$ for some i $(1 \leq i \leq \widetilde{N})$. Thus, $(\boldsymbol{C} \odot \boldsymbol{Q}_F)_{ij} = 1$ indicates that $(\boldsymbol{Q}_{\widetilde{F}} \odot \boldsymbol{C})_{ij} = 1$. Accordingly, we have $(\boldsymbol{C})_{lj} = (\boldsymbol{Q}_{\widetilde{F}})_{il} = 1$, for some l $(1 \leq l \leq \widetilde{N})$. This means that, for $\boldsymbol{x} = \boldsymbol{\delta}_N^j$, there is $\boldsymbol{\delta}_{\widetilde{N}}^l$ satisfying $(\boldsymbol{\delta}_N^j, \boldsymbol{\delta}_{\widetilde{N}}^l) \in R$ since $(\boldsymbol{C})_{lj} = 1$. In addition, $(\boldsymbol{Q}_{\widetilde{F}})_{il} = 1$ suggests that there is a one-step transition from $\boldsymbol{\delta}_{\widetilde{N}}^l$ to $\boldsymbol{\delta}_{\widetilde{N}}^i$, or the probability $P\{\boldsymbol{z}(1) = \boldsymbol{\delta}_{\widetilde{N}}^i | \boldsymbol{z}(0) = \boldsymbol{\delta}_{\widetilde{N}}^l, \boldsymbol{v}(0) = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{\delta}_{\widetilde{N}}^l\} > 0$. Since $(\boldsymbol{C})_{it} = 1$, it follows that $(\boldsymbol{\delta}_N^t, \boldsymbol{\delta}_{\widetilde{N}}^i) \in R$. Thus, condition (2) in Definition 3 is satisfied, and we conclude that R is a simulation relation of Σ by $\widetilde{\Sigma}$.

Finally, we must prove that R^{-1} is a simulation relation of $\widetilde{\Sigma}$ by Σ. Condition (1) in Definition 3 is obviously satisfied, since $\boldsymbol{C}$ is a surjection. It suffices to show that for any $(\boldsymbol{z}, \boldsymbol{x}) \in R^{-1}$ and any $\widetilde{\boldsymbol{L}}_j$ with the input $\boldsymbol{v} = \widetilde{\boldsymbol{H}} \ltimes z$, there exist $\boldsymbol{L}_i$ and $\boldsymbol{u} = \boldsymbol{H} \ltimes \boldsymbol{x}$ such that $(\widetilde{\boldsymbol{L}}_j \ltimes \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{\Phi}_{\widetilde{N}} \ltimes \boldsymbol{z}, \boldsymbol{L}_i \ltimes \boldsymbol{H} \ltimes \boldsymbol{\Phi}_N \ltimes \boldsymbol{x}) \in R^{-1}$. Let $\boldsymbol{z} = \boldsymbol{\delta}_{\widetilde{N}}^l$ and $(\boldsymbol{Q}_{\widetilde{F}})_{il} = 1$ for some i and l $(1 \leq i, l \leq \widetilde{N})$. It follows that there is a one-step transition from $\boldsymbol{\delta}_{\widetilde{N}}^l$ to $\boldsymbol{\delta}_{\widetilde{N}}^i$ or $P\{\boldsymbol{z}(1) = \boldsymbol{\delta}_{\widetilde{N}}^i | \boldsymbol{z}(0) = \boldsymbol{\delta}_{\widetilde{N}}^l, \boldsymbol{v}(0) = \widetilde{\boldsymbol{H}} \times \boldsymbol{\delta}_{\widetilde{N}}^l\} > 0$. For this l, there exists some $\boldsymbol{x} \in \mathcal{S}_l$. Supposing that $\boldsymbol{\delta}_N^j \in \mathcal{S}_l$ $(1 \leq j \leq N)$, then $(\boldsymbol{C})_{lj} = 1$ and $(\boldsymbol{\delta}_{\widetilde{N}}^l, \boldsymbol{\delta}_N^j) \in R^{-1}$. Hence, we have $(\boldsymbol{Q}_{\widetilde{F}} \odot \boldsymbol{C})_{ij} = 1$. It follows from Eq. (11) that $(\boldsymbol{C} \odot \boldsymbol{Q}_F)_{ij} = 1$. Thus, we can find some t $(1 \leq t \leq N)$ so that $(\boldsymbol{C})_{it} = (\boldsymbol{Q}_F)_{tj} = 1$. When $(\boldsymbol{Q}_F)_{tj} = 1$, the state transition probability $P\{\boldsymbol{x}(1) = \boldsymbol{\delta}_N^t \mid \boldsymbol{x}(0) = \boldsymbol{\delta}_N^j, \boldsymbol{u}(0) = \boldsymbol{H} \ltimes \boldsymbol{\delta}_N^j\} > 0$. That is to say, Σ has a state transition from $\boldsymbol{\delta}_N^j$ to $\boldsymbol{\delta}_N^t$ in one step. In addition, $(\boldsymbol{\delta}_{\widetilde{N}}^i, \boldsymbol{\delta}_N^t) \in R^{-1}$ since $(\boldsymbol{C})_{it} = 1$.

Remark 1 Given a complex PBCN and a predetermined surjection between the state sets, the structure matrix of the simpler PBCN can be constructed. Let $\boldsymbol{M} = \boldsymbol{C} \odot \boldsymbol{Q}_F$. Then $\boldsymbol{Q}_{\widetilde{F}}$ can be obtained through the relation $\boldsymbol{Q}_{\widetilde{F}} \odot \boldsymbol{C} = \boldsymbol{M}$. If we denote $\boldsymbol{C} = \boldsymbol{\delta}_{\widetilde{N}}[\alpha_1, \alpha_2, \ldots, \alpha_N]$, where $\alpha_i \leq \widetilde{N}$ and $i = 1, 2, \ldots, N$, then the multiplying law indicates that $\boldsymbol{M} = [\mathbf{Col}_{\alpha_1}(\boldsymbol{Q}_{\widetilde{F}}), \mathbf{Col}_{\alpha_2}(\boldsymbol{Q}_{\widetilde{F}}), \ldots, \mathbf{Col}_{\alpha_N}(\boldsymbol{Q}_{\widetilde{F}})]$. Note that $\boldsymbol{C}$ is nonsingular since it is a surjection, which implies that the matrix $\boldsymbol{M}$ contains all the columns in $\boldsymbol{Q}_{\widetilde{F}}$. Hence, $\boldsymbol{Q}_{\widetilde{F}}$ can be recovered.

