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Review:

A survey on complex dynamical networks with impulsive effects*

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Abstract: We review the research on complex dynamical networks (CDNs) with impulsive effects. We provide a comprehensive and intuitive overview of the fundamental results and recent progress of CDNs with impulsive effects, where impulsive effects are considered from two aspects, i.e., impulsive control and impulsive perturbation. Five aspects of CDNs with impulsive effects are surveyed, including synchronizing impulses, desynchronizing impulses, adaptive-impulsive synchronization, pinning impulsive synchronization, and CDNs with stochastic and impulsive effects. Finally, conclusions and some future research directions are briefly addressed.

Key words: Complex dynamical networks; Synchronizing impulses; Desynchronizing impulses; Pinning control; Time delay

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1 Introduction

Complex networks are found in many practical fields as diverse as neural networks, cellular and metabolic networks, biological networks, computer networks, the World Wide Web, co-authorship and citation networks, and social networks (Huberman and Adamic, 1999; Buchanan, 2002). In the past decades, there has been much research on complex networks such as applications of complex networks, modeling and statistical analysis, big-data diagnosis in complex networks, dynamics and structures of complex dynamical networks (CDNs), and control and synchronization of CDNs.

Synchronization, as an interesting collective behavior of CDNs, has been extensively studied in various fields (Hong et al., 2002; Wang XF and Chen, 2002b; Tang et al., 2014). In the earlier years, most research focused on synchronization of continuous-time networks without delays. In practice, time delay may appear in the state of the system, control input, or the measurement (Guan XP et al., 2007; Wang HL and Chen, 2015; Lv et al., 2018b; Yang D et al., 2018; Yang XY and Li, 2018), and it can sometimes induce instability and poor performance. Because signal traveling speeds are limited, time delay is a common phenomenon in networks, and it is important to study the synchronization of networks with time delay. Numerous sufficient conditions for synchronization of various kinds of CDNs have been obtained, such as time-varying coupling CDNs, directed and weighted couplings, and switching couplings. The relationship between network structure and the ability of network synchronization has also been studied.

It is costly to control every node in a large and complex network. A pinning control strategy was introduced to cope with this situation (Wang XF

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and Chen, 2002a; Li X et al., 2004), where just a small number of nodes were directly controlled. This method can reduce the number of controllers, and reduce the communication frequency between nodes. To improve efficiency and save resources, some methods have been combined with pinning control strategies to ensure global synchronization, such as fuzzy control, finite-time control, adaptive control, event-triggered control, and impulsive control (Zhang XY et al., 2017b, 2018; Xing et al., 2019; Huang et al., 2020).

In real applications, a system (such as signal processing systems, computer networks, automatic control systems, and telecommunications) may be affected by impulsive effects due to instantaneous perturbations at instant moments. As a kind of discontinuous control, impulsive control is often seen as an efficient and economic method for networks. Then the studies of CDNs with impulsive effects are very important for understanding the dynamics of most complex real-world networks. In the past decades, impulsive theory has been studied deeply and results of many impulsive differential systems have been derived (Khadra et al., 2009; Chang et al., 2010; Li XD and Bohner, 2012; Amato et al., 2013; Li XD et al., 2015b; Xu F et al., 2015; Zhang XY et al., 2017a). Meanwhile, the synchronization problem of CDNs can be widely solved by impulsive control methods (Chen YS et al., 2010; Wu B et al., 2012; Chen J et al., 2013; Li XD et al., 2015a). The stability and ability to synchronize CDNs can be guaranteed by designing small impulses at some impulsive instants, which can increase the robustness and reduce information redundancy in the processing of transmitted signals. Some results on the exponential synchronization of discrete-time impulsive networks with time-varying delays were presented (Li XD et al., 2015a), involving an average impulsive interval. With the development of impulse research, adaptive-impulsive synchronization of systems and uncertain general CDNs were derived (Li K and Lai, 2008; Chen YS et al., 2010).

As far as we know, networks are not isolated, but rather inter-dependent or inter-connected. In many scenarios, a few sets of nodes interact within their own layers and across different layers, where each layer represents a different edge type, and the connectivity pattern on each layer can be unique. Similarly, the interactions of predators differ entirely from that of prey; that is, the topology structures of the two communities are different. Therefore, synchronization between two different complex networks is a more practical and significant problem (Um et al., 2011). Synchronization in multi-layer networks has been extensively investigated in the past few years (Rakshit et al., 2017; Mei et al., 2018; Wei et al., 2018). There are three typical types of synchronization in multi-layer networks, that is, complete synchronization, intra-layer synchronization, and inter-layer synchronization.

The remainder of the paper reviews the main results concerning CDNs with impulse effects.

2 Preliminaries

Several notations used throughout the paper are listed in Table 1. For any interval $J \subseteq \mathbb{R}$, set $S \subseteq \mathbb{R}^k$ $(1 \le k \le n)$, $C(J,S) = \{\boldsymbol{\varphi} : J \to S$ is continuous$\}$, and $C^1(J,S) = \{\boldsymbol{\varphi} : J \to S$ is continuously differentiable$\}$. $\Lambda = \{1,2,\ldots,N\}$, where N denotes the number of nodes in a network. Function $\alpha : [0,\infty) \to [0,\infty)$ is of class $\mathcal{K}$ if α is continuous, strictly increasing, and $\alpha(0)=0$. In addition, if α is unbounded, it is of class $\mathcal{K}_\infty$. Function

Table 1 Notations

Notation	Description
$\mathbb{R}$	Set of real numbers
$\mathbb{R}_+$	Set of positive numbers
$\mathbb{R}^n$	n-dimensional real spaces equipped with the Euclidean norm $\|\cdot\|$
$\mathbb{R}^{n\times m}$	$n \times m$-dimensional real spaces
$\mathbb{Z}_+$	Set of positive integers
$\mathbb{N}$	Set of nonnegative integers
$\boldsymbol{M} > 0$	Matrix $\boldsymbol{M}$ is symmetric and positive definite
$\boldsymbol{M} < 0$	Matrix $\boldsymbol{M}$ is symmetric and negative definite
$\boldsymbol{M} > \boldsymbol{N}$	$\boldsymbol{M} - \boldsymbol{N}$ is positive definite for symmetric matrices $\boldsymbol{M}$ and $\boldsymbol{N}$
$\lambda_{\max}(\boldsymbol{M})$	The maximum eigenvalue of matrix $\boldsymbol{M}$
$\lambda_{\min}(\boldsymbol{M})$	The minimum eigenvalue of matrix $\boldsymbol{M}$
$\boldsymbol{I}$	Identity matrix with appropriate dimensions
$\star$	Symmetric block in a symmetric matrix
$\overline{\mathcal{A}}$	Closure of set $\mathcal{A}$
$N(t_0,t)$	Number of impulse times in interval $[t_0,t)$
$x(t^+) = \lim\limits_{s\to t^+} x(s)$	Right-hand limit
$x(t^-) = \lim\limits_{s\to t^-} x(s)$	Left-hand limit

$\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is of class $\mathcal{KL}$ if $\beta(\cdot,t)$ is of class $\mathcal{K}$ for each fixed $t \ge 0$ and $\beta(r,t)$ decreases to 0 as $t \to \infty$ for each fixed $r \ge 0$. For any interval $J \subseteq \mathbb{R}$, set $V \subseteq \mathbb{R}^k$ $(1 \le k \le n)$, $\mathrm{PC}(J,V) = \{\boldsymbol{\varphi} : J \to V$ is continuous everywhere except at a finite number of points t, at which $\boldsymbol{\varphi}(t^+)$, $\boldsymbol{\varphi}(t^-)$ exist and $\boldsymbol{\varphi}(t^+) = \boldsymbol{\varphi}(t)\}$. For given $\tau > 0$, set $C_\tau = C([-\tau,0],\mathbb{R}^n)$, $\mathrm{PC}_\tau = \mathrm{PC}([-\tau,0],\ \mathbb{R}^n)$ with norm $||\cdot||_\tau$ defined by $||\boldsymbol{\varphi}||_\tau = \sup\{||\boldsymbol{\varphi}(s)|| : s \in [-\tau,0]\}$, and $\mathrm{PCB}(J,V) = \{\boldsymbol{\varphi} \in \mathrm{PC}(J,V) : \boldsymbol{\varphi}$ is bounded$\}$. In particular, let $\mathbb{PCB}_+ = \mathrm{PCB}([0,+\infty),\mathbb{R})$, $\mathbb{PCB}_- = \mathrm{PCB}((-\infty,0],\mathbb{R})$, and for each $\boldsymbol{\varphi} \in \mathbb{PCB}_+(\mathbb{PCB}_-)$, the norm is defined by $||\boldsymbol{\varphi}||_B = \sup_{s\ge 0(s\le 0)} |\boldsymbol{\varphi}(s)|$.

2.1 Complex dynamical networks

Numerous inter-connected nodes can compose a CDN, where each node is a dynamical system. The linearly coupled ordinary differential equations of CDNs can be described as

$$\dot{\boldsymbol{x}}_i(t) = \boldsymbol{f}_i(\boldsymbol{x}_i(t),t) + \alpha \sum_{j=1}^{N} a_{ij}\boldsymbol{\Gamma}\boldsymbol{x}_j(t),\ i \in \Lambda, \qquad (1)$$

where $\boldsymbol{x}_i(t) \in \mathbb{R}^n$ are the state vectors of the i^{th} node, $\alpha > 0$ is the coupling strength, $\boldsymbol{\Gamma}$ is the inner coupling matrix, and functions $\boldsymbol{f}_i : \mathbb{R}^n \to \mathbb{R}^n$ are continuously differentiable. Specifically, $\boldsymbol{f}_i$ can be chosen as $\boldsymbol{f}_1 = \boldsymbol{f}_2 = \ldots = \boldsymbol{f}_N = \boldsymbol{f}$, that is, the CDNs with identical dynamics of nodes. $\boldsymbol{A} = (a_{ij}) \in \mathbb{R}^{N\times N}$ is the coupling configuration matrix representing the topology structures. The elements of $\boldsymbol{A}$ are defined as follows: $a_{ij} = a_{ji} = 1$ if nodes i and j are connected for $i \ne j$; otherwise, $a_{ij} = a_{ji} = 0$. It is assumed that $\sum_{j=1}^{N} a_{ij} = 0$ for $i \in \Lambda$. Fig. 1 displays a CDN with 20 nodes. There is an edge between nodes 1 and 2 if $a_{12} = a_{21} = 1$.

During the last several decades, control of CDNs has attracted much interest, and the equations can be shown by

$$\begin{aligned}\dot{\boldsymbol{x}}_i(t) &= \boldsymbol{f}_i(\boldsymbol{x}_i(t),t) + \alpha \sum_{j=1}^{N} a_{ij}\boldsymbol{\Gamma}\boldsymbol{x}_j(t) \\ &\quad +\boldsymbol{u}_i(t),\ i \in \Lambda,\end{aligned} \qquad (2)$$

where $\boldsymbol{u}_i(t)$ is the control input. Many control methods, such as impulsive control, continuous feedback control, adaptive control, and pinning control, are based on different control inputs. Impulsive control, as a discontinuous input in the system, is regarded as an effective control method, and has been widely applied to synchronize CDNs.

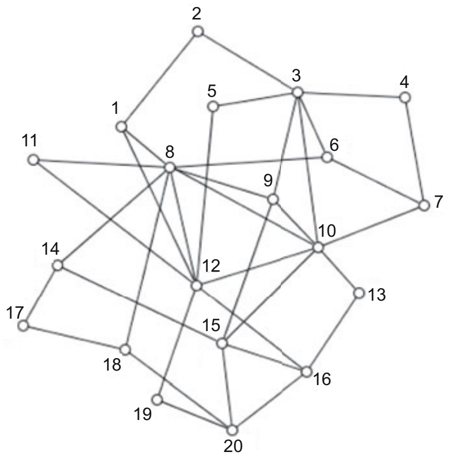

Fig. 1 A complex dynamical network (CDN) with 20 nodes

Viewing system (1) as a drive network, the following can be viewed as a response network:

$$\dot{\boldsymbol{y}}_i(t) = \boldsymbol{f}_i(\boldsymbol{y}_i(t),t) + \alpha \sum_{j=1}^{N} a_{ij}\boldsymbol{\Gamma}\boldsymbol{y}_j(t) + \boldsymbol{u}_i(t), \quad (3)$$

where $\boldsymbol{y}_i(t) = (y_{i1}(t), y_{i2}(t), \ldots, y_{in}(t))^{\mathrm{T}} \in \mathbb{R}^n, i \in \Lambda$.

Synchronization is a typical and important collective dynamical behavior of CDNs. Much effort has been made to study the subject (Lv et al., 2018a). Below we review some definitions of synchronization of CDNs.

Definition 1 Complex dynamical networks (1) are said to achieve synchronization if $\boldsymbol{x}_1(t) = \boldsymbol{x}_2(t) = \ldots = \boldsymbol{x}_N(t) = \boldsymbol{s}(t)$ as $t \to \infty$, where $\boldsymbol{s}(t)$ is the state variable of equations for an isolated node of CDNs, i.e., $\dot{\boldsymbol{s}} = \boldsymbol{f}(\boldsymbol{s}(t),t)$ for all $t \in \mathbb{R}_+$.

