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Unified construction of two *n*-order circuit networks with diodes

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Abstract: In this paper, two different *n*-order topological circuit networks are connected by diodes to establish a unified network model, which is a previously unexplored problem. The network model includes not only five resistive elements but also diode devices, so the network contains many different network types. This problem can be solved through three main steps: First, the network is simplified into two different equivalent circuit models. Second, the nonlinear difference equation model is established by applying Kirchhoff's law. Finally, the two equations with similar structures are processed uniformly, and the general solutions of the nonlinear difference equations are obtained by using the transformation technique. As an example, several interesting specific results are deduced. Our study on the network model has significant value, as it can be applied to relevant interdisciplinary research.

Key words: Complex networks; Equivalent transform; Nonlinear difference equation; Equivalent resistance
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1 Introduction

In 1845, Kirchhoff established the node current law and loop voltage law as the first thorough mathematical description of electrical circuits, and laid the theoretical basis for research on large-scale circuits (Kirchhoff, 1847). Since then, more and more scholars have studied the resistor network and investigated its various forms, such as square, triangular, honeycomb, and hypercube (Aitchison, 1964; Venezian, 1994; Atkinson and van Steenwijk, 1999; Cserti, 2000). Currently, the research on resistor network models is no longer limited to the field of circuits, but has been extended to many other areas. The calculation of resistance in electric circuit theory can be used to solve many abstract and complex scientific and engineering problems, such as random walks (Doyle and Snell, 1984), first-passage processes (Redner, 2001), graphene properties (Kimouche et al., 2015), electronic plants (Stavrinidou et al., 2015), metagratings (Xu et al., 2021), reflectarray antennas (Hum and Du, 2017), and topological properties (Albert et al., 2015). Therefore, the construction and application of resistor network models have gradually become the basic solution methods for a series of scientific problems, playing an important role in the natural and engineering fields.

In the past 170 years, much progress has been made in the study of resistor network models, providing solutions for a series of resistor network problems (Brayton and Moser, 1964a, 1964b; Desoer and Wu, 1974; Bianco et al., 2000; Bianco and Giordano, 2003). Some circuits including diodes have also been discussed; for example, electrical characterization of random networks and mixtures was investigated (Bianco et al., 2000; Bianco and Giordano, 2003). General small-scale circuit problems can be summarized into solving difference equations governed by Ohm's and Kirchhoff's laws. However, when the circuit becomes a complex large-scale network structure, it is not enough to use Kirchhoff's law alone. Therefore, researchers have proposed several new methods. Cserti

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(2000) proposed the Green function technique, which opened the door to the study of infinite resistor networks. He calculated the equivalent resistance between any two points of several infinite lattice resistance structures using this method (Cserti et al., 2002). He also applied this method to the problem of a perturbed lattice, in which one of the bonds is missing (Cserti et al., 2011). Asad used this method in several networks and found it to be a useful tool for studying capacitor networks (Asad et al., 2013; Asad, 2013a, 2013b). Hijjawi et al. (2008) discussed the Green function of anisotropic diamond lattice, and the analytical properties of the Green function in all dimensions were investigated by Guttmann (2010). Giordano (2007) applied Green function technique to the study of two-dimensional anisotropic random lattices (a class of anisotropic infinite networks), which is a new research progress. Green function technique has been widely applied in research on infinite networks. As is well known, an infinite network is a kind of ideal model, yet the finite network is a problem occurring in real life. Therefore, Green function technique is not suitable for studying finite resistor networks. Wu (2004) developed a theory called the Laplace matrix method, which can calculate the resistance between arbitrary nodes for a finite lattice of resistors. Since then, this method has been further developed and applied to impedance networks (Tzeng and Wu, 2006; Izmailian et al., 2014; Essam et al., 2014, 2015). For example, corner-to-corner resistance and its asymptotic expansion for free boundary conditions were obtained by Essam and Wu (2009). Izmailian and Huang (2010) also calculated the resistance for other boundary conditions. However, the Laplace matrix method has some limitations on the finite networks with complex boundary conditions; therefore, Tan ZZ (2011, 2015a, 2015b, 2015c, 2015d, 2017) set up a new method called the recursion-transform (RT) method, and the original theory has been further improved (Tan ZZ and Tan, 2020a, 2020b, 2020c). This approach is based on recursive techniques and variable transformation techniques. He has applied this method to solve the resistor network problems of various resistor network models, such as various basic and applied research of this theory in the literature (Tan ZZ and Zhang, 2015; Tan ZZ, 2015a, 2015b, 2016; Zhou et al., 2017; Tan Z and Tan, 2018; Tan Z et al., 2018a, 2018b). Nowadays, the RT method is still applied to study n-order networks in different situations, and a large number of complex networks have been solved. For example, Tan ZZ et al. (2017) studied the multi-purpose n-order resistor network model and a series of other complex n-order circuit network models (Chen et al., 2019, 2020; Chen and Tan, 2020; Chen and Yang, 2020; Fang and Tan, 2022; Tan ZZ, 2022). The above-mentioned studies indicate that the RT method can be applied to many different types of resistor network models, and a series of interesting conclusions have been drawn.

Although many breakthroughs and much progress have been made in resistor network research, there are still several complicated resistor network models that have not been solved. In this work, two different n-order topological circuit networks (Fig. 1) are considered. On one hand, the resistor network model can be used to simulate the properties of some materials to carry out theoretical innovation research. On the other hand, the two different resistor networks above can be unified into a single network model due to the unidirectional conductivity of diodes. Although the electrical properties of the two resistor networks are different, they can be described in a unified form, which is an interesting and novel aspect of the relevant research. Even the analysis and derivation of these unified networks can be applied to LC networks. The unified network model constructed in this paper contains not only five different resistor elements but also diode devices (Fig. 2). Our study belongs to theoretical research; therefore, its main purpose is to provide a theoretical basis for future related research in different fields, such as physics, mathematics, biology, and engineering.