We then introduce a practical example to illustrate the validity of Theorem 1.

Example 1 Let us consider a BCN. It is a reduced model of the lac operon in the *Escherichia coli*. The model contains the following logical variables: $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, and $\boldsymbol{X}_3$ are state variables representing the lac mRNA, the lactose in high concentrations, and the lactose in medium concentrations, respectively. $\boldsymbol{U}_1$, $\boldsymbol{U}_2$, and $\boldsymbol{U}_3$ are input variables denoting the extracellular glucose, the high extracellular lactose, and the medium extracellular lactose, respectively. The model is expressed as

$$\Sigma:\begin{cases}\boldsymbol{X}_1(t+1)=\neg\boldsymbol{U}_1(t)\wedge(\boldsymbol{X}_2(t)\vee\boldsymbol{X}_3(t)),\\ \boldsymbol{X}_2(t+1)=\neg\boldsymbol{U}_1(t)\wedge\boldsymbol{U}_2(t)\wedge\boldsymbol{X}_1(t),\\ \boldsymbol{X}_3(t+1)=\neg\boldsymbol{U}_1(t)\wedge\Big(\boldsymbol{U}_2(t)\vee\big(\boldsymbol{U}_3(t)\wedge\boldsymbol{X}_1(t)\big)\Big).\end{cases}\tag{18}$$

If we assume that the state variable $\boldsymbol{X}_3(t)$ in Eq. (18) may be constant at some time and let the input variable $\boldsymbol{U}_1=0$, then the model can be described as a PBCN. The PBCN is given as

$$\Sigma:\begin{cases}\boldsymbol{X}_1(t+1)=\boldsymbol{X}_2(t)\vee\boldsymbol{X}_3(t),\\ \boldsymbol{X}_2(t+1)=\boldsymbol{U}_1(t)\wedge\boldsymbol{X}_1(t),\\ \boldsymbol{X}_3(t+1)=\begin{cases}\boldsymbol{U}_1(t)\vee\big(\boldsymbol{U}_2(t)\wedge\boldsymbol{X}_1(t)\big),P=\alpha,\\ \boldsymbol{X}_3(t),\ P=1-\alpha,\end{cases}\end{cases}\tag{19}$$

where $P=1-\alpha$ denotes the probability that $\boldsymbol{X}_3$ remains intact at each time point, $0<\alpha<1$, and $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ are used to signify the high extracellular lactose and the medium extracellular lactose, respectively. Then consider another PBCN as

$$\widetilde{\Sigma}:\begin{cases}P=\beta:\\ \qquad\begin{cases}\boldsymbol{Z}_1(t+1)=\neg\big(\boldsymbol{V}(t)\wedge\boldsymbol{Z}_2(t)\big),\\ \boldsymbol{Z}_2(t+1)=\boldsymbol{Z}_1(t)\wedge\boldsymbol{Z}_2(t).\end{cases}\\ P=1-\beta:\\ \qquad\begin{cases}\boldsymbol{Z}_1(t+1)=\neg\boldsymbol{V}(t)\vee\big(\boldsymbol{Z}_1(t)\wedge\neg\boldsymbol{Z}_2(t)\big),\\ \boldsymbol{Z}_2(t+1)=\big(\neg\boldsymbol{Z}_1(t)\vee\boldsymbol{Z}_2(t)\big)\wedge\boldsymbol{V}(t),\end{cases}\end{cases}\tag{20}$$

where $0<\beta<1$, $\boldsymbol{Z}(t)_1$, $\boldsymbol{Z}(t)_2\in\{0,1\}$ are state variables, and $\boldsymbol{V}(t)\in\{0,1\}$ is an input variable.

Vector forms of these logical variables are used to define $\boldsymbol{x}(t)=\boldsymbol{x}_1(t)\ltimes\boldsymbol{x}_2(t)\ltimes\boldsymbol{x}_3(t)$, $\boldsymbol{u}(t)=\boldsymbol{u}_1(t)\ltimes\boldsymbol{u}_2(t)$, and $\boldsymbol{z}(t)=\boldsymbol{z}_1(t)\ltimes\boldsymbol{z}_2(t)$ with $\boldsymbol{x}_1(t),\boldsymbol{x}_2(t),\boldsymbol{x}_3(t),\boldsymbol{u}_1(t),\boldsymbol{u}_2(t),\boldsymbol{z}_1(t),\boldsymbol{z}_2(t),\boldsymbol{v}(t)\in\Delta_2$. Then by Lemma 1, the algebraic forms of Eqs. (19) and (20) can be obtained:

$$\Sigma:\begin{cases}\boldsymbol{x}(t+1)=\boldsymbol{L}_1\ltimes\boldsymbol{u}(t)\ltimes\boldsymbol{x}(t),\ P=\alpha,\\ \boldsymbol{x}(t+1)=\boldsymbol{L}_2\ltimes\boldsymbol{u}(t)\ltimes\boldsymbol{x}(t),\ P=1-\alpha,\end{cases}\tag{21}$$

where $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are structure matrices chosen with a given probability distribution.

$$\boldsymbol{L}_1=\boldsymbol{\delta}_8[1,1,1,5,3,3,3,7,1,1,1,5,3,3,3,7,\\ 3,3,3,7,4,4,4,8,4,4,4,8,4,4,4,8],$$

and

$$\boldsymbol{L}_2=\boldsymbol{\delta}_8[1,2,1,6,3,4,3,8,1,2,1,6,3,4,3,8,\\ 3,4,3,8,3,4,3,8,3,4,3,8,3,4,3,8].$$