Definition 2 Complex dynamical networks (1) are synchronized to the average state of CDNs if $\boldsymbol{x}_1(t) = \boldsymbol{x}_2(t) = \ldots = \boldsymbol{x}_N(t) = \boldsymbol{s}(t)$ as $t \to \infty$, where $\boldsymbol{s} = \dfrac{1}{N}\sum_{i=1}^{N}\boldsymbol{x}_i$.

Definition 3 The drive CDNs (1) and response CDNs (3) are said to achieve drive-response synchronization, if $\lim\limits_{t\to\infty} \|\boldsymbol{y}_i(t) - \boldsymbol{x}_i(t)\| = 0,\ i \in \Lambda$.

Most results on synchronization of CDNs are based on the above definitions.

2.2 Impulsive systems

The concept of impulsive system and its mathematical theories are called impulsive differential equations, or differential equations with impulse effects (Guan ZH et al., 2005; Zhang Y et al., 2009;

Wang L and Li, 2013; Li XD et al., 2015c, 2015d). Generally speaking, an impulsive system consists of three elements:

1. a continuous-time differential equation, which governs the motion of the system between resetting events:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t,\ \boldsymbol{x}),\ (t,\ \boldsymbol{x}) \notin \varSigma;$$

2. a difference equation (jumping operator), which governs the way that the states are instantaneously changed:

$$\Delta\boldsymbol{x} = \boldsymbol{x}(t) - \boldsymbol{x}(t^-) = J\boldsymbol{x}(t^-),\ (t,\ \boldsymbol{x}) \in \varSigma;$$

3. a criterion (switching set) that determines when the states of the system are to be reset $\varSigma \subseteq \mathbb{R}_+ \times \varOmega$ ($\varOmega$ is called the state space).

Consider the following general impulsive functional differential equation:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t,\ \boldsymbol{x}_t),\ t > t_0,\ t \neq t_k, \\ \Delta\boldsymbol{x} = \boldsymbol{I}_k(t,\ \boldsymbol{x}(t^-)),\ t = t_k,\ k \in \mathbb{N}, \\ \boldsymbol{x}_{t_0} = \boldsymbol{\phi}, \end{cases} \tag{4}$$

where $\boldsymbol{x}_t \in \mathrm{PC}_\tau$ is defined as $\boldsymbol{x}_t(s) = \boldsymbol{x}(t+s), s \in [-\tau, 0]$, $\boldsymbol{f},\ \boldsymbol{I}_k : \mathbb{R}_+ \times \mathrm{PC}_\tau \mapsto \mathbb{R}^n$, $\boldsymbol{\phi} \in \mathrm{PC}_\tau$, $\Delta\boldsymbol{x} = \boldsymbol{x}(t_k) - \boldsymbol{x}(t_k^-)$. The impulse times $\{t_k\}$ satisfy $0 \leq t_0 < \cdots < t_k < \cdots,\ \lim\limits_{k\to\infty} t_k = \infty$.

Definition 4 Assume that $\boldsymbol{x}(t)$ is the solution of system (4). Then the trivial solution of system (4) is said to be the following:

(A1) stable, if for any $\sigma \geq t_0$ and $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that any $\boldsymbol{x}_0 : |\boldsymbol{x}_0| < \delta(\varepsilon, t_0)$ implies $|\boldsymbol{x}(t, t_0, x_0)| < \varepsilon$, for all $t > t_0$;

(A2) uniformly stable, if the δ in (A1) is independent of t_0;

(A3) uniformly asymptotically stable, if (A2) holds and for any $t_0 \geq 0$ and $\varepsilon > 0$, there exist $\sigma(t_0) > 0$ and $T(\varepsilon, \boldsymbol{x}_0) > 0$ such that $|\boldsymbol{x}_0| < \delta$ implies that $|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)| < \varepsilon$, for all $t > t_0 + T$, i.e., $\lim_{t\to+\infty} \boldsymbol{x}(t, t_0, \boldsymbol{x}_0) = 0$;

(A4) globally stable (globally asymptotically stable), if (A1) holds and for any δ there exist $\lambda > 0$ and $k(\delta) > 0$, when $|\boldsymbol{x}_0| < \delta$ we have $|\boldsymbol{x}(t, t_0, x_0)| < k(\delta)\mathrm{e}^{-\lambda(t-t_0)},\ t > t_0$.

Definition 5 (Yang T, 2001) Function $V : [T_0 - \tau, \infty) \times \mathbb{R}^n \mapsto \mathbb{R}_+$ belongs to class v_0 if:

(1) for set $[t_{k-1}, t_k) \times \mathbb{R}_+$, V is continuous and $\lim_{(t,u)\to(t_k^-,v)} V(t,u) = V(t_k^-, v)$;

(2) for $t \geq t_0$, $V(t,x)$ is locally Lipschitz with x and $V(t,0) \equiv 0$.

Definition 6 (Yang T, 2001) Let $V \in v_0$. The upper right-hand derivative of V with respect to system (4) is defined by

$$\begin{aligned} D^+V(t, \psi(0)) = \lim_{h\to0^+} \sup \frac{1}{h}[V(t+h, \boldsymbol{\psi}(0) \\ + hf(t, \boldsymbol{\psi})) - V(t, \boldsymbol{\psi}(0))], \end{aligned}$$

for $(t, \boldsymbol{\psi}) \in [t_{k-1}, t_k) \times \mathrm{PC}_\tau$.

Definition 7 (Yang XS et al., 2011b) An impulsive sequence $\{t_k, k \in \mathbb{N}_+\}$ is said to have an average impulsive interval T_a if there exist a positive integer ζ and a positive constant T_a such that

$$\frac{T-t}{T_\mathrm{a}} - \zeta \leq N(T,t) \leq \frac{T-t}{T_\mathrm{a}} + \zeta,\ \forall T \geq t \geq 0, \tag{5}$$

where $N(T,t)$ denotes the number of impulsive occurrences of the impulsive sequence $\{t_k, k \in \mathbb{N}_+\}$ on interval (t,T).

Meanwhile, impulsive control, as an effective control method, is based on impulsive systems. Generally, impulsive control systems (ICSs) can be classified into three types according to impulsive effects and impulsive instances (Yang XY et al., 2018). Up to now, the existing results are divided into two aspects, i.e., impulsive control and impulsive perturbation (Guan ZH and Chen, 1999; Guan ZH and Zhang, 2008; Li XD, 2012; Li XD et al., 2013; Lin et al., 2013; Liu J and Li, 2013). For instance, if a given system is stable without impulsive effects, then it can still be stable even with impulsive interference, which is viewed as an impulsive perturbation issue (Li XD et al., 2018). The following is an example to illustrate the fact that impulsive perturbation may affect the dynamical behaviors of CDNs:

Example 1 Consider the following two-dimensional (2D) Hopfield neural networks:

$$\begin{cases} \dot{x}_1(t) = 0.01\sin tx_1(t) + 0.2\cos t\tanh(x_1(t)) \\ \qquad + \tanh(x_1(t)) - \tanh(x_2(t)) - 0.3x_1(t) \\ \qquad +0.5\sin t\tanh(x_2(t)) + \cos t, \\ \dot{x}_2(t) = 0.02\cos tx_2(t) - 0.2\sin t\tanh(x_1(t)) \\ \qquad -0.4x_2(t) + \tanh(x_1(t)) + \tanh(x_2(t)) \\ \qquad +0.1\cos t\tanh(x_2(t)) + \sin t, \\ (x_1(t_k), x_2(t_k))^\mathrm{T} = \boldsymbol{K}(x_1(t_k^-), x_2(t_k^-))^\mathrm{T},\ k \in \mathbb{Z}_+. \end{cases} \tag{6}$$

By simulation, if $\boldsymbol{K} = \boldsymbol{I}$, then the solution of system (6) is not periodic (Figs. 2c and 2d). However, if we choose $\boldsymbol{K} = \mathrm{diag}(0.3, 0.2)$, $t_k = k\pi/8,\ k \in$

ℕ, then the solution of system (6) is 2π-periodic and globally attractive (Figs. 2a and 2b). Therefore, it is significant and indispensable to study ICSs.

Additionally, if a given system is unstable or stable without impulsive effects, then it can become uniformly stable or uniformly asymptotically stable

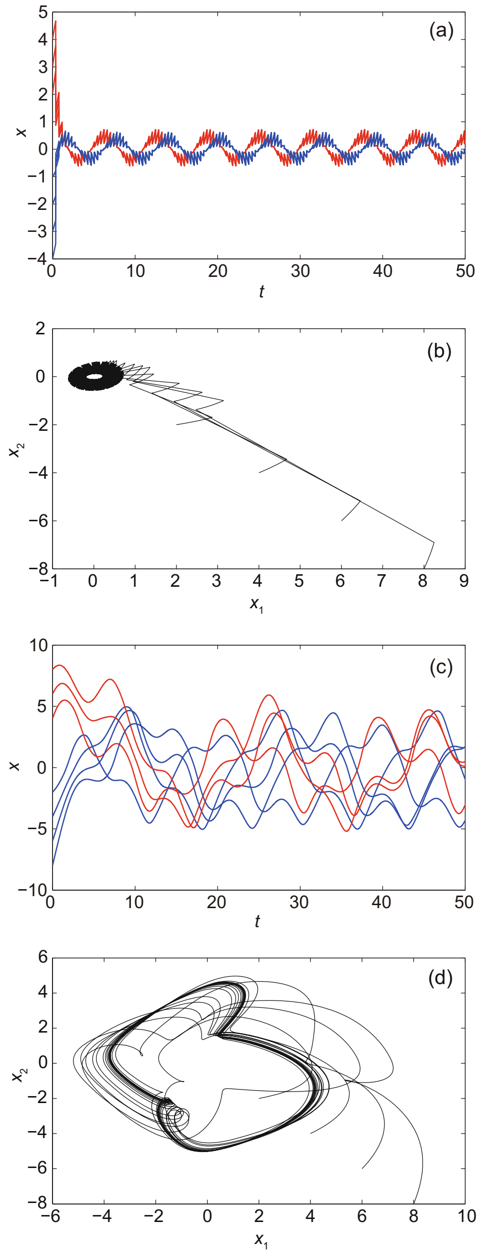

Fig. 2 State trajectories of system (6) with impulsive effect (a), phase portraits of 2π-periodic solutions of system (6) (b), state trajectories of system (6) without impulses (c), and phase portraits of system (6) without impulses (d). Reprinted from Li XD et al. (2015b), Copyright 2014, with permission from Elsevier Ltd.

even exponentially stable, with a proper impulsive control, which is deemed an impulsive stabilization issue (Li XD and Fu, 2012). The following example illustrates this problem:

Example 2 Consider the following 2D neural networks:

$$\begin{cases} \dot{x}_1(t) = -2x_1(t) + (-0.1 + 0.2\sin(\pi t))\sin(x_1(t)) \\ \qquad +0.15\cos(x_2(t)) - \dfrac{5}{2}\cos(\pi t),\ t \neq t_k, \\ \dot{x}_2(t) = -3x_2(t) - 0.6\sin(x_1(t)) + \dfrac{4}{3}\sin(\pi t) \\ \qquad +(0.15 + 0.05\cos(\pi t))\cos(x_2(t)),\ t \neq t_k, \\ x_1(t_k) = \mathrm{e}^{0.5}x_1(t_k^-), k \in \mathbb{Z}_+, \\ x_2(t_k) = \mathrm{e}^{0.6}x_2(t_k^-), k \in \mathbb{Z}_+. \end{cases} \tag{7}$$

By simulation, if we choose $t_k = 0.5k,\ k \in \mathbb{Z}_+$, then the solution of system (7) is 2π-periodic and globally attractive (Figs. 3a and 3b).

At certain time, the nodes may be affected by instantaneous perturbations or experience an abrupt change of state. In other words, the dynamic behavior, in terms of synchronization of CDNs, may be affected by impulsive effects, which can be classified into two types (synchronizing impulses and desynchronizing impulses). In the following sections, we review further research on the two kinds of impulses.

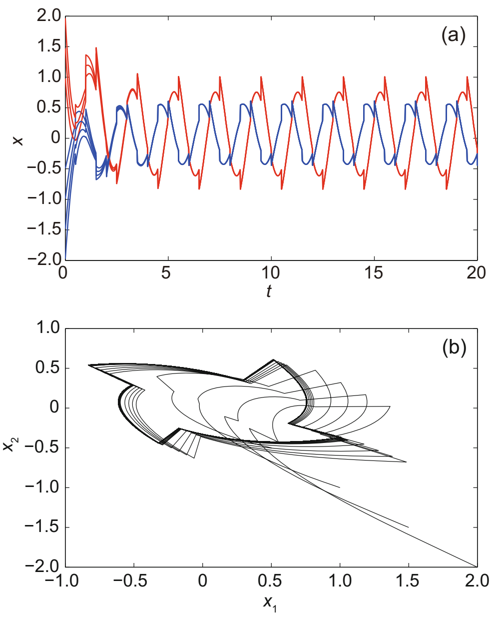

Fig. 3 State trajectories (a) and phase portraits of 2π-periodic solutions (b) of system (7). Reprinted from Li XD et al. (2015b), Copyright 2014, with permission from Elsevier Ltd.