With regard to Eq. (20), its algebraic representation is

$$\widetilde{\Sigma}:\begin{cases}\boldsymbol{z}(t+1)=\widetilde{\boldsymbol{L}}_1\ltimes\boldsymbol{v}(t)\ltimes\boldsymbol{z}(t),\ P=\beta,\\ \boldsymbol{z}(t+1)=\widetilde{\boldsymbol{L}}_2\ltimes\boldsymbol{v}(t)\ltimes\boldsymbol{z}(t),\ P=1-\beta,\end{cases}\tag{22}$$

where $\widetilde{\boldsymbol{L}}_1=\boldsymbol{\delta}_4[3,2,4,2,1,2,2,2]$, $\widetilde{\boldsymbol{L}}_2=\boldsymbol{\delta}_4[3,2,3,3,2,2,2,2]$.

Since the control systems of Σ and $\widetilde{\Sigma}$ are state feedback control systems with the current state variables being the input, as previously discussed, these control systems can also be described in their algebraic forms as in Eqs. (4) and (5).

Let $\boldsymbol{H}$ and $\widetilde{\boldsymbol{H}}$ be the structure matrices of Eqs. (4) and (5), respectively. Suppose that $\boldsymbol{H}=\boldsymbol{\delta}_4[1,4,1,4,4,4,4,3]$ and $\widetilde{\boldsymbol{H}}=\boldsymbol{\delta}_2[1,1,2,1]$. Thus, Eqs. (21) and (22) can be further converted to

$$\Sigma:\begin{cases}\boldsymbol{x}(t+1)=\boldsymbol{L}_1\boldsymbol{H}\boldsymbol{\Phi}_3\boldsymbol{x}(t)=\boldsymbol{F}_1\boldsymbol{x}(t),\ P=\alpha,\\ \boldsymbol{x}(t+1)=\boldsymbol{L}_2\boldsymbol{H}\boldsymbol{\Phi}_3\boldsymbol{x}(t)=\boldsymbol{F}_2\boldsymbol{x}(t),\ P=1-\alpha.\end{cases}$$

$$\widetilde{\Sigma}:\begin{cases}\boldsymbol{z}(t+1)=\widetilde{\boldsymbol{L}}_1\widetilde{\boldsymbol{H}}\boldsymbol{\Phi}_2\boldsymbol{z}(t)=\widetilde{\boldsymbol{F}}_1\boldsymbol{z}(t),\ P=\beta,\\ \boldsymbol{z}(t+1)=\widetilde{\boldsymbol{L}}_2\widetilde{\boldsymbol{H}}\boldsymbol{\Phi}_2\boldsymbol{z}(t)=\widetilde{\boldsymbol{F}}_2\boldsymbol{z}(t),\ P=1-\beta.\end{cases}$$

According to Eqs. (9) and (10), the two PBCNs' skeleton matrices, denoted by $\boldsymbol{Q}_F$ and $\boldsymbol{Q}_{\widetilde{F}}$, can be calculated:

$$\boldsymbol{Q}_F=[\boldsymbol{\delta}_8^1,\ \boldsymbol{\delta}_8^4,\ \boldsymbol{\delta}_8^1,\ \boldsymbol{\delta}_8^8,\ \boldsymbol{\delta}_8^3+\boldsymbol{\delta}_8^4,\ \boldsymbol{\delta}_8^4,\ \boldsymbol{\delta}_8^3+\boldsymbol{\delta}_8^4,\ \boldsymbol{\delta}_8^8],\\ \boldsymbol{Q}_{\widetilde{F}}=[\boldsymbol{\delta}_4^3,\ \boldsymbol{\delta}_4^2,\ \boldsymbol{\delta}_4^2,\ \boldsymbol{\delta}_4^2+\boldsymbol{\delta}_4^3].$$

Based on the skeleton matrices, we depict their state transition graphs as shown in Figs. 1 and 2. Note that numbers on these arrows are the corresponding transition probabilities. Let $\boldsymbol{C}=\boldsymbol{\delta}_4[2,1,2,3,4,1,4,2]$. By Eq. (6), relation R is

$$R=\big\{(\boldsymbol{\delta}_8^1,\boldsymbol{\delta}_4^2),(\boldsymbol{\delta}_8^2,\boldsymbol{\delta}_4^1),(\boldsymbol{\delta}_8^3,\boldsymbol{\delta}_4^2),(\boldsymbol{\delta}_8^4,\boldsymbol{\delta}_4^3),(\boldsymbol{\delta}_8^5,\boldsymbol{\delta}_4^4),\\ (\boldsymbol{\delta}_8^6,\boldsymbol{\delta}_4^1),(\boldsymbol{\delta}_8^7,\boldsymbol{\delta}_4^4),(\boldsymbol{\delta}_8^8,\boldsymbol{\delta}_4^2)\big\}.$$

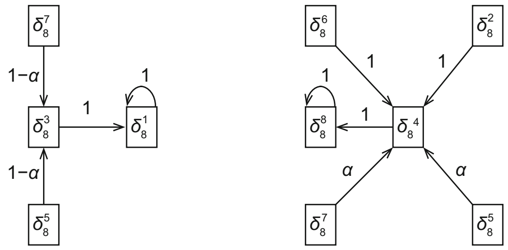

Fig. 1 State transitions of PBCN $\boldsymbol{\Sigma}$

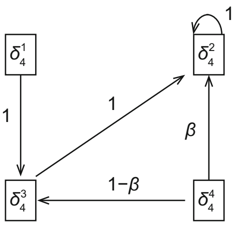

Fig. 2 State transitions of PBCN $\widetilde{\boldsymbol{\Sigma}}$

We then have

$$\boldsymbol{C} \odot \boldsymbol{Q}_F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \boldsymbol{Q}_{\widetilde{F}} \odot \boldsymbol{C}.$$