3 Synchronizing impulses

In the past decades, synchronization of impulsive CDNs has attracted wide attention (Sun JT et al., 2004; Yang ZC and Xu, 2005; Cai et al., 2008; Liu B et al., 2008; Lu JG and Chen, 2009; Yang XS et al., 2011b; Liu ZW et al., 2012; Li XD and Rakkiyappan, 2013; Wang X et al., 2017a; Xu ZL et al., 2019; Yang JJ et al., 2020). These studies focus on two aspects of synchronization of impulsive CDN, i.e., impulsive synchronization and impulsive desynchronization. We will introduce mainly some works involving impulsive synchronization. A given CDN is synchronous or asynchronous when there are no impulsive effects, but it can become synchronous, globally asymptotically synchronous, and even exponentially synchronous by designing a proper impulsive controller. Such impulses are said to be synchronizing impulses and are viewed as impulsive controllers. Generally speaking, impulsive controllers have simple structures and the advantages of low cost, effectiveness, and robustness. Much impulsive control research about synchronization of various CDNs has been developed (Lu JG and Chen, 2009; Yang XS et al., 2011b; Li XD and Rakkiyappan, 2013; Wang X et al., 2017a; Xu ZL et al., 2019). For instance, Lu JG and Chen (2009) considered the chaotic neural networks $\mathcal{M}$ and $\mathcal{S}$ as a master-slave system:

$$\begin{cases} \mathcal{M}:\begin{cases}\dot{\boldsymbol{x}}=-\boldsymbol{E}\boldsymbol{x}+\boldsymbol{W}f(\boldsymbol{x})+\boldsymbol{I},\\ \boldsymbol{k}=\boldsymbol{H}\boldsymbol{x},\end{cases}\\ \mathcal{S}:\begin{cases}\dot{\boldsymbol{z}}=-\boldsymbol{E}\boldsymbol{z}+\boldsymbol{W}f(\boldsymbol{z})+\boldsymbol{I},\ t\neq t_i,\\ \boldsymbol{l}=\boldsymbol{H}\boldsymbol{z},\end{cases}\\ \mathcal{C}:\Delta\boldsymbol{z}=\boldsymbol{H}(\boldsymbol{k}-\boldsymbol{l}),\ t=t_i.\end{cases} \tag{8}$$

The synchronization error is defined as $\boldsymbol{e}=\boldsymbol{x}-\boldsymbol{z}$; thus, the error dynamics can be described by

$$\mathcal{E}:\begin{cases}\dot{\boldsymbol{e}}=-\boldsymbol{E}\boldsymbol{e}+\boldsymbol{W}\boldsymbol{\eta}(\boldsymbol{e};\boldsymbol{z}),\ t\neq t_i,\\ \Delta\boldsymbol{e}=-\boldsymbol{K}\boldsymbol{H}\boldsymbol{e},\ t=t_i.\end{cases} \tag{9}$$

For simplicity, we omit the details of the abovementioned system; details can be found in Lu JG and Chen (2009).

Definition 8 (Lu JG and Chen, 2009) The master system $\mathcal{M}$ and slave system $\mathcal{S}$ in Eq. (8) are said to be globally asymptotically synchronized if the synchronization error system (9) is globally asymptotically stable.

Theorem 1 (Lu JG and Chen, 2009) Given the impulsive interval $T_{\rm a}>0$, if there exist a diagonal and positive definite matrix $\boldsymbol{\lambda}=\mathrm{diag}\{\lambda_i\}\in\mathbb{R}^{n\times n}$, a symmetric and positive definite matrix $\boldsymbol{Q}\in\mathbb{R}^{n\times n}$, and a constant matrix $\boldsymbol{M}\in\mathbb{R}^{n\times l}$ such that

$$\begin{bmatrix}\boldsymbol{\Pi} & \boldsymbol{Q}-\boldsymbol{C}^{\rm T}\boldsymbol{M}^{\rm T} & T_{\rm a}\boldsymbol{Q}\boldsymbol{W}\\ * & -\boldsymbol{Q} & \boldsymbol{0}\\ * & * & -T_{\rm a}\boldsymbol{\lambda}\end{bmatrix}<0,$$

where $\boldsymbol{\Pi}=T_{\rm a}(-\boldsymbol{E}^{\rm T}\boldsymbol{Q}-\boldsymbol{Q}\boldsymbol{E}+\boldsymbol{\Sigma}\boldsymbol{\lambda}\boldsymbol{\Sigma})-\boldsymbol{Q}$, $\boldsymbol{\Sigma}=\mathrm{diag}(\sigma_1,\sigma_2,\ldots,\sigma_n)$, then networks $\mathcal{M}$ and $\mathcal{S}$ in Eq. (8) will achieve global asymptotic synchronization under controller $\mathcal{C}$ with $\boldsymbol{K}=\boldsymbol{P}^{-1}\boldsymbol{Y}$.

Lu JG and Chen (2009) have considered two chaotic neural networks as a master-slave system and proposed a sufficient condition for global asymptotic synchronization of this master-slave system. Based on output feedback impulsive control, a linear matrix inequality (LMI) based approach was developed to design the linear impulsive control law. For this method, the linear output feedback impulsive controller can be easily obtained. In addition, an upper bound of the impulsive interval was estimated. Many results were based on the fact that the impulsive interval is limited. As far as impulsive controllers are concerned, the goal is to make the impulsive interval as small as possible. However, to obtain the main results, the maximum of impulsive intervals was limited (Lu JG and Chen, 2009).

Actually, due to the inherent property of complex networks, time delay should be considered and it is significant to investigate complex networks with time delay. Based on Lu JG and Chen (2009), synchronization of neural networks with mixed delay was investigated (Yang XS et al., 2011b; Li XD and Rakkiyappan, 2013; Li XD et al., 2019b), including time-varying delays and distributed delays.

By the impulsive control method, Yang XS et al. (2011b) considered a class of switched neural networks with time-varying delays and unbounded distributed delays. Several sufficient conditions for global exponential synchronization were obtained, and are given in the following.

Consider the following two switched systems as a master-slave system:

$$\begin{cases}\dot{\boldsymbol{x}}(t)=-\boldsymbol{C}_\sigma\boldsymbol{x}(t)+\boldsymbol{A}_\sigma\boldsymbol{f}_\sigma(\boldsymbol{x}(t))\\ \qquad +\boldsymbol{B}_\sigma\boldsymbol{f}_\sigma(\boldsymbol{x}(t-\tau_\sigma(t)))\\ \qquad +\boldsymbol{D}_\sigma\int_{-\infty}^{t}p_\sigma(t-s)\boldsymbol{f}_\sigma(x(s))\mathrm{d}s\\ \qquad +\boldsymbol{I}_\sigma(t),\ t>0,\\ \boldsymbol{x}(s)=\boldsymbol{\phi}(s),\end{cases} \tag{10}$$

$$\begin{cases} \dot{\boldsymbol{y}}(t) = -\boldsymbol{C}_\sigma \boldsymbol{y}(t) + \boldsymbol{A}_\sigma \boldsymbol{f}_\sigma(\boldsymbol{y}(t)) \\ \qquad +\boldsymbol{B}_\sigma \boldsymbol{f}_\sigma(\boldsymbol{y}(t-\tau_\sigma(t))) \\ \qquad +\boldsymbol{D}_\sigma \int_{-\infty}^{t} p_\sigma(t-s)\boldsymbol{f}_\sigma(\boldsymbol{y}(s))\mathrm{d}s \\ \qquad +\boldsymbol{I}_\sigma(t),\ t>0, \\ \Delta \boldsymbol{y}(t_k) = \sum\limits_{\sigma=1}^{m} \boldsymbol{E}_\sigma^k \boldsymbol{z}(t_k^-),\ t=t_k,\ k\in\mathbb{N}_+, \\ \boldsymbol{y}(s) = \boldsymbol{\varphi}(s), \end{cases} \tag{11}$$

where $\boldsymbol{z}(t) = \boldsymbol{y}(t) - \boldsymbol{x}(t)$ is the synchronization error, $\sigma := \sigma(t) : [0,+\infty) \to \mathfrak{M} = \{1,2,\dots,m\}$ is a switching signal, and $0 < \tau_\sigma(t) \le \tau_\sigma$ (τ_σ are constants). The real-valued continuous functions $p_\sigma : [0,+\infty) \to [0,+\infty)$ satisfy $\int_0^{+\infty} p_\sigma(s)\mathrm{d}s = \alpha_\sigma$, where α_σ are positive constants. $\boldsymbol{f}_\sigma(\cdot) = (f_\sigma^1(\cdot),\dots,f_\sigma^n(\cdot))^{\mathrm{T}}$ satisfies $|f_\sigma^i(x) - f_\sigma^i(y)| \le f_\sigma^i|x-y|$, $\forall x,y\in\mathbb{R}, x\ne y$. Detailed information can be found in Yang XS et al. (2011b).

Theorem 2 (Yang XS et al., 2011b) Assume that the impulsive sequence $\{t_k, k\in\mathbb{N}_+\}$ satisfies constraint (5) with elasticity number ζ and average impulsive interval T_{a}. If for each $\sigma\in\mathfrak{M}$ there hold

$$\begin{cases} -c_\sigma + \|\boldsymbol{A}_\sigma\|F_\sigma + \dfrac{F_\sigma}{b_\sigma^{0.5\zeta}}(\|\boldsymbol{B}_\sigma\| + \alpha_\sigma\|\boldsymbol{D}_\sigma\|) + \dfrac{\ln b_\sigma}{2T_{\mathrm{a}}} < 0, \\ \|\boldsymbol{I}_n + \boldsymbol{E}_\sigma^k\|^2 \le b_\sigma < 1,\ k\in\mathbb{N}_+, \end{cases} \tag{12}$$

where $\boldsymbol{C}_\sigma = \mathrm{diag}(c_\sigma^1, c_\sigma^2, \dots, c_\sigma^N)$, $c_\sigma = \min\{c_\sigma^i : i\in\Lambda\}$, $F_\sigma = \max\{f_\sigma^i : i\in\Lambda\}$, then global exponential stabilization for the zero solution of the hybrid impulsive and switched error system can be derived under any switching law.

Based on the Lyapunov stability theory, Yang XS et al. (2011b) obtained a new synchronization criterion for a switched neural network model with mixed delays. Compared with the results in Lu JG and Chen (2009), the switching law can be arbitrary and the synchronization criterion is less conservative than those based on a maximum of impulsive intervals. The results in Lu JG and Chen (2009), Yang XS et al. (2011b), and Li XD and Rakkiyappan (2013) are mainly on chaotic neural networks with mixed delays based on the impulsive control method. Some conditions were presented and easily checked by the MATLAB LMI toolbox.

Consider the delayed drive-response system as follows:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = -\boldsymbol{E}\boldsymbol{x}(t) + \boldsymbol{C}_1\boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{C}_2\boldsymbol{f}(\boldsymbol{x}(t-\tau(t))) \\ \qquad +\boldsymbol{W}\int_{-\infty}^{t}\boldsymbol{h}(t-s)\boldsymbol{f}(\boldsymbol{x}(s))\mathrm{d}s + \boldsymbol{I},\ t>0, \\ \boldsymbol{x}(s) = \boldsymbol{\theta}_1(s), \end{cases} \tag{13}$$

$$\begin{cases} \dot{\boldsymbol{y}}(t) = -\boldsymbol{E}\boldsymbol{y}(t) + \boldsymbol{C}_1\boldsymbol{f}(\boldsymbol{y}(t)) + \boldsymbol{C}_2\boldsymbol{f}(\boldsymbol{y}(t-\tau(t))) \\ \qquad +\boldsymbol{W}\int_{-\infty}^{t}\boldsymbol{h}(t-s)\boldsymbol{f}(\boldsymbol{y}(s))\mathrm{d}s + \boldsymbol{I},\ t\ne t_k, \\ \Delta\boldsymbol{y}(t_k) = \boldsymbol{y}(t_k) - \boldsymbol{y}(t_k^-) = -\boldsymbol{H}(\boldsymbol{y}(t_k^-) - \boldsymbol{x}(t_k^-)),\ k\in\mathbb{Z}_+, \\ \boldsymbol{y}(s) = \boldsymbol{\theta}_2(s),\ s\in(-\infty,0], \end{cases} \tag{14}$$

where $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \mathrm{PC}((-\infty,0],\mathbb{R}^n)$, $\boldsymbol{x}(t) = (x_1(t), x_2(t), \dots, x_N(t))^{\mathrm{T}}$ is the neuron state vector of the neural network, $\boldsymbol{E}$ is a diagonal matrix, $\boldsymbol{C}_1$ is the connection weight matrix, $\boldsymbol{C}_2$ is the delayed weight matrix, and $\boldsymbol{W}$ is the distributively delayed connection weight matrix. $0\le\tau(t)\le\delta$, where δ is a positive constant. $\boldsymbol{f}(\boldsymbol{x}(\cdot)) = \mathrm{diag}(f_1(x_1(\cdot)),\dots,f_j(x_j(\cdot)),\dots,f_N(x_N(\cdot)))$ $(j\in\Lambda)$ represents the neuron activation function and satisfies

$$l_j^- \le \frac{f_j(\alpha) - f_j(\beta)}{\alpha-\beta} \le l_j^+$$

for any $\alpha,\ \beta\in\mathbb{R},\ \alpha\ne\beta$, where l_j^- and l_j^+ are real constants. $\boldsymbol{h}(\cdot) = \mathrm{diag}(h_1(\cdot),\dots,h_j(\cdot),\dots,h_N(\cdot))$ $(j\in\Lambda)$ is the delay kernel function and satisfies $h_j(t)\le\mathbb{H}(t)$,

$$\int_0^\infty h_j(s)\mathrm{d}s = 1,\ j\in\Lambda,\ \int_0^\infty \mathbb{H}(s)\mathrm{e}^{\varrho s}\mathrm{d}s = \mathbb{H}^\star < \infty,$$

where $\mathbb{H}(t)$ represents a nonnegative function defined in $[0,+\infty)$, and constants ϱ and $\mathbb{H}^\star$ are two positive numbers.

Theorem 3 (Li XD and Rakkiyappan, 2013) The exponential synchronization of systems (13) and (14) can be guaranteed, if there exist five constants $\gamma\in(0,\varrho)$, $\beta_i > 0$ $(i=1,2,3)$, $\beta_4\in(0,1)$, an $n\times n$ matrix $\boldsymbol{P}>0$, and three $n\times n$ diagonal matrices $\boldsymbol{Q}_i > 0$ $(i=1,2,3)$ such that

$$\begin{cases} \boldsymbol{\mathcal{F}}_1 \le 0, \\ \boldsymbol{\mathcal{F}}_2 \ge 0, \\ \boldsymbol{L}\boldsymbol{Q}_2\boldsymbol{L} - \beta_1\boldsymbol{P} \le 0, \\ \boldsymbol{L}\boldsymbol{Q}_3\boldsymbol{L} - \beta_2\boldsymbol{P} \le 0, \\ \beta_3\tau + \ln\beta_4 < 0, \end{cases} \tag{15}$$

where

$$\boldsymbol{\mathcal{F}}_1 = \begin{bmatrix} \Delta & \boldsymbol{PA} & \boldsymbol{PB} & \boldsymbol{PW} \\ * & -\boldsymbol{Q}_1 & \boldsymbol{0} & \boldsymbol{0} \\ * & * & -\boldsymbol{Q}_2\beta_4 & \boldsymbol{0} \\ * & * & * & -\boldsymbol{Q}_3\beta_4 \end{bmatrix} < 0,$$

$$\boldsymbol{\mathcal{F}}_2 = \begin{bmatrix} \beta_4\boldsymbol{P} & (\boldsymbol{I}-\boldsymbol{H})^{\mathrm{T}}\boldsymbol{P} \\ * & \boldsymbol{P} \end{bmatrix} < 0,$$

$\varDelta = -\boldsymbol{PC} - \boldsymbol{CP} + \boldsymbol{LQ}_1\boldsymbol{L} + \beta_1 \mathrm{e}^{\gamma\delta}\boldsymbol{P} + \beta_2 \mathbb{H}^\star \boldsymbol{P} - \beta_3 \boldsymbol{P}$, $\tau = \max_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\}$, $\boldsymbol{L} = \mathrm{diag}(l_1, l_2, \ldots, l_N)$ ($l_j = \max\{|l_j^-|, |l_j^+|\}$).

In what follows, an example is given to illustrate the existence and attractiveness of the periodic solution for an impulsive control system.

Example 3 (Li XD and Song, 2013) Consider the following neural networks with distributed delays:

$$\begin{cases} \dot{x}_1(t) = -\alpha_1(t)x_1(t) + \displaystyle\sum_{j=1}^{2} b_{1j}(t)f_j(x_j(t-10)) \\ \qquad + \displaystyle\sum_{j=1}^{2} w_{1j}(t)\int_0^\infty \mathrm{e}^{-s} f_j(x_j(t-s))\mathrm{d}s \\ \qquad + \sin\left(\dfrac{2\pi}{\omega}t\right),\ t \neq t_k, \\ \dot{x}_2(t) = -\alpha_2(t)x_2(t) + \displaystyle\sum_{j=1}^{2} b_{2j}(t)f_j(x_j(t-10)) \\ \qquad + \displaystyle\sum_{j=1}^{2} w_{2j}(t)\int_0^\infty \mathrm{e}^{-s} f_j(x_j(t-s))\mathrm{d}s \\ \qquad + \cos\left(\dfrac{2\pi}{\omega}t\right),\ t \neq t_k, \end{cases} \tag{16}$$

subject to impulses

$$x_i(t_k) = \frac{1}{\rho}x_i(t_k^-),\ k \in \mathbb{Z}_+,\ i = 1, 2,$$

where $f_1 = f_2 = 0.5(|x+1| - |x-1|)$, and $\omega > 0$ and $\rho > 1$ are real constants,

$$\begin{cases} \alpha_1(t) = 0.3 + 0.1\cos\left(\dfrac{2\pi}{\omega}t\right), \\ \alpha_2(t) = 0.3 - 0.1\sin\left(\dfrac{2\pi}{\omega}t\right), \\ b_{1j}(t) = 0.29 - 0.1\cos\left(\dfrac{2\pi}{\omega}(t+1+j)\right), \\ b_{2j}(t) = 0.1 + 0.01\sin\left(\dfrac{2\pi}{\omega}(t+2+j)\right), \\ w_{1j}(t) = 0.2 - 0.08\cos\left(\dfrac{2\pi}{\omega}(t+1+j)\right), \\ w_{2j}(t) = 0.2 + 0.02\sin\left(\dfrac{2\pi}{\omega}(t+2+j)\right). \end{cases}$$

In Figs. 4a–4d, it can be seen that the solution of system (16) with $\omega = 6$ or 35 is not only periodic, but also globally exponentially stable without impulsive effects. However, by designing a suitable impulsive controller, a unique periodic solution of system (16) is obtained and it is globally exponentially stable. Additionally, the periodic solution (Li XD and Song, 2013) can be held, if we choose $\omega = 6, \rho = 2, \mu = 0.3$ or $\omega = 35,\ \rho = 2,\ \mu = 0.35$ (Figs. 5a–5d).

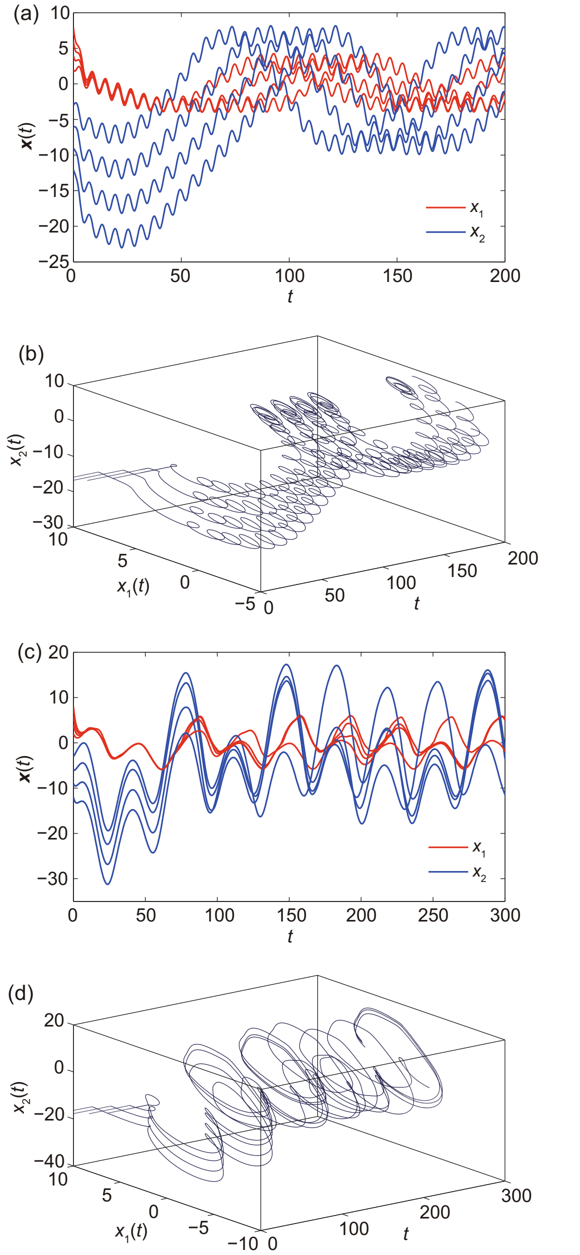

Fig. 4 State trajectories of system (16) with ω=6 and without impulsive control (a), phase portraits of system (16) with ω=6 and without impulsive control (b), state trajectories of system (16) with ω=35 and without impulsive control (c), and phase portraits of system (16) with ω=35 and without impulsive control (d). Reprinted from Li XD and Song (2013), Copyright 2013, with permission from IEEE

There are many other scholars who have studied CDNs' synchronization problems with impulsive

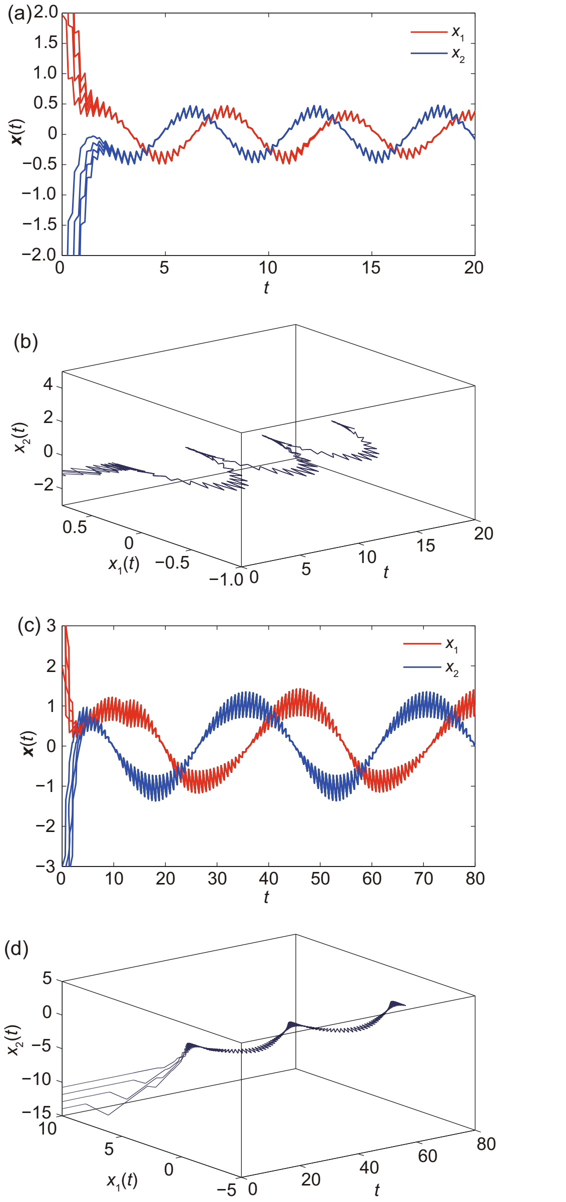

Fig. 5 State trajectories of system (16) with $\omega = 6$, $\rho = 2, \mu = 0.3$ (a), phase portraits of system (16) with $\omega = 6$, $\rho = 2$, $\mu = 0.3$ (b), state trajectories of system (16) with $\omega = 35$, $\rho = 2$, $\mu = 0.35$ (c), and phase portraits of system (16) with $\omega = 35$, $\rho = 2$, $\mu = 0.35$ (d). Reprinted from Li XD and Song (2013), Copyright 2013, with permission from IEEE

effects. Their works have developed the control theory greatly, and are not repeated here.

4 Desynchronizing impulses

In this section, we present mainly research on the impulsive desynchronization problem. A given CDN is synchronous when there are no impulsive effects, but synchronization is suppressed under certain impulsive interference, which can be viewed as an impulsive desynchronization problem. Thus, there are many interesting results on desynchronizing impulses. By establishing an impulsive delay differential inequality, a general neural network with time-varying delays and impulsive effects was considered (Yang ZC and Xu, 2005). It has been shown that the stability depends on not only delays but also impulses. With the development of impulsive control theory, synchronization of coupled dynamical systems under impulsive effects has gained more attention due to potential applications in many fields. Zhou et al. (2007) have considered the issues of synchronization dynamics of complex delayed dynamical networks with impulsive effects. It has been shown that the impulsive effects of the connecting configuration play a significant role in network synchronization. Consider a linearly coupled delayed dynamical network with N nodes as follows:

$$\begin{aligned} Dx_i(t) &= f(t, x_i(t), x_i(t-\tau)) \\ &\quad + \alpha \sum_{j=1}^{N} b_{ij} \Gamma x_j(t) Dw_j(t),\ i \in \Lambda. \end{aligned} \tag{17}$$

For detailed information, please refer to Zhou et al. (2007); we omit it here.

Theorem 4 (Zhou et al., 2007) Let the coupling matrix $\boldsymbol{B} = (b_{ij})_{N\times N}$ be a Laplacian matrix; its eigenvalues can be ordered as

$$0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_N.$$

Assume that the following conditions are satisfied for all $i = 1, 2, \ldots, n$ and $k \in \mathbb{Z}_+$:

(A1) There exist n positive numbers $\delta_1, \delta_2, \ldots, \delta_n$, a positive number $\varepsilon \in [0, 1]$, and denote

$$\begin{cases} p_i = \delta_i - \dfrac{1}{2}\sum_{s=1}^{n}\left(2k_{is}^{2\varepsilon} + k_{si}^{2(1-\varepsilon)}\right), \\ q_i = \dfrac{1}{2}\sum_{s=1}^{n} k_{si}^{2(1-\varepsilon)}, \end{cases}$$

such that $p = \min_{1\leq i\leq n}\{2p_i\} > q = \max_{1\leq i\leq n}\{2q_i\}$ and $c\gamma_i\lambda(\gamma_i) + \delta_i \leq 0$, where

$$\lambda(\gamma_i) = \begin{cases} \lambda_2,\ \gamma_i > 0, \\ 0,\ \gamma_i = 0, \\ \lambda_N,\ \gamma_i < 0. \end{cases} \tag{18}$$

(A2) Let $\mu > 0$ satisfy $\mu - p + q\mathrm{e}^{\mu\tau}$, and

$$\begin{cases} \theta_k = \max\left\{1, \dfrac{1}{1-2cu_k\gamma_i\lambda(u_k\gamma_i)}\right\}, \\ \theta = \sup\left(\dfrac{\ln\theta_k}{t_k - t_{k-1}}\right), \end{cases}$$

such that $2cu_k\gamma_i\lambda(u_k\gamma_i) < 1$ and $\theta < \mu$. Then the impulsive coupled delayed dynamical network (17) is globally exponentially synchronized.