By Theorem 1, R should be a bisimulation relation between Σ and $\widetilde{\Sigma}$. According to Figs. 1 and 2, it is easy to verify that R is a bisimulation between Σ and $\widetilde{\Sigma}$. Taking $(\boldsymbol{\delta}_8^1, \boldsymbol{\delta}_4^2)$ for instance, there is a single one-step transition from $\boldsymbol{\delta}_8^1$ to $\boldsymbol{\delta}_8^1$; i.e., as for any $\boldsymbol{L}_j$ $(j \in \{1,2\})$, $P\{\boldsymbol{x}(1) = \boldsymbol{\delta}_8^1 | \boldsymbol{x}(0) = \boldsymbol{\delta}_8^1, \boldsymbol{u}(0) = \boldsymbol{H} \ltimes \boldsymbol{\delta}_8^1\} = 1$. Obviously, there exists a unique one-step transition from $\boldsymbol{\delta}_4^2$, such that $P\{\boldsymbol{z}(1) = \boldsymbol{\delta}_4^2 | \boldsymbol{z}(0) = \boldsymbol{\delta}_4^2, \boldsymbol{v}(0) = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{\delta}_4^2\} = 1$. Then for any $\boldsymbol{F}_j$, we have $(\boldsymbol{F}_j \ltimes \boldsymbol{H} \ltimes \boldsymbol{\delta}_8^1, \widetilde{\boldsymbol{F}}_i \ltimes \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{\delta}_4^2) = (\boldsymbol{\delta}_8^1, \boldsymbol{\delta}_4^2) \in R$. A similar pattern can be seen in other pairs belonging to R, which implies that R is a simulation relation of Σ by $\widetilde{\Sigma}$. Similarly, it can be certified that R^{-1} is a simulation relation of $\widetilde{\Sigma}$ by Σ as well.

4 Propagation of stabilization with probability one

After obtaining the necessary and sufficient conditions for a relation R to be a bisimulation relation, we may want to go further and use the property of bisimulation relation to propagate some features of one network to another. In this section, we will explore whether two bisimilar PBCNs exhibit a uniform stabilization property and whether they can be stabilizable with probability one.

Definition 5 Consider Σ as in Eq. (2) with the control system as in Eq. (4). Σ is stabilizable to $\boldsymbol{x}^* \in \Delta_N$ with probability one if for every $\boldsymbol{x} \in \Delta_N$ and a positive integer K, every possible t-step transition from $\boldsymbol{x}$ can reach the state $\boldsymbol{x}^*$. Expressed in a mathematical way, the condition below must be satisfied:

$$\sum_{\boldsymbol{x}(1),\boldsymbol{x}(2),\ldots,\boldsymbol{x}(t-1)} P\big\{\boldsymbol{x}(t) = \boldsymbol{x}^* | \boldsymbol{x}(0) = \boldsymbol{x}, \boldsymbol{u}_0 = \boldsymbol{H} \ltimes \boldsymbol{x}, \boldsymbol{u}_1 = \boldsymbol{H} \ltimes \boldsymbol{x}(1), \ldots, \boldsymbol{u}_{t-1} = \boldsymbol{H} \ltimes \boldsymbol{x}(t-1)\big\} = 1,$$

whenever $t \geq K$. $(\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{t-1})$ is the corresponding control sequence.

Definition 6 Consider Σ as in Eq. (2) with the control system as in Eq. (4). Let $\mathcal{W}$ be a subset of Δ_N. Σ is stabilizable to $\mathcal{W}$ with probability one if for every $\boldsymbol{x} \in \Delta_N$ and a positive integer K, every t-step transition from $\boldsymbol{x}$ will fall in the state set $\mathcal{W}$. More formally,

$$\sum_{\boldsymbol{x}(1),\boldsymbol{x}(2),\ldots,\boldsymbol{x}(t-1)} P\big\{\boldsymbol{x}(t) \in \mathcal{W} \, | \boldsymbol{x}(0) = \boldsymbol{x}, \boldsymbol{u}_0 = \boldsymbol{H} \ltimes \boldsymbol{x}, \boldsymbol{u}_1 = \boldsymbol{H} \ltimes \boldsymbol{x}(1), \ldots, \boldsymbol{u}_{t-1} = \boldsymbol{H} \ltimes \boldsymbol{x}(t-1)\big\} = 1,$$

for $t \geq K$. $(\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{t-1})$ is the control sequence.

Theorem 2 Given PBCNs Σ and $\widetilde{\Sigma}$ as in Eqs. (2) and (3) with their state feedback control systems as given in Eqs. (4) and (5), respectively. Let $\boldsymbol{z}^* \in \Delta_{\widetilde{N}}$ and $\mathcal{W} = \{\boldsymbol{x} \in \Delta_N : \boldsymbol{C} \ltimes \boldsymbol{x} = \boldsymbol{z}^*\}$. Suppose that the relation R defined in Eq. (6) is a bisimulation relation between Σ and $\widetilde{\Sigma}$. $\boldsymbol{C} \in \mathcal{L}_{\widetilde{N} \times N}$ is a surjection. Then, Σ is stabilizable to $\mathcal{W}$ with probability one if and only if $\widetilde{\Sigma}$ is stabilizable to $\boldsymbol{z}^*$ with probability one.