(A1) and (A2) are two important conditions for global synchronization of the impulsive coupled delayed dynamical network (17). From the above mentioned conditions, we can see that the strength values θ_k (or u_k) of impulsive effects and the impulsive interval $t_k - t_{k-1}$ may have some impact on this result. Therefore, it is hugely meaningful to study synchronization of impulsive coupled delayed dynamical networks.

In the past decades, the problem of synchronization of CDNs with impulsive effects has been extensively investigated (Liu B et al., 2005, 2008; Zhang G et al., 2007; Zhou et al., 2007; Cai et al., 2008; Zhang QJ and Lu, 2009; Lu JQ et al., 2010; Yang XS et al., 2011a; Wang JL and Wu, 2012; Yang XS and Lu, 2016; Hu et al., 2018) from the desynchronizing impulse view point. For instance, some sufficient conditions for synchronization of uncertain dynamical networks have been established based on the impulsive control theory (Liu B et al., 2005). In addition, it is worth mentioning that time delay may occur in the implementation of electronic networks and it is unavoidable. For this reason, it is important to consider time delay in the synchronization of CDNs. The output synchronization of a class of impulsive CDNs with time-varying delay was studied (Wang JL and Wu, 2012). By constructing suitable Lyapunov functions, the local and global exponential output synchronizations have been studied under condition $\mu\tau < \inf\{t_k - t_{k-1}\}$, where $\mu > 0$ and τ is some positive constant. Generally speaking, to guarantee the synchronization of CDNs with impulsive perturbation, it may be necessary to restrict the lower bound of the impulse interval. However, based on the concept of the average impulsive interval, the global exponential synchronizations for linearly coupled neural networks with time-varying delay and impulsive disturbances have been derived, where the lower bound of the impulse interval has not been restricted (Lu JQ et al., 2011).

Sometimes, the impulsive perturbation may affect the dynamics of CDNs. For example, a stable equilibrium may become a periodic attractor under impulsive perturbation. In the following, an example is given to show this observation.

Example 4 Consider the following impulsive interval neural networks with discrete and distributed time-varying delays:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = -\alpha(\boldsymbol{x}(t)) + \widetilde{\boldsymbol{A}}f(\boldsymbol{x}(t)) + \widetilde{\boldsymbol{B}}f(\boldsymbol{x}(t-\tau(t))) \\ \qquad +\widetilde{\boldsymbol{W}}\int_{t-\mu(t)}^{t} f(\boldsymbol{x}(s))\mathrm{d}s + \boldsymbol{J}(t), \\ \Delta\boldsymbol{x}(t_k) = \boldsymbol{x}(t_k) - \boldsymbol{x}(t_k^-) = -\boldsymbol{D}_k\boldsymbol{x}(t_k^-),\ k\in\mathbb{Z}_+, \\ \boldsymbol{x}(s) = \boldsymbol{\phi}(s), \quad s\in[-\rho, 0], \end{cases} \tag{19}$$

where

$$\begin{cases} \widetilde{\boldsymbol{A}} = \begin{pmatrix} 1.0 & -0.32 \\ 0.64 & 0.9 \end{pmatrix},\ \widetilde{\boldsymbol{B}} = \begin{pmatrix} 1.40 & -0.10 \\ -0.25 & 1.10 \end{pmatrix}, \\ \widetilde{\boldsymbol{W}} = \begin{pmatrix} -0.77 & 0.59 \\ -0.53 & 0.81 \end{pmatrix},\ \boldsymbol{L}^\delta = \begin{pmatrix} 7 & 0 \\ 0 & 9 \end{pmatrix}, \\ \boldsymbol{D}_k = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix},\ k\in\mathbb{Z}_+, \end{cases}$$

$\alpha(\boldsymbol{x}) = \boldsymbol{L}^\delta\boldsymbol{x},\ f(\boldsymbol{x}) = \tanh(\boldsymbol{x}),\ \tau(t) = \mu(t) = 0.5$, $\boldsymbol{J}(t) = \left[5\sin\left(\dfrac{2\pi}{\omega}t\right), 6\cos\left(\dfrac{2\pi}{\omega}t\right)\right]^{\mathrm{T}}$, and $t_k = 0.2k\omega,\ k\in\mathbb{Z}_+,\ \omega\in\mathbb{R}_+$.

The numerical results (Figs. 6 and 7) show that system (19) has a unique ω-periodic solution that is globally exponentially stable, which matches the obtained theoretical results perfectly.

5 Adaptive-impulsive synchronization

In general, most of the early results on synchronization of CDNs consider a kind of network, in which the dynamics of every node and couplings are assumed to be known in advance. Nevertheless, little information about the network structure can be derived in practice, and it is almost impossible to obtain the exact value of the coupling strength. The state of the network after achieving synchronization is often not known and cannot be changed in the design. Some works have been derived on general uncertain CDNs, where the coupling strength of the network is unknown, and the topology structure of the network is partially known or completely unknown. Therefore, an adaptive-impulsive synchronization method for uncertain CDNs was proposed,

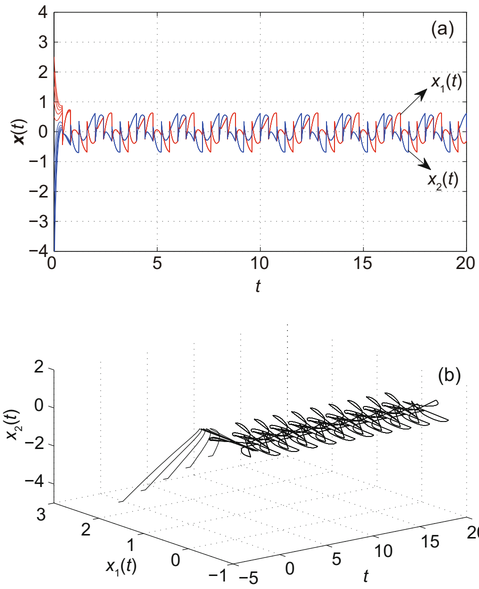

Fig. 6 State trajectories (a) and phase plots (b) of system (19) with $\omega = 2$. Reprinted from Li XD and Shen (2010), Copyright 2010, with permission from IEEE

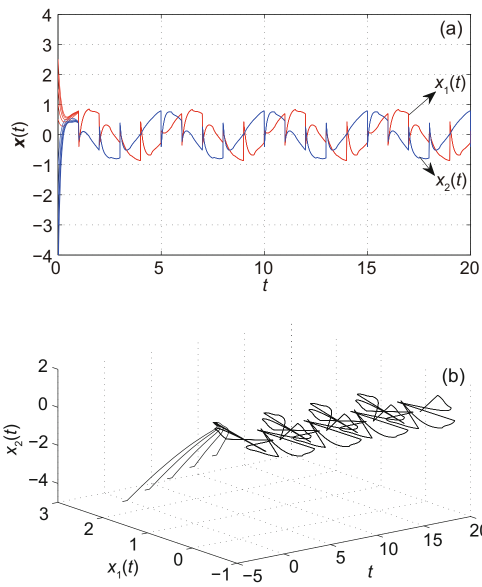

Fig. 7 State trajectories (a) and phase plots (b) of system (19) with $\omega = 5$. Reprinted from Li XD and Shen (2010), Copyright 2010, with permission from IEEE

which has the advantages of the adaptive method and impulsive method (Li K and Lai, 2008).

Nonlinearly coupled CDNs with unknown functions are described by the following equations, consisting of N identical nonlinear oscillators (Li K and Lai, 2008):

$$\dot{\boldsymbol{x}}_i(t) = \boldsymbol{F}(\boldsymbol{x}_i, t) + \boldsymbol{\phi}_i(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N),\ i \in \varLambda, \quad (20)$$

where the dynamics of isolated node i is $\dot{\boldsymbol{x}}_i = \boldsymbol{F}(\boldsymbol{x}_i, t)$, $\boldsymbol{x}_i = (x_{i1}, x_{i2}, \cdots, x_{in})^{\mathrm{T}} \in \mathbb{R}^n$ is the state variable of node i, $\boldsymbol{F} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is a smooth nonlinear vector field, and $\boldsymbol{\phi}_i : \mathbb{R}^m \to \mathbb{R}^n$ $(i \in \varLambda)$ are unknown coupling functions. Note that they are all nonlinear smooth functions, where $m = nN$.

Define errors of synchronization as $\boldsymbol{e}_i(t) = \boldsymbol{x}_i(t) - \boldsymbol{s}(t)$, $i \in \varLambda$. An adaptive-feedback controller $\boldsymbol{u}_i$ and an impulsive controller $\boldsymbol{U}_{ik}(\boldsymbol{x}_i - \boldsymbol{s}, t)$ are designed such that the state of the dynamical network (20) synchronizes with the state of the isolated node, implying $\lim\limits_{t\to\infty}[\boldsymbol{x}_i(t) - \boldsymbol{s}(t)] = 0$. The state of CDNs controlled by the adaptive-impulsive method is given by

$$\begin{cases} \dot{\boldsymbol{x}}_i(t) = \boldsymbol{F}(\boldsymbol{x}_i, t) + \boldsymbol{\phi}_i(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N) \\ \qquad\quad +\boldsymbol{u}_i,\ t \neq t_k, \\ \Delta \boldsymbol{x}_i = \boldsymbol{x}_i(t_k^+) - \boldsymbol{x}_i(t_k^-) \\ \qquad = \boldsymbol{U}_{ik}(\boldsymbol{x}_i - s, t),\ t = t_k, \end{cases} \quad (21)$$

where $i \in \varLambda$, $\boldsymbol{x}_i(t_k^+)$ and $\boldsymbol{x}_i(t_k^-)$ are defined in Table 1, and the impulsive sequence $\{t_k\}$ satisfies $0 \leq t_1 < t_2 < \ldots$. Two classes of controllers are given in CDNs. The first is a continuous feedback controller ($\boldsymbol{u}_i$), and the second is a controller with impulsive inputs ($\boldsymbol{U}_{ik}$). The adaptive-feedback controller $\boldsymbol{u}_i$ is defined as $\boldsymbol{u}_i = d_i \boldsymbol{e}_i$, $i \in \varLambda$ and its updating laws are $d_i = k_i \boldsymbol{e}_i^{\mathrm{T}} \boldsymbol{e}_i = k_i \|\boldsymbol{e}_i\|^2$, $i \in \varLambda$, where $k_i > 0$ are arbitrary constants. If synchronization of CDNs can be achieved, d_i will tend to an ultimate constant strength $\hat{d}_i$. A linear impulsive controller is chosen as $\boldsymbol{U}_{ik}(\boldsymbol{x}_i - \boldsymbol{s}, t_k) = \boldsymbol{B}_{ik}(\boldsymbol{x}_i - \boldsymbol{s}, t_k)$, where each $\boldsymbol{B}_{ik}$ is an $n \times n$ constant matrix. Because the adaptive-feedback controller is added in the impulsive control scheme, the impulsive interval in this method can be larger than that in the ordinary impulsive control method.

The linearly coupled CDNs (1) are a particular example of Eq. (20) because $h_i(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N) = \sum\limits_{j=1}^{N} a_{ij} \boldsymbol{\varGamma} \boldsymbol{x}_j$ for $i \in \varLambda$. Then, the CDN with adaptive-impulsive control can be rewritten as follows (Li K and Lai, 2008):

$$\begin{cases} \dot{\boldsymbol{x}}_i = f(\boldsymbol{x}_i, t) + \sum\limits_{j=1}^{N} a_{ij}\boldsymbol{\Gamma}\boldsymbol{x}_j + \boldsymbol{u}_i,\ t \neq t_k, \\ \Delta\boldsymbol{x}_i = \boldsymbol{U}_{ik}(\boldsymbol{x}_i - \boldsymbol{s}, t),\ t = t_k,\ k = 1, 2, ... \end{cases} \tag{22}$$

It is well known that ring-coupled networks are difficult to synchronize. The coupling coefficient c must satisfy $c = O(N^2)$. However, CDNs can easily achieve synchronization locally or globally asymptotically to the synchronous state $\boldsymbol{s}(t)$ by the adaptive-impulsive method, where $\boldsymbol{s}(t)$ has the same meaning as in Definition 1.

In this study, CDNs are composed of N identical nonlinear oscillators. It is impractical in engineering to assume that the topology structures of CDNs are identical. Then some works would be considered on CDNs with nonidentical nodes and nonidentical topology structures. Zhang QJ et al. (2013) considered a more general CDN model. The CDN consists of N different dynamical nodes, different from the CDNs in Li K and Lai (2008), and it has unknown parameters. The drive system is described by

$$\dot{\boldsymbol{x}}_i = F_i(\boldsymbol{x}_i, t, \Theta_i) + \sum_{j=1}^{N} a_{ij}\boldsymbol{\Gamma}\boldsymbol{x}_j, \tag{23}$$

where Θ_i are unknown parameters and $\boldsymbol{A} = (a_{ij})_{N\times N}$ is the coupling configuration matrix including coupling strengths and the topology structure of CDNs. Another response system with impulsive control is given by

$$\begin{cases} \dot{\boldsymbol{y}}_i = F_i(\boldsymbol{y}_i, t, \hat{\Theta}_i) + \sum\limits_{j=1}^{N} a_{ij}\boldsymbol{\Gamma}\boldsymbol{y}_j + \boldsymbol{u}_i,\ t \neq t_k, \\ \Delta\boldsymbol{y}_i(t^+) = \boldsymbol{D}_{ik}(\boldsymbol{y}_i(t) - \boldsymbol{x}_i(t)),\ t = t_k,\ k = 1, 2, \ldots, \\ \boldsymbol{y}_i(t_0^+) = \boldsymbol{y}_{i0}, \end{cases}$$

where $\boldsymbol{y}_i(t) \in \mathbb{R}^n$ $(i \in \Lambda)$ is the state variable of node i and $\hat{\Theta}_i$ is the estimate of the unknown Θ_i for $i \in \Lambda$. $\boldsymbol{D}_{ik} \in \mathbb{R}^{n\times n}$ are the impulsive feedback matrices. Several synchronization criteria for uncertain CDNs have been obtained from the adaptive-impulsive method. Specifically, uncertain parameters can be identified when synchronization occurs.

CDNs are composed of N nonidentical nonlinear oscillators (Zhang QJ et al., 2013). The synchronization of CDNs is given in Definition 3, but the topology structures of response CDNs are identical to those of the drive CDNs. Li HL et al. (2015) extended the work to uncertain CDNs with nonidentical topology structures by adaptive-impulsive control. The response CDNs are given by

$$\begin{cases} \dot{\boldsymbol{y}}_i = f_i(\boldsymbol{y}_i, t, \hat{\Theta}_i) + \sum\limits_{j=1}^{N} b_{ij}\boldsymbol{\Gamma}\boldsymbol{y}_j + \boldsymbol{u}_i,\ t \neq t_k, \\ \Delta\boldsymbol{y}_i(t^+) = \boldsymbol{D}_{ik}(\boldsymbol{y}_i(t) - \boldsymbol{x}_i(t)),\ t = t_k,\ k \in \mathbb{Z}_+, \\ \boldsymbol{y}_i(t_0^+) = \boldsymbol{y}_{i0}, \end{cases} \tag{24}$$

where $\boldsymbol{B} = (b_{ij})_{N\times N}$ is the topology structure matrix of the response CDNs. It can be different from the topology structure matrix $\boldsymbol{A}$ in Eq. (23).

The synchronization of drive CDNs (23) and response CDNs (24) is more common and generous because it is composed of N different node dynamics and with different structures. The CDN will degenerate into a cluster CDN if nodes can be classified into several classes according to their dynamics.

The variables are real values in the above results. A general model was considered in Wu ZY (2015). The complex hybrid synchronization of CDNs with the complex coupling matrix and complex-variable coupling function was investigated via adaptive-impulsive control. The definition of complex hybrid synchronization was given in Wu ZY (2015). For a given complex-variable matrix $\boldsymbol{H}$, CDNs achieve the complex hybrid synchronization with respect to matrix $\boldsymbol{H}$, if $\lim\limits_{t\to\infty} \|\boldsymbol{X}_k(t) - \boldsymbol{H}\boldsymbol{s}(t)\| = 0,\ k \in \Lambda$, where $\boldsymbol{X}_k(t)$ is an n-dimensional complex variable and $\boldsymbol{s}(t)$ is an n-dimensional manifold.

CDNs coupled with complex-variable functions were given by

$$\dot{\boldsymbol{X}}_k(t) = \boldsymbol{F}(\boldsymbol{X}_k(t)) + \sum_{l=1}^{N} a_{kl}\boldsymbol{\Gamma}\boldsymbol{X}_l(t). \tag{25}$$

A proper impulsive controller is designed to realize the complex hybrid synchronization of CDNs (25). Network (25) with impulsive controllers can be written as

$$\begin{cases} \dot{\boldsymbol{X}}_k(t) = \boldsymbol{F}(\boldsymbol{X}_k(t)) + \sum\limits_{l=1}^{N} a_{kl}\boldsymbol{\Gamma}\boldsymbol{X}_l(t),\ t \neq t_\sigma, \\ \Delta\boldsymbol{X}_k(t_\sigma) = b_\sigma(\boldsymbol{X}_k(t) - \boldsymbol{H}\boldsymbol{s}(t)),\ t = t_\sigma. \end{cases} \tag{26}$$

The result says that when vector function $\boldsymbol{F}(\boldsymbol{X}_k(t))$ satisfies appropriate conditions, complex hybrid synchronization of the network, coupled with complex-variable chaotic systems, can still be obtained using impulsive control.

Assumption 1 (Wu ZY, 2015) Assuming that there is a positive constant L, the vector function $\boldsymbol{F}(\boldsymbol{X}_k(t))$ satisfies $\overline{(\boldsymbol{X}_k(t)-\boldsymbol{H}\boldsymbol{s}(t))^{\mathrm{T}}(\boldsymbol{F}(\boldsymbol{X}_k(t))-\boldsymbol{H}\boldsymbol{F}(\boldsymbol{s}(t)))}$ + $\overline{(\boldsymbol{F}(\boldsymbol{X}_k(t))-\boldsymbol{H}\boldsymbol{F}(\boldsymbol{s}(t)))^{\mathrm{T}}(\boldsymbol{X}_k(t)-\boldsymbol{H}\boldsymbol{s}(t))} \leq L(\boldsymbol{X}_k(t)-\boldsymbol{H}\boldsymbol{s}(t))^{\mathrm{T}} \cdot \overline{(\boldsymbol{X}_k(t)-\boldsymbol{H}\boldsymbol{s}(t))}$, where $k \in \Lambda$.

It was assumed that the vector function $\boldsymbol{F}(\boldsymbol{X}_k(t))$ satisfies the conditions in Assumption 1 (Wu ZY, 2015). If there exists a constant $\alpha > 0$ such that inequality $\ln\beta_\sigma + \alpha + (L+\lambda_1)\tau_\sigma < 0$ holds, then CDNs (25) can achieve synchronization for complex-variable CDNs.

Liu DF et al. (2014) also investigated the structure identification of uncertain CDNs (25) via adaptive-impulsive control. It is comparable to the most general results (Wu ZY, 2015). The identification of the structure was improved by Zhang QJ et al. (2013). The adaptive-impulsive method has the advantages of the adaptive method and impulsive method. It can be widely used to deal with other CDN problems.

6 Pinning impulsive control

While investigating the synchronization of CDNs, many useful control methods have been developed. A strategy of pinning control was introduced by Wang XF and Chen (2002a), where only a few network nodes were directly controlled (Li X et al., 2004). The CDN pinning control model (Wang XF and Chen, 2002a) can be described by

$$\dot{\boldsymbol{x}}_{i_k} = \begin{cases} f(\boldsymbol{x}_{i_k}) + c\sum\limits_{j=1}^{N} a_{i_k j}\boldsymbol{\Gamma}\boldsymbol{x}_j \\ \qquad -cd\boldsymbol{\Gamma}(\boldsymbol{x}_{i_k}-\boldsymbol{s}),\ k=1,2,\ldots,l, \\ f(\boldsymbol{x}_{i_k}) + c\sum\limits_{j=1}^{N} a_{i_k j}\boldsymbol{\Gamma}\boldsymbol{x}_j, \\ \qquad k=l+1,l+2,\ldots,N, \end{cases} \tag{27}$$

where $i_1, i_2, \ldots, i_l$ are the nodes being controlled and $1 \leq l < N$. The homogeneous stationary state has been defined in Definition 1, that is, $\boldsymbol{x}_1 = \boldsymbol{x}_2 = \ldots = \boldsymbol{x}_N = \boldsymbol{s}(t)$ and $f(\boldsymbol{s}(t)) = 0$. Then CDNs can be pinning controlled to their synchronization state by a linear feedback controller (Wang XF and Chen, 2002a). The performance of pinning synchronization of CDNs is strongly influenced by several factors, such as the containing networks' structures, dynamics characteristics of nodes, coupling strengths, and pinning strategies. Chen TP et al. (2007) proved that a single state feedback controller can pin the linearly coupled CDN to its homogenous solution. The number of controlled nodes is one. Note that if the coupling strength c is large enough, CDNs must be pinned to the solution of the uncoupled system by a single controller. However, this is not the best strategy. It is clear that when the number of controllers is larger, it is much easier to pin CDNs.

Later, the pinning stabilization problem was discussed by Lu WL et al. (2010). They focused on undirected and connected graphs, in which the corresponding Laplacian matrix is symmetric and irreducible. Qin et al. (2011) improved the above works; the pinning synchronization ability of complex networks with an arbitrary topology structure was discussed. The topology structure of CDNs can be a weighted and directed graph with an asymmetric and reducible Laplacian matrix, which is the extension of the results in Lu WL et al. (2010).

Because of the effectiveness of impulsive control, pinning synchronization for CDNs with impulsive control was considered in the following works. Lu JQ et al. (2012a, 2012b, 2013, 2015) and Wang YQ et al. (2019) developed pinning impulsive control of CDNs. Lu JQ et al. (2012b) focused on introducing a pinning impulsive strategy into synchronization of nonlinear stochastic CDNs. Based on the above results, a single impulsive controller can be used to pin a CDN to global exponential synchronization (Lu JQ et al., 2013). As an extension, the work by Lu JQ et al. (2015) is the outer synchronization for drive-response partially coupled networks via pinning impulsive controllers. Pinning impulsive synchronization of CDNs with various delays was discussed in Wang X et al. (2017b), and the time delay is time-varying. A CDN with N identical time-delayed nodes is diffusively coupled linearly (Wang X et al., 2017b):

$$\dot{\boldsymbol{x}}_i(t) = f(\boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau(t))) + c\sum_{j=1}^{N} b_{ij}\boldsymbol{\Gamma}\boldsymbol{x}_j(t) + \tilde{c}\sum_{j=1}^{N} g_{ij}\boldsymbol{\Gamma}\boldsymbol{x}_j(t-\tau(t)),\ i \in \Lambda, \tag{28}$$

where $\tau(t)$ is the time-varying delay in the coupling function satisfying $0 \leq \tau(t) \leq \tau,\ \dot{\tau}(t) \leq \mu < 1$ for some positive scalars τ and μ.

The response network is characterized by

$$\dot{\boldsymbol{y}}_i(t)=f(t,\boldsymbol{y}_i(t),\boldsymbol{y}_i(t-\tau(t)))+c\sum_{j=1}^{N}b_{ij}\boldsymbol{y}_j(t)+\tilde{c}\sum_{j=1}^{N}g_{ij}\boldsymbol{\Gamma}\boldsymbol{y}_j(t-\tau(t))+\boldsymbol{u}_i(t), \quad (29)$$

where $\boldsymbol{y}_i(t)\in\mathbb{R}^n,\ i\in\Lambda$. To achieve synchronization of CDNs in Definition 2, several impulsive controllers are designed for the response CDNs (29); thus, the controlled response network is

$$\begin{cases}\dot{\boldsymbol{y}}_i(t)=f(\boldsymbol{y}_i(t),\boldsymbol{y}_i(t-\tau(t)))+c\sum\limits_{j=1}^{N}b_{ij}\boldsymbol{\Gamma}\boldsymbol{y}_j(t),\\ \qquad +\tilde{c}\sum\limits_{j=1}^{N}g_{ij}\boldsymbol{\Gamma}\boldsymbol{y}_j(t-\tau(t)),\ t\neq t_k,\\ \boldsymbol{y}_i(t_k^+)=\boldsymbol{y}_i(t_k^-)+\boldsymbol{B}_i(t_k)(\boldsymbol{y}_i(t_k^-)-\boldsymbol{x}_i(t_k^-)),\ t=t_k.\end{cases} \quad (30)$$

The assumption (Wang X et al., 2017b) is that there are two constants, $L_1\geq 0$ and $L_2\geq 0$, which can be used to guarantee that the following inequality of nonlinear function $f(\boldsymbol{x},\bar{\boldsymbol{x}})$ holds: $(\boldsymbol{x}-\boldsymbol{y})^{\mathrm{T}}(f(\boldsymbol{x},\bar{\boldsymbol{x}})-f(\boldsymbol{y},\bar{\boldsymbol{y}}))\leq L_1(\boldsymbol{x}-\boldsymbol{y})^{\mathrm{T}}(\boldsymbol{x}-\boldsymbol{y})+L_2(\bar{\boldsymbol{x}}-\bar{\boldsymbol{y}})^{\mathrm{T}}(\bar{\boldsymbol{x}}-\bar{\boldsymbol{y}})$, for any $\boldsymbol{x},\ \boldsymbol{y},\ \bar{\boldsymbol{x}},\ \bar{\boldsymbol{y}}\in\mathbb{R}^n$.