Proof Necessity ($\Longrightarrow$): Suppose that Σ is stabilizable to $\mathcal{W}$ with probability one. Let $\boldsymbol{z} \in \Delta_{\widetilde{N}}$. Then there exists $\boldsymbol{x} \in \Delta_N$ such that $(\boldsymbol{x}, \boldsymbol{z}) \in R$ since $\boldsymbol{C}$ is a surjection. As Σ is stabilizable to $\mathcal{W}$ with probability one, we can find a positive integer K such that

$$\sum_{\boldsymbol{x}(1),\boldsymbol{x}(2),\ldots,\boldsymbol{x}(t-1)} P\big\{\boldsymbol{x}(t) \in \mathcal{W} | \boldsymbol{x}(0) = \boldsymbol{x}, \boldsymbol{u}_0 = \boldsymbol{H} \ltimes \boldsymbol{x}, \boldsymbol{u}_1 = \boldsymbol{H} \ltimes \boldsymbol{x}(1), \ldots, \boldsymbol{u}_{t-1} = \boldsymbol{H} \ltimes \boldsymbol{x}(t-1)\big\} = 1, \quad (23)$$

for all $t \geq K$. Since R is a simulation relation, there exists a t-step transition from $\boldsymbol{z}$ to $\boldsymbol{z}(t)$, i.e.,

$$P\big\{\boldsymbol{z}(t) | \boldsymbol{z}(0) = \boldsymbol{z}, \boldsymbol{v}_0 = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}, \boldsymbol{v}_1 = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}(1), \ldots, \boldsymbol{v}_{t-1} = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}(t-1)\big\} > 0,$$

where $(\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{t-1})$ is a control sequence in accordance with the trajectory and satisfies $\big(\boldsymbol{x}(t), \boldsymbol{z}(t)\big) \in R$.

It follows from Eq. (6) that $\boldsymbol{z}(t) = \boldsymbol{C} \ltimes \boldsymbol{x}(t) = \boldsymbol{z}^*$, as $\boldsymbol{x}(t) \in \mathcal{W}$. Then, we have to show that for all possible t-step transitions from $\boldsymbol{z}$,

$$\sum_{\boldsymbol{z}(1),\boldsymbol{z}(2),\ldots,\boldsymbol{z}(t-1)} P\big\{\boldsymbol{z}(t) = \boldsymbol{z}^* | \boldsymbol{z}(0) = \boldsymbol{z}, \boldsymbol{v}_0 = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}, \boldsymbol{v}_1 = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}(1), \ldots, \boldsymbol{v}_{t-1} = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}(t-1)\big\} = 1.$$

If $\sum_{\boldsymbol{z}(1),\boldsymbol{z}(2),\ldots,\boldsymbol{z}(t-1)} P\big\{\boldsymbol{z}(t) = \boldsymbol{z}^* | \boldsymbol{z}(0) = \boldsymbol{z}, \boldsymbol{v}_0 = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}, \boldsymbol{v}_1 = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}(1), \ldots, \boldsymbol{v}_{t-1} = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}(t-1)\big\} \neq 1$, there must be some trajectory from $\boldsymbol{z}$ to $\boldsymbol{z}'$, e.g., $\boldsymbol{z}' \in \Delta_{\widetilde{N}}$ with $P\big\{\boldsymbol{z}(t) = \boldsymbol{z}' | \boldsymbol{z}(0) = \boldsymbol{z}, \boldsymbol{v}_0 = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}, \boldsymbol{v}_1 = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}(1), \ldots, \boldsymbol{v}_{t-1} = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}(t-1)\big\} > 0$. Since R^{-1} is a simulation relation of $\widetilde{\Sigma}$ by Σ and $(\boldsymbol{z}, \boldsymbol{x}) \in R^{-1}$, it follows that there is a t-step transition from $\boldsymbol{x}$ to $\boldsymbol{x}'$ so that $(\boldsymbol{z}', \boldsymbol{x}') \in R^{-1}$ with $P\big\{\boldsymbol{x}(t) = \boldsymbol{x}' | \boldsymbol{x}(0) = \boldsymbol{x}, u_0 = \boldsymbol{H} \ltimes \boldsymbol{x}, \boldsymbol{u}_1 = \boldsymbol{H} \ltimes \boldsymbol{x}(1), \ldots, \boldsymbol{u}_{t-1} = \boldsymbol{H} \ltimes \boldsymbol{x}(t-1)\big\} > 0$. However, $\boldsymbol{x}' \notin \mathcal{W}$, since $\boldsymbol{z}' \neq \boldsymbol{z}^*$ is in contradiction with Eq. (23). Thus, the relation

$$\sum_{\boldsymbol{z}(1),\boldsymbol{z}(2),\ldots,\boldsymbol{z}(t-1)} P\big\{\boldsymbol{z}(t) = \boldsymbol{z}^* | \boldsymbol{z}(0) = \boldsymbol{z}, \boldsymbol{v}_0 = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}, \boldsymbol{v}_1 = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}(1), \ldots, \boldsymbol{v}_{t-1} = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}(t-1)\big\} = 1$$

has been proved.

Sufficiency ($\Longleftarrow$): Assume that $\widetilde{\Sigma}$ is stabilizable to $\boldsymbol{z}^*$ with probability one. Let $\boldsymbol{x} \in \Delta_N$ and $\boldsymbol{z} = \boldsymbol{C} \ltimes \boldsymbol{x}$. Then $(\boldsymbol{z}, \boldsymbol{x}) \in R^{-1}$. With regard to this $\boldsymbol{z}$, we can find a positive integer K such that

$$\sum_{\boldsymbol{z}(1),\boldsymbol{z}(2),\ldots,\boldsymbol{z}(t-1)} P\big\{\boldsymbol{z}(t) = \boldsymbol{z}^* | \boldsymbol{z}(0) = \boldsymbol{z}, \boldsymbol{v}_0 = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}, \boldsymbol{v}_1 = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}(1), \ldots, \boldsymbol{v}_{t-1} = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}(t-1)\big\} = 1 \quad (24)$$

for $t \geq K$. Since R^{-1} is a simulation relation of $\widetilde{\Sigma}$ by Σ, Σ admits a t-step transition from $\boldsymbol{x}$ to $\boldsymbol{x}(t)$, satisfying $\big(\boldsymbol{z}^*, \boldsymbol{x}(t)\big) \in R^{-1}$. Then, we have $P\big\{\boldsymbol{x}(t) | \boldsymbol{x}(0) = \boldsymbol{x}, \boldsymbol{u}_0 = \boldsymbol{H} \ltimes \boldsymbol{x}, \boldsymbol{u}_1 = \boldsymbol{H} \ltimes \boldsymbol{x}(1), \ldots, \boldsymbol{u}_{t-1} = \boldsymbol{H} \ltimes \boldsymbol{x}(t-1)\big\} > 0$. According to Eq. (6), we have $\boldsymbol{C} \ltimes \boldsymbol{x}(t) = \boldsymbol{z}^*$, and thus $\boldsymbol{x}(t) \in \mathcal{W}$. Next, we must demonstrate that