From the above result, it can be concluded that synchronization of CDNs can be achieved by impulsive control of a small number of network nodes. For the meaning of symbols in the following theorem, please refer to Wang X et al. (2017b). If there exists a positive constant α such that $\ln(\rho_k+2\omega\tau)+(\alpha+\hat{L})(t_k-t_{k-1})\leq 0, k=1,2,\ldots$, then the drive CDNs (28) can synchronize to the response CDNs (30) with the pinning impulsive control in Definition 3.

Moreover, pinning adaptive-impulsive synchronization of fractional-order CDNs and CDNs with various time-varying delays were considered. Later, pinning and impulsive synchronization control of CDNs with nonderivative and derivative coupling were reported by Zheng (2017). The network (Zheng, 2017) consists of N identical delayed nodes with nonderivative and derivative coupling, which can be given by $\dot{\boldsymbol{x}}_i(t)=\boldsymbol{f}(t,\boldsymbol{x}_i(t),\boldsymbol{x}_i(t-\tau(t)))+c_1\sum\limits_{j=1}^{N}a_{ij}\boldsymbol{x}_j(t)+c_2\sum\limits_{j=1}^{N}b_{ij}\dot{\boldsymbol{x}}_j(t)$, where $\boldsymbol{x}_i(t)$ is the state vector of the i^{th} node ($i\in\Lambda$), and $\boldsymbol{f}:[0,+\infty)\times\mathbb{R}^n\times\mathbb{R}^n\rightarrow\mathbb{R}^n$ is a continuous nonlinear vector function that describes the dynamical behaviors of the isolated node. The time-varying delay $\tau(t)$ may be unknown, but is bounded by a constant, that is, $0<\tau(t)\leq\tau$.

The response network is characterized by

$$\dot{\boldsymbol{y}}_i(t)=\boldsymbol{f}(t,\boldsymbol{y}_i(t),\boldsymbol{y}_i(t-\tau(t)))+c_1\sum_{j=1}^{N}a_{ij}\boldsymbol{y}_j(t)+c_2\sum_{j=1}^{N}b_{ij}\dot{\boldsymbol{y}}_j(t)+\boldsymbol{u}_i(t), \quad (31)$$

where $\boldsymbol{y}_i(t)=(y_{i1}(t),y_{i2}(t),\ldots,y_{in}(t))^{\mathrm{T}}\in\mathbb{R}^n$ is the state vector of the i^{th} node and $\boldsymbol{u}_i(t)$ is a hybrid controller on node i .

The models (Zheng, 2017) can describe the interrelated networks. The spatial position of node i and the velocity of the process of transmitting information can be expressed by variables $\boldsymbol{x}_i(t)$ and $\dot{\boldsymbol{x}}_i(t)$, respectively. The hybrid control inputs $\boldsymbol{u}_i(t)$ can be chosen as $\boldsymbol{u}_i(t)=\boldsymbol{v}_i(t)+\boldsymbol{\omega}_i(t)$ $(1\leq i\leq N)$, where $\boldsymbol{v}_i(t)$ are linear pinning control and $\boldsymbol{\omega}_i(t)$ are impulsive pinning control on p nodes. They are designed as

$$\boldsymbol{v}_i(t)=\begin{cases}-c_1d_i\boldsymbol{e}_i(t), & 1\leq i\leq p,\\ 0, & p+1\leq i\leq N,\end{cases} \quad (32)$$

$$\boldsymbol{\omega}_i(t)=\begin{cases}\sum\limits_{k=1}^{\infty}\boldsymbol{B}_i(t_k)\boldsymbol{e}_i(t)\delta(t-t_k),\ 1\leq i\leq p,\ t=t_k,\\ 0,\ \ p+1\leq i\leq N,\ t\neq t_k,\end{cases} \quad (33)$$

where d_i are positive constants, $\delta(\cdot)$ is the Dirac delta function, $\boldsymbol{B}_i(t)\in\mathbb{R}^{n\times n}$ are the state impulse gain matrices at $t=t_k$, and $\boldsymbol{B}_i(t)=\mathbf{0}$ for $t\neq t_k,\ i\in\Lambda,\ k=1,2,\ldots$.

In general, it is supposed that the first p $(1\leq p<N)$ nodes are controlled. Based on theories of stability and some analysis methods, several novel sufficient conditions of global synchronization were obtained (Zheng, 2017).

It is assumed that there exist two constants $L_3>0$ and $L_4>0$ for the time-varying delay function $\boldsymbol{f}(t,\boldsymbol{x}(t),\boldsymbol{x}(t-\tau(t)))$, and the following condition $(\boldsymbol{x}(t)-\boldsymbol{y}(t))^{\mathrm{T}}[\boldsymbol{f}(t,\boldsymbol{x}(t),\boldsymbol{x}(t-\tau(t)))-\boldsymbol{f}(t,\boldsymbol{y}(t),\boldsymbol{y}(t-\tau(t)))]\leq L_3(\boldsymbol{x}(t)-\boldsymbol{y}(t))^{\mathrm{T}}(\boldsymbol{x}(t)-\boldsymbol{y}(t))+L_4(\boldsymbol{x}(t-\tau(t))-\boldsymbol{y}(t-\tau(t)))^{\mathrm{T}}(\boldsymbol{x}(t-\tau(t))-\boldsymbol{y}(t-\tau(t)))$ is satisfied. Then we have the following theorem:

Theorem 5 (Zheng, 2017) Suppose that the assumptions for function $\boldsymbol{f}$ hold. If there exist positive constants δ and γ such that the following inequalities hold:

$$L_1\boldsymbol{I}_N - \frac{\delta}{2}(\boldsymbol{I}_N - c_2\boldsymbol{B}) + c_1(\boldsymbol{A}^s - \boldsymbol{D}) < 0, \quad (34)$$

$$2L_2\gamma < 0, \quad (35)$$

$$\frac{\ln\eta}{T_a} + \delta + \eta^{-N_0}\gamma < 0, \quad (36)$$

where $\boldsymbol{A}^s = \dfrac{\boldsymbol{A}+\boldsymbol{A}^{\mathrm{T}}}{2}$, $\boldsymbol{D} = \mathrm{diag}(d_1, \ldots, d_p, 0, \ldots, 0)$, $\hat{\beta}_k = \lambda_{\max}(\boldsymbol{I}+\beta_k)^{\mathrm{T}}(\boldsymbol{I}+\beta_k)$, $\rho(t) = 1-(1-\hat{\beta}_k)p/N$ for $t = t_k$, and $\rho(t) = 1$ for $t \neq t_k$, $\eta(t) = \lambda_{\boldsymbol{I}_N - c_2\boldsymbol{B}}\rho(t)$, $0 < \eta(t_k) < 1$, $\eta = \sup\{\eta t_k\}$, $k = 1, 2, \ldots$, then the drive CDNs and response CDNs can achieve outer synchronization by hybrid control.

As an extension, Li YY et al. (2018) and Wang YQ et al. (2019) presented mainly brief conditions to achieve pinning synchronization of a Takagi-Sugeno fuzzy CDN under partial and discrete-time communication, respectively. Then a hybrid pinning and impulsive control strategy was developed. The nonlinearly coupled CDN with N identical Lur'e dynamical systems was considered (Wang YQ et al., 2019), where the structure of the network is strongly connected:

$$\dot{\boldsymbol{x}}_i(t) = \boldsymbol{A}\boldsymbol{x}_i(t) + \boldsymbol{B}\tilde{\boldsymbol{f}}(\boldsymbol{C}\boldsymbol{x}_i(t)) + c\sum_{j=1}^{N} l_{ij}\boldsymbol{\Gamma}\tilde{h}(\boldsymbol{x}_j(t)), \quad (37)$$

where function $\tilde{\boldsymbol{f}}(\boldsymbol{C}\boldsymbol{x}_i) = (\tilde{f}_1(\boldsymbol{C}\boldsymbol{x}_i), \tilde{f}_2(\boldsymbol{C}\boldsymbol{x}_i), \ldots, \tilde{f}_m(\boldsymbol{C}\boldsymbol{x}_i)) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a memoryless nonlinear vector-valued function and function $\tilde{h}(\boldsymbol{x}_i) = (\tilde{h}(x_{i1}), \tilde{h}(x_{i2}), \ldots, \tilde{h}(x_{in}))^{\mathrm{T}}$ satisfies $\dfrac{\tilde{h}(u)-\tilde{h}(v)}{u-v} \geq \beta > 0$ for any $u \neq v$.

The Lur'e CDNs with impulsive dynamics are described by

$$\begin{cases} \dot{\boldsymbol{x}}_i(t) = \boldsymbol{A}\boldsymbol{x}_i(t) + \boldsymbol{B}\tilde{\boldsymbol{f}}(\boldsymbol{C}\boldsymbol{x}_i(t)) + c\sum\limits_{j=1}^{N} l_{ij}\boldsymbol{\Gamma}\tilde{\boldsymbol{h}}(\boldsymbol{x}_j(t)), \\ \qquad\qquad t \neq t_k, \\ \boldsymbol{x}_j(t_k^+) - \boldsymbol{x}_i(t_k^+) = \mu_k(\boldsymbol{x}_j(t_k^-) - \boldsymbol{x}_i(t_k^-)),\ t = t_k, \end{cases} \quad (38)$$

where $\zeta = \{t_i\}_{i=1,2,\ldots}$ is an impulsive time, and μ_k is the corresponding impulsive gain which may contain desynchronizing impulses and synchronizing impulses simultaneously. The controller presented in Wang YQ et al. (2019) is a pinning impulsive one with $\tilde{\mu}_k < 1$.

Assumption 2 (Wang YQ et al., 2019) For nonlinear function $\tilde{f}_i(\cdot)$, it is assumed that there exist some constants $\pi_{ik} > 0$ (i, $k = 1, 2, \ldots, m$) for any $\boldsymbol{y}_1$, $\boldsymbol{y}_2 \in \mathbb{R}^m$; one has $|f_i(\boldsymbol{y}_1) - f_i(\boldsymbol{y}_2)| \leq \sum\limits_{k=m}^{n} \pi_{ik}|\boldsymbol{y}_{1k} - \boldsymbol{y}_{2k}|$.

Definition 9 (Wang YQ et al., 2019) The Lur'e CDN (38) is called to achieve global exponential synchronization if there exist constants $\eta > 0$, $T > 0$, and $\theta > 0$ such that for arbitrary initial values and $t > T$, $|\boldsymbol{x}_i(t) - \boldsymbol{s}(t)| \leq \theta \mathrm{e}^{-\eta t}$ holds for $i \in \mathit{\Lambda}$.

Then there is the theorem for $T_a < \infty$. The case of $T_a = \infty$ is another theorem similar to Theorem 6.

Theorem 6 (Wang YQ et al., 2019) Suppose that Assumption 2 holds and $T_a < \infty$. The impulsively controlled Lur'e CDN (38) is globally exponentially synchronized to the isolated node $\boldsymbol{s}(t)$ with convergence rate η_1 if $\eta < \eta_1 < 0$, where $\eta = \dfrac{2\ln\mu}{T_a} + \delta$, $\delta = \lambda_{\max}(\boldsymbol{A}+\boldsymbol{A}^{\mathrm{T}}+\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}+\boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Pi}^{\mathrm{T}}\boldsymbol{\Pi}\boldsymbol{C}-c\omega\boldsymbol{\Gamma})$, $\omega = -\beta\lambda_2(\boldsymbol{Q})/\lambda_{\max}(\boldsymbol{R})$, $\boldsymbol{Q} = \boldsymbol{\Xi}\boldsymbol{L}+\boldsymbol{L}^{\mathrm{T}}\boldsymbol{\Xi}$, $\boldsymbol{R} = \boldsymbol{\Xi}-\boldsymbol{\xi}\boldsymbol{\xi}^{\mathrm{T}}$, $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_N)$ is the left eigenvalue of matrix $\boldsymbol{L}$ with respect to eigenvalue 0 satisfying $\sum\limits_{i=1}^{N}\xi_i = 1$, and $\boldsymbol{\Xi} = \mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_N)$.

A single controller with impulse is chosen to investigate the synchronization of a Lur'e network (Wang YQ et al., 2019). The pinning impulsive strategy may be used in a CDN with nonidentical nodes or the issues of stochastic CDNs.

7 Complex dynamical networks with stochastic and impulsive effects

Impulsive effects and stochastic effects exist in CDNs during signal transmission between dynamical nodes. The phenomena of switching, the changes of the frequency, and other sudden noise can cause impulsive effects in a system's state at certain instants, and some results have been derived for synchronization of CDNs with stochastic and impulsive effects. Robust exponential stability of uncertain impulsive stochastic neural networks with delayed impulses was discussed (Li CX et al., 2011; Vinodkumar et al., 2018; Li XD et al., 2019a). Synchronization of delayed CDNs with impulsive and stochastic effects was discussed by Yang XS et al. (2011a). Synchronization of chaotic delayed neural networks with stochastic perturbation using the impulsive control method was considered by Li XD and Song (2014). Stabilization of probabilistic Boolean networks via the pinning control strategy was discussed by Huang et al. (2020). Stability analyses of complex-valued stochastic delayed networks with Markovian switching and impulsive effects were displayed in Li Z et al. (2017). Synchronization of stochastic discrete-time complex networks with partial mixed impulsive effects was studied by Li F and Sun (2011).

The concept of "average impulsive interval" was presented by Yang XS et al. (2011a). Exponential synchronization of impulsive CDNs was studied, and a unified synchronization criterion was derived for CDNs with synchronizing or desynchronizing impulses.