$$\sum_{\boldsymbol{x}(1),\boldsymbol{x}(2),\ldots,\boldsymbol{x}(t-1)} P\big\{\boldsymbol{x}(t) \in \mathcal{W} | \boldsymbol{x}(0) = \boldsymbol{x}, \boldsymbol{u}_0 = \boldsymbol{H} \ltimes \boldsymbol{x}, \boldsymbol{u}_1 = \boldsymbol{H} \ltimes \boldsymbol{x}(1), \ldots, \boldsymbol{u}_{t-1} = \boldsymbol{H} \ltimes \boldsymbol{x}(t-1)\big\} = 1,$$

concerning all the t-step transitions from $\boldsymbol{x}$.

$\sum_{\boldsymbol{x}(1),\boldsymbol{x}(2),\ldots,\boldsymbol{x}(t-1)} P\big\{\boldsymbol{x}(t) \in \mathcal{W} | \boldsymbol{x}(0) = \boldsymbol{x}, u_0 = \boldsymbol{H} \ltimes \boldsymbol{x}, \boldsymbol{u}_1 = \boldsymbol{H} \ltimes \boldsymbol{x}(1), \ldots, \boldsymbol{u}_{t-1} = \boldsymbol{H} \ltimes \boldsymbol{x}(t-1)\big\} \neq 1$ indicates that there is a t-step transition from $\boldsymbol{x}$ to some $\boldsymbol{x}'$ ($\boldsymbol{x}' \notin \mathcal{W}$). Since R is a bisimulation relation, we have $(\boldsymbol{x}, \boldsymbol{z}) \in R$ with R being a simulation relation of Σ by $\widetilde{\Sigma}$. Thus, there is a t-step transition from $\boldsymbol{z}$ to some $\boldsymbol{z}'$, where $\boldsymbol{z}' \in \Delta_{\widetilde{N}}$ or $P\big\{\boldsymbol{z}(t) = \boldsymbol{z}' | \boldsymbol{z}(0) = \boldsymbol{z}, \boldsymbol{v}_0 = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}, \boldsymbol{v}_1 = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}(1), \ldots, \boldsymbol{v}_{t-1} = \widetilde{\boldsymbol{H}} \ltimes \boldsymbol{z}(t-1)\big\} > 0$ such that $(\boldsymbol{x}', \boldsymbol{z}') \in R$. As $\boldsymbol{x}' \notin \mathcal{W}$, then $\boldsymbol{z}' \neq \boldsymbol{z}^*$ contradicts Eq. (24).

Next, we set about considering the maximum number of steps needed by a PBCN to realize stabilization with probability one. Analogous to Cheng et al. (2010b), we redefine the transient period of a PBCN, which indicates the maximum number of requisite steps (all the states) for the whole network to achieve stabilization. In a similar manner, the transient period of a given state is the maximum number of steps needed to shift this state to stabilization. Then, we will test whether two bisimilar PBCNs are able to achieve synchronous stabilization.

Corollary 1 Given PBCNs Σ and $\widetilde{\Sigma}$ as in Eqs. (2) and (3) with their state feedback control systems as given in Eqs. (4) and (5), respectively. Let $\boldsymbol{z}^* \in \Delta_{\widetilde{N}}$ and $\mathcal{W} = \{\boldsymbol{x} \in \Delta_N : \boldsymbol{C} \ltimes \boldsymbol{x} = \boldsymbol{z}^*\}$ with $\boldsymbol{z}^*$ being the state to which $\widetilde{\Sigma}$ stabilizes. Suppose that the relation R defined in Eq. (6) is a bisimulation relation between Σ and $\widetilde{\Sigma}$. $\boldsymbol{C} \in \mathcal{L}_{\widetilde{N} \times N}$ is a surjection. Then, the transient periods of Σ and $\widetilde{\Sigma}$ are identical.

Proof For any $(\boldsymbol{x}, \boldsymbol{z}) \in R$, let the transient period of $\boldsymbol{x}$, denoted by $\mathcal{T}(\boldsymbol{x})$, be T_0. Then, it suffices to prove that the transient period of $\boldsymbol{z}$, denoted by $\mathcal{T}(\boldsymbol{z})$, equals T_0.

Assume that $\mathcal{T}(\boldsymbol{z}) = \widetilde{T_0}$. Since $(\boldsymbol{x}, \boldsymbol{z}) \in R$ and $\mathcal{T}(\boldsymbol{x}) = T_0$, this implies that there is a T_0-step transition from $\boldsymbol{x}$ to $\boldsymbol{x}(T_0)$ that belongs to the steady state set $\mathcal{W}$. Hence, there also exists a T_0-step transition from $\boldsymbol{z}$ to $\boldsymbol{z}(T_0)$ satisfying $\boldsymbol{C} \ltimes \boldsymbol{x}(T_0) = \boldsymbol{z}(T_0) = \boldsymbol{z}^*$. Thus, we can infer that $\mathcal{T}(\boldsymbol{z}) \leq T_0$.

Because R^{-1} is a simulation relation of $\widetilde{\Sigma}$ by Σ and $(\boldsymbol{z}, \boldsymbol{x}) \in R^{-1}$, for any $\widetilde{T_0}$-step transition from $\boldsymbol{z}$, denoted by $\boldsymbol{z}(\widetilde{T_0})$, we can find a $\widetilde{T_0}$-step transition from $\boldsymbol{x}$, denoted by $\boldsymbol{x}(\widetilde{T_0})$, such that $\big(\boldsymbol{z}(\widetilde{T_0}), \boldsymbol{x}(\widetilde{T_0})\big) \in R^{-1}$. Similarly, since $\boldsymbol{z}(\widetilde{T_0}) = \boldsymbol{z}^*$, we have $\boldsymbol{x}(\widetilde{T_0}) \in \mathcal{W}$. Thus, $\mathcal{T}(\boldsymbol{x}) = T_0 \leq \mathcal{T}(\boldsymbol{z}) = \widetilde{T_0}$. To conclude, $\mathcal{T}(\boldsymbol{x}) = \mathcal{T}(\boldsymbol{z})$.