A linearly coupled CDN composed of N identical nodes was considered which has impulsive and stochastic effects (Yang XS et al., 2011a). It can be described as follows:

$$\begin{cases} \mathrm{d}\boldsymbol{x}_i(t) = [\boldsymbol{A}\boldsymbol{x}_i(t) + \boldsymbol{f}(t, \boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau_1(t))) \\ \quad + \sum\limits_{j=1}^{N} g_{ij}\boldsymbol{\phi}\boldsymbol{x}_j(t) + \sum\limits_{j=1}^{N} b_{ij}\boldsymbol{\Gamma}\boldsymbol{x}_j(t-\tau_2(t))]\mathrm{d}t \\ \quad + \boldsymbol{\sigma}(t, \boldsymbol{x}_t(t), \boldsymbol{x}_i(t-\tau_3(t)))\mathrm{d}\boldsymbol{\omega}(t), \ t \neq t_k, \ i \in \Lambda, \\ \Delta\boldsymbol{x}_i(t_k) = \boldsymbol{x}_i(t_k) - \boldsymbol{x}_i(t_k^-) = \sum\limits_{j=1}^{N} u_{ij}^k \Gamma^k \boldsymbol{x}_j(t_k^-), \\ \qquad t = t_k, \ k \in \mathbb{N}_+, \end{cases} \tag{39}$$

where $\boldsymbol{x}_i(t)$ displays the state variable of the i^{th} node in CDNs at time t, $\boldsymbol{A}_{n\times n}$ is a constant matrix, $\boldsymbol{\phi}, \ \boldsymbol{\Gamma} \in \mathbb{R}^{n\times n}$ are non-delayed and delayed inner coupling matrices of the CDN, describing the individual couplings between two subsystems at time t and $t-\tau_2(t)$, respectively. $\tau_l(t)$ $(l=1,2,3)$ are time delays satisfying $0<\tau_l(t)\leq\tau_l$, where τ_l $(l=1,2,3)$ are positive constants, $\boldsymbol{f}:\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n\rightarrow\mathbb{R}^n$ is a continuously vector-valued function, $\boldsymbol{G}=(g_{ij})_{N\times N}$ and $\boldsymbol{B}=(b_{ij})_{N\times N}$ are the outer coupling matrices of the whole network, and $\boldsymbol{U}^k=(u_{ij}^k)_{N\times N}$ and Γ^k $(k\in\mathbb{N}_+)$ are outer and inner couplings of the network at time t_k, respectively. Matrices $\boldsymbol{G}$, $\boldsymbol{B}$, and $\boldsymbol{U}^k$ satisfy the following diffusive conditions:

$$\begin{cases} g_{ij} \geq 0 \ (i\neq j), \ g_{ii} = -\sum\limits_{j=1,j\neq i}^{N} g_{ij}, \\ b_{ij} \geq 0 \ (i\neq j), \ b_{ii} = -\sum\limits_{j=1,j\neq i}^{N} b_{ij}, \\ u_{ij}^k \geq 0 \ (i\neq j), \ u_{ii}^k = -\sum\limits_{j=1,j\neq i}^{N} u_{ij}^k, \end{cases} \tag{40}$$

where $i,j\in\Lambda, \ k\in\mathbb{N}_+$. $\boldsymbol{\omega}(t)=(\omega_1(t),\omega_2(t),\ldots,\omega_n(t))^{\mathrm{T}}$ is an n-dimensional Weiner process defined on $(\Omega,\mathfrak{F},\{\mathfrak{F}_i\}_{t\geq0},P)$. The white noise $\mathrm{d}\omega_i(t)$ is not dependent on $\mathrm{d}\omega_j(t)$ for $i\neq j$, and $\boldsymbol{\sigma}:\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n\rightarrow\mathbb{R}^{n\times n}$ is the noise intensity function matrix. The stochastic perturbation can be viewed as a result of the random uncertainties in the nodes. The initial condition of Eq. (39) is given by $\boldsymbol{x}_i(t)=\boldsymbol{\varphi}_i(t)\in C([-\tau,0], \ \mathbb{R}^n), \ i\in\Lambda$, where $\tau=\max\{\tau_1,\tau_2,\tau_3\}$. Yang XS et al. (2011a) assumed that at least one of the matrices $\boldsymbol{G}$ and $\boldsymbol{U}^k$ is irreducible; that is, there is no isolated node in the connected graph.

Definition 10 (Yang XS et al., 2011a) The dynamical network with impulsive and stochastic effects (39) is said to be globally exponentially synchronized in mean square if there exist constants $\alpha>0$ and $\gamma>0$ such that for any initial values $E\|\boldsymbol{x}_i(t)-\boldsymbol{x}_j(t)\|^2\leq\gamma\mathrm{e}^{-\alpha t}, \ i\in\Lambda$ hold for $t\geq0$.

The following assumptions are needed:

Assumption 3 (Yang XS et al., 2011a)

(H1) Assume that there exist two nonnegative constants K and L such that $\|\boldsymbol{f}(t,\boldsymbol{x},\boldsymbol{y})-\boldsymbol{f}(t,\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})\|\leq K\|\boldsymbol{x}-\bar{\boldsymbol{x}}\|+L\|\boldsymbol{y}-\bar{\boldsymbol{y}}\|$, where $\boldsymbol{x}, \ \bar{\boldsymbol{x}}, \ \boldsymbol{y}, \ \bar{\boldsymbol{y}}\in\mathbb{R}^n, \ t\in\mathbb{R}_+$.

(H2) There exist nonnegative constants ρ_1 and ρ_2 such that $\mathrm{tr}\{[\boldsymbol{\sigma}(t,\boldsymbol{x},\boldsymbol{y})-\boldsymbol{\sigma}(t,\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})]^{\mathrm{T}}[\boldsymbol{\sigma}(t,\boldsymbol{x},\boldsymbol{y})-\boldsymbol{\sigma}(t,\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})]\}\leq\rho_1\|\boldsymbol{x}-\bar{\boldsymbol{x}}\|^2+\rho_2\|\boldsymbol{y}-\bar{\boldsymbol{y}}\|^2$, $\boldsymbol{x}, \ \bar{\boldsymbol{x}}, \ \boldsymbol{y}, \ \bar{\boldsymbol{y}}\in\mathbb{R}^n, \ t\in\mathbb{R}_+$.

Theorem 7 (Yang XS et al., 2011a) Suppose that (H1) and (H2) in Assumption 3 are satisfied, that the impulsive sequence $\varsigma=\{t_i\}_{i=1,2,\ldots}$ satisfies $\dfrac{T-t}{T_{\mathrm{a}}}-\xi\leq N_\varsigma(T,t)\leq\xi+\dfrac{T-t}{T_{\mathrm{a}}}$ with some positive integer ξ and constant $T_{\mathrm{a}}>0$, and that there exist an $(N-1)\times N$ matrix $\boldsymbol{M}\in\boldsymbol{M}_2$ and positive numbers ξ and ϵ such that inequality $a-R\left(\dfrac{L^2}{\epsilon}+\dfrac{1}{\xi}\|\tilde{\boldsymbol{B}}\|^2+\rho_2\right)>0$ holds, where $a=-\left(\lambda_{\max}(\boldsymbol{A}_1+\boldsymbol{A}_1^{\mathrm{T}}+\tilde{\boldsymbol{G}}+\tilde{\boldsymbol{G}}^{\mathrm{T}})+\xi+2K+\epsilon+\rho_1+\dfrac{\ln\theta}{T_{\mathrm{a}}}\right)$, $\lambda_{\max}(\boldsymbol{I}_{(N-1)n}+(\tilde{\boldsymbol{U}}^k)^{\mathrm{T}}(\boldsymbol{I}_{(N-1)n}+\tilde{\boldsymbol{U}}^k))=\theta_k\leq0, \ k\in\mathbb{N}_+, \ R=\max\{\theta^{-\varsigma},1,\theta^{\varsigma}\}$, θ_k and θ are constants. Then CDN (39) is globally exponentially synchronized in mean square.

Li XD and Song (2014) considered the following recurrent network:

$$\begin{cases} \mathrm{d}\boldsymbol{x}(t) = [-\boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{A}\boldsymbol{f}(\boldsymbol{x}(t)) \\ \qquad + \boldsymbol{B}\boldsymbol{f}(\boldsymbol{x}(t-\tau(t))) + \boldsymbol{J}]\mathrm{d}t, \ t>0, \\ \boldsymbol{x}(s) = \boldsymbol{\phi}(s), \ s\in[-\tau,0]. \end{cases} \tag{41}$$

The response is given by

$$\begin{cases} \mathrm{d}\boldsymbol{y}(t) = [-\boldsymbol{C}\boldsymbol{y}(t) + \boldsymbol{A}\boldsymbol{f}(\boldsymbol{y}(t)) \\ \qquad\qquad +\boldsymbol{B}\boldsymbol{f}(\boldsymbol{y}(t-\tau(t))) + \boldsymbol{J}]\mathrm{d}t \\ \qquad\qquad +\boldsymbol{\sigma}(t, \boldsymbol{e}(t), \boldsymbol{e}(t-\tau(t)))\mathrm{d}\boldsymbol{\omega}(t),\ t \in [t_{k-1}, t_k), \\ \Delta \boldsymbol{y}(t_k) = \boldsymbol{y}(t_k) - \boldsymbol{y}(t_k^-) = -\boldsymbol{H}\boldsymbol{u}(t_k^-),\ k \in \mathbb{Z}_+, \\ \boldsymbol{y}(s) = \boldsymbol{\phi}(s),\ s \in [-\tau, 0], \end{cases} \tag{42}$$

where model $\boldsymbol{\omega}(t) = (\omega_1(t), \omega_2(t), \ldots, \omega_m(t))^{\mathrm{T}}$ is an m-dimensional Brownian motion. It is defined in a complete probability space $(\Omega, \mathfrak{F}, \mathfrak{P})$. Several conditions ensure the global exponential stability of the error system, and thus the drive CDN (41) synchronizes to the response CDN (42).

Biological systems are generally difficult to model because of their complexity and nonlinearity. First, Kauffman (1969) proposed the Boolean network (BN). This was developed by Shmulevich et al. (2002) and Farrow et al. (2004). As a particular CDN with stochastics, BNs have been studied extensively. In recent years, it has been found that BNs seem especially suitable for modeling genetic regulation networks and biological systems. Moreover, BNs can be used for modeling other cellular processes. Boolean control networks with impulsive effects were studied (Xu XJ et al., 2018a, 2018b). Li BW et al. (2019) gave the robust invariant set analysis of BNs, and then discussed robust control in variance of probabilistic Boolean control networks via event-triggered control (Liu Y et al., 2017; Tong et al., 2018; Zhu QX et al., 2018, 2019; Zhu SY et al., 2018). The stability-based Lyapunov and construction of Lyapunov functions for BNs were discussed (Li HT and Wang, 2017; Li YY et al., 2019; Sun L et al., 2020). The synchronization of switched BNs and output tracking control of Boolean control networks with impulsive effects were reported in the above studies.

8 Conclusions and future work

Impulsive control for CDNs is a very important research area with wide applications. Synchronization is a typical collective dynamical behavior of CDNs. In this paper, we have presented a survey of CDNs with impulse effects involving the following topics:

1. We have introduced the CDNs and reviewed some definitions of synchronization for such systems. Moreover, we have described the general input-to-state stability, in which impulse effects can be classified as impulsive synchronization and impulsive desynchronization. Two examples have been illustrated to show the effects of impulses.

2. We have reviewed related research results for the impulsive synchronization problem. Some sufficient conditions have been established for CDNs based on the impulsive control theory. An example has been given to illustrate the existence and attractiveness of the periodic solution for neural networks with impulsive control.

3. We have reviewed the impulsive desynchronization problem for CDNs and presented some sufficient conditions for the addressed system. In addition, we have given an example to show the effects of impulsive perturbation on neural networks. From the example, we can see that a stable equilibrium becomes a periodic attractor under impulsive perturbation.

4. In view of the results obtained on impulsive effects in the last decades, we have discussed the adaptive-impulsive synchronization of CDNs, pinning control of CDNs with impulsive effects, and CDNs with stochastic and impulsive effects.

There are many possible extensions of CDNs with impulsive effects. For future work, we are considering the following:

1. The nonlinearly coupled CDN with more general connected topology structures can be considered. The dynamics of nodes are nonidentical..

2. With the development of multi-layer networks (Um et al., 2011; Rakshit et al., 2017; Mei et al., 2018; Wei et al., 2018), many impulsive control strategies can be applied, such as studying complete synchronization, intra-layer synchronization, and inter-layer synchronization. The work of CDNs with time delays and impulse effects can also be extended to multi-layer CDNs.

3. Because an impulsive control strategy is simple in structure and requires only discrete control to achieve the desired performance, event-triggered impulsive control as a combination of event-triggered control and impulsive control has received more and more attention in recent years (Tan et al., 2019). However, the existing results on event-triggered impulsive control can only be applied to some specific systems, leading to a promising research direction for neural networks.

Contributors

Xiao-di LI designed the research. Xiu-ping HAN and Yong-shun Zhao processed the data. Xiu-ping HAN drafted the manuscript. Xiao-di LI helped organize the manuscript. Yong-shun ZHAO and Xiu-ping HAN revised and finalized the manuscript.

Compliance with ethics guidelines

Xiu-ping HAN, Yong-shun ZHAO, and Xiao-di LI declare that they have no conflict of interest.

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