Then we will give an illustration of the application of Theorem 2 and Corollary 1.

Example 2 Consider PBCNs and the logical matrix $\boldsymbol{C}$ as in Example 1. According to Fig. 2, it is not difficult to find out that $\widetilde{\Sigma}$ is stabilizable to $\boldsymbol{\delta}_4^2$ with probability one. For example, let $\boldsymbol{z}_0 = \boldsymbol{\delta}_4^4$ and $\boldsymbol{z}_0$ has a couple of two-step transitions from $\boldsymbol{\delta}_4^4$:

$$\begin{aligned}
&P\{\boldsymbol{z}(2)=\boldsymbol{\delta}_4^2|\boldsymbol{z}(0)=\boldsymbol{\delta}_4^4,\boldsymbol{v}(\tau)=\widetilde{\boldsymbol{H}}\ltimes\boldsymbol{z}(\tau),\ \tau=0,1\}\\
&=P\{\boldsymbol{z}(1)=\boldsymbol{\delta}_4^2|\boldsymbol{z}(0)=\boldsymbol{\delta}_4^4,\boldsymbol{v}(0)=\widetilde{\boldsymbol{H}}\ltimes\boldsymbol{\delta}_4^4\}\\
&\quad\cdot P\{\boldsymbol{z}(2)=\boldsymbol{\delta}_4^2|\boldsymbol{z}(1)=\boldsymbol{\delta}_4^2,\boldsymbol{v}(1)=\widetilde{\boldsymbol{H}}\ltimes\boldsymbol{\delta}_4^2\}\\
&\quad+P\{\boldsymbol{z}(1)=\boldsymbol{\delta}_4^3|\boldsymbol{z}(0)=\boldsymbol{\delta}_4^4,\boldsymbol{v}(0)=\widetilde{\boldsymbol{H}}\ltimes\boldsymbol{\delta}_4^4\}\\
&\quad\cdot P\{\boldsymbol{z}(2)=\boldsymbol{\delta}_4^2|\boldsymbol{z}(1)=\boldsymbol{\delta}_4^3,\boldsymbol{v}(1)=\widetilde{\boldsymbol{H}}\ltimes\boldsymbol{\delta}_4^3\}\\
&=\beta\times 1+(1-\beta)\times 1\\
&=1.
\end{aligned}$$

Since $\boldsymbol{\delta}_4^2$ stabilizes to itself with probability one, $\boldsymbol{\delta}_4^4$ is stabilizable to $\boldsymbol{\delta}_4^2$ with probability one after two steps. Similarly, we can verify that the remaining states of $\widetilde{\Sigma}$ are stabilizable to $\boldsymbol{\delta}_4^2$ with probability one. We list the probabilities $\big(P\{\boldsymbol{z}(T)=\boldsymbol{\delta}_4^2|\boldsymbol{z}(0)=\boldsymbol{z}_0,\boldsymbol{v}(\tau)=\widetilde{\boldsymbol{H}}\boldsymbol{z}(\tau)\}\big)$ of all the state transitions in $\widetilde{\Sigma}$ to $\boldsymbol{\delta}_4^2$ in Table 3, from which we can conclude that $\widetilde{\Sigma}$ reaches stabilization after two steps.

By Theorem 2, if $\widetilde{\Sigma}$ is stabilizable to $\boldsymbol{\delta}_4^2$, Σ should be stabilizable to a state set $\mathcal{W}=\{\boldsymbol{x}\in\Delta_8: \boldsymbol{C}\ltimes\boldsymbol{x}=\boldsymbol{\delta}_4^2\}=\{\boldsymbol{\delta}_8^1,\boldsymbol{\delta}_8^8\}$. It has been proved in Li R et al. (2014b) that Σ is stabilizable to $\{\boldsymbol{\delta}_8^1,\boldsymbol{\delta}_8^8\}$. Here, we give the probabilities $\big(P\{\boldsymbol{x}(T)\in\{\boldsymbol{\delta}_8^1,\boldsymbol{\delta}_8^8\}|\boldsymbol{x}(0)=\boldsymbol{x}_0,\boldsymbol{u}(\tau)=\boldsymbol{H}\boldsymbol{x}(\tau)\}\big)$ of all the state transitions in Σ to $\{\boldsymbol{\delta}_8^1,\boldsymbol{\delta}_8^8\}$ in Table 4.

Table 3 State transition probabilities of $\widetilde{\boldsymbol{\Sigma}}$ to $\boldsymbol{\delta}_4^2$

Initial state $\boldsymbol{z}_0$	State transition probability	
	$T=1$	$T=2$
$\boldsymbol{\delta}_4^1$	0	1
$\boldsymbol{\delta}_4^2$	1	1
$\boldsymbol{\delta}_4^3$	1	1
$\boldsymbol{\delta}_4^4$	β	1

Table 4 State transition probabilities of $\boldsymbol{\Sigma}$ to $\{\boldsymbol{\delta}_8^1,\boldsymbol{\delta}_8^8\}$

Initial state $\boldsymbol{x}_0$	State transition probability	
	$T=1$	$T=2$
$\boldsymbol{\delta}_8^1$	1	1
$\boldsymbol{\delta}_8^2$	0	1
$\boldsymbol{\delta}_8^3$	1	1
$\boldsymbol{\delta}_8^4$	1	1
$\boldsymbol{\delta}_8^5$	0	1
$\boldsymbol{\delta}_8^6$	0	1
$\boldsymbol{\delta}_8^7$	0	1
$\boldsymbol{\delta}_8^8$	1	1

Note that both Σ and $\widetilde{\Sigma}$ achieve stabilization after two steps, which implies that if two PBCNs are matched by a bisimulation relation, they not only can realize stabilization with probability one in company with each other, but also can be stabilizable simultaneously, which means that they share the same transient period. In addition, if $\widetilde{\Sigma}$ does not stabilize to a point, e.g., $\boldsymbol{\delta}_4^1$, Σ can never be stabilizable to the set $\{\boldsymbol{x}:\boldsymbol{C}\ltimes\boldsymbol{x}=\boldsymbol{\delta}_4^1\}=\{\boldsymbol{\delta}_8^2,\boldsymbol{\delta}_8^6\}$. This supports the converse proposition of Theorem 2.

So far, we have discussed the definitions of (bi)similar PBCNs with a state feedback controller. The necessary and sufficient conditions for two PBCNs to be bisimilar and the problem of propagating stabilization between two bisimilar PBCNs have been produced. In fact, when we consider the control input to PBCNs as a constant variable, a new kind of network can be obtained.

Definition 7 A PBN can be described mathematically as

$$\boldsymbol{x}(t+1)=\boldsymbol{L}_i\ltimes\boldsymbol{x}(t), t=0,1,\ldots,\ i=1,2,\ldots,r,$$

where $\boldsymbol{L}_i$ is the structure matrix selected from $\boldsymbol{L}_1,\boldsymbol{L}_2,\ldots,\boldsymbol{L}_r$ with probability $p_i>0$ at time t. The equation $\sum_{i=1}^r p_i=1$ must be satisfied. $\boldsymbol{x}(t)$ is the state variable at time t, which is an n-dimensional column vector that takes value in Δ_n. Consider two PBNs as

$$\Omega:\ \boldsymbol{x}(t+1)=\boldsymbol{L}_i\ltimes\boldsymbol{x}(t), \tag{25}$$

$$\widetilde{\Omega}:\ \boldsymbol{z}(t+1)=\widetilde{\boldsymbol{L}}_j\ltimes\boldsymbol{z}(t), \tag{26}$$

where $\boldsymbol{x}(t)\in\Delta_N$, $\boldsymbol{z}(t)\in\Delta_{\widetilde{N}}$, $\boldsymbol{L}_i\in\mathcal{L}_{N\times N}$, $\widetilde{\boldsymbol{L}}_j\in\mathcal{L}_{\widetilde{N}\times\widetilde{N}}$, $i=1,2,\ldots,r$, and $j=1,2,\ldots,k$. Then define their skeleton matrices as $\boldsymbol{F}=\boldsymbol{L}_1\vee\boldsymbol{L}_2\vee\ldots\vee\boldsymbol{L}_r$, and $\widetilde{\boldsymbol{F}}=\widetilde{\boldsymbol{L}}_1\vee\widetilde{\boldsymbol{L}}_2\vee\ldots\vee\widetilde{\boldsymbol{L}}_k$.

Since PBNs are special cases of PBCNs, the two theorems proposed above are also satisfied by PBN.

Corollary 2 Consider PBNs Ω and $\widetilde{\Omega}$ as in Eqs. (25) and (26), respectively. Let $\boldsymbol{F}$ and $\widetilde{\boldsymbol{F}}$ be the skeleton matrices of Ω and $\widetilde{\Omega}$, respectively. Let $\boldsymbol{C}\in\mathcal{L}_{\widetilde{N}\times N}$. Then, the relation R defined in Eq. (6) is a bisimulation relation between Ω and $\widetilde{\Omega}$ if and only if

$$\boldsymbol{C}\odot\boldsymbol{F}=\widetilde{\boldsymbol{F}}\odot\boldsymbol{C}.$$

Corollary 3 Consider PBNs Ω and $\widetilde{\Omega}$ as in Eqs. (25) and (26), respectively. $\boldsymbol{z}^*\in\Delta_{\widetilde{N}}$ and $\mathcal{W}=\{\boldsymbol{x}\in\Delta_N:\ \boldsymbol{C}\ltimes\boldsymbol{x}=\boldsymbol{z}^*\}$. Suppose that the relation R defined in Eq. (6) is a bisimulation relation between Ω and $\widetilde{\Omega}$. $\boldsymbol{C}\in\mathcal{L}_{\widetilde{N}\times N}$ is a surjection.

Then, Ω is stabilizable to $\sqsupseteq$ with probability one if and only if $\widetilde{\Omega}$ is stabilizable to $\boldsymbol{z}^*$ with probability one. In addition, they have the identical transient period.

5 Conclusions

In this study, PBCNs with state feedback controllers have been investigated. To begin with, a bisimulation relation of PBCNs has been proposed. As far as we know, this is the first time this has been done. After converting PBCNs to their algebraic forms using STP, the skeleton matrices, which can be understood as the state transient matrices, have been constructed, through which necessity and sufficiency have been observed to examine bisimulation relations. Moreover, the propagation of stabilization with probability one between bisimilar PBCNs comes to the center of our attention. We have proved our hypothesis that if two PBCNs are coupled with a bisimulation relation, then one can achieve stabilization to a state set with probability one if and only if the other can realize stabilization to a fixed point with probability one. Additionally, these two bisimilar PBCNs have been verified to possess the same transient period, which represents the maximum number of steps required to reach stabilization. To show the effectiveness of our results, a real-world example has been given. Finally, two other corollaries have been revealed to generalize our results to PBNs. However, testing the bisimulation relation cannot be accomplished in polynomial time. This cast a cloud on the results obtained. We hope the problem will be solved in future work.

Contributors

Chi HUANG formulated the research goals. Yao CHEN and Nan JIANG proposed the method and obtained the final theorems. Jürgen KURTHS and Nan JIANG built the model for verification. Nan JIANG wrote the original draft. Nan JIANG, Chi HUANG, Yao CHEN, and Jürgen KURTHS revised and edited the final version.

Compliance with ethics guidelines

Nan JIANG, Chi HUANG, Yao CHEN, and Jürgen KURTHS declare that they have no conflict of interest.

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