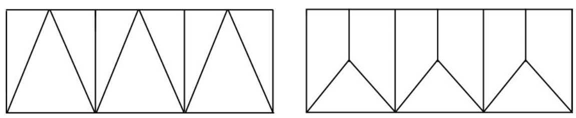

Fig. 1 Two circuit network models with different topologies

The RT method is used to study the new network model. It can be divided into four steps: step 1 is to build an equivalent circuit model, step 2 is to create a nonlinear difference equation model using Kirchhoff's laws, step 3 is to construct the method of equivalent transformation to obtain the general solution of the nonlinear difference equation, and step 4 is to discuss special situations.

Fig. 1 presents two circuit network models with two different topologies and different resistance elements distributed on the branches. The two circuits may look very similar, but they have different internal structures. In this paper, the two models in Fig. 1 are unified into the model depicted in Fig. 2.

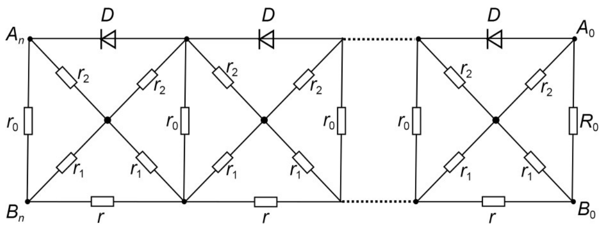

Fig. 2 A multi-functional *n*-order resistor network with X circuits and diodes, which contains five different resistor elements (including R_0) and ideal diodes in the upper boundary

Fig. 2 presents a type of n-order resistor network model with ideal diodes and X circuits. The characteristics of such a network are as follows: the number of network elements is n, the upper boundary element is an ideal diode, the element of lower boundary resistance is r, the element of resistance in the direction of vertical section is r_0, the resistors on the X-type cross line are r_1 and r_2, and the load resistor on the right end is R_0. The network is composed of five different resistance elements and a diode, which means that it is a multi-functional network. This paper focuses on the analytical expression of the resistance between A_n and B_n in the network in Fig. 1. Two equivalent resistance values are given, as well as their derivation and proof processes.

2 Total equivalent resistance formulae

In Fig. 1, let A_k be the k^{th} arbitrary node between A_0 and A_n of the upper axis (counted from the right end, the first node on the right end is A_0). B_k is the k^{th} arbitrary node between B_0 and B_n of the lower axis. The ideal diode D (meaning its absolute unidirectional conductivity) is connected to the upper boundary and X circuits in the network, and all parameters are shown in Fig. 2. The two main results are as follows:

Result 1 When the current I is input from A_n and output from B_n as shown in Fig. 2, the equivalent resistance can be written as

$$R_n(A_n \to B_n) = \frac{(R_0 - \lambda_1)F_{n+1}^{(1)} + F_{n+2}^{(1)}}{(R_0 - \lambda_1)F_n^{(1)} + F_{n+1}^{(1)}} - \frac{a_1 + r_0}{b_1}, \tag{1}$$

where

$$F_n^{(1)} = \frac{\alpha_1^n - \beta_1^n}{\alpha_1 - \beta_1}, \lambda_1 = \frac{r_0}{b_1}, \tag{2}$$

$$\begin{cases} \alpha_1 = \dfrac{1}{2b_1}\left(a_1 + 2r_0 + \sqrt{(a_1 + 4r_0 b_1)a_1}\right), \\ \beta_1 = \dfrac{1}{2b_1}\left(a_1 + 2r_0 - \sqrt{(a_1 + 4r_0 b_1)a_1}\right). \end{cases} \tag{3}$$

In addition,

$$\begin{cases} a_1 = \dfrac{r(r_2 + r_1) + 2r_1 r_2}{(r + r_1)(r_1 + r_2) + r_1 r_2}(r_1 + r_2), \\ b_1 = 1 + \dfrac{(r + 2r_1)r_0}{(r + r_1)(r_1 + r_2) + r_1 r_2}. \end{cases} \tag{4}$$

Result 2 When the current I is input from B_n and output from A_n as shown in Fig. 2, the equivalent resistance can be written as

$$R_n(B_n \to A_n) = \frac{(R_0 - \lambda_2)F_{n+1}^{(2)} + F_{n+2}^{(2)}}{(R_0 - \lambda_2)F_n^{(2)} + F_{n+1}^{(2)}} - \frac{a_2 + r_0}{b_2}, \tag{5}$$

where

$$F_n^{(2)} = \frac{\alpha_2^n - \beta_2^n}{\alpha_2 - \beta_2}, \lambda_2 = \frac{r_0}{b_2}, \tag{6}$$

$$\begin{cases} \alpha_2 = \dfrac{1}{2b_2}\left(a_2 + 2r_0 + \sqrt{(a_2 + 4r_0 b_2)a_2}\right), \\ \beta_2 = \dfrac{1}{2b_2}\left(a_2 + 2r_0 - \sqrt{(a_2 + 4r_0 b_2)a_2}\right). \end{cases} \tag{7}$$

In addition,

$$\begin{cases} a_2 = \dfrac{2rr_1(r_1 + r_2)}{2(r_1 + r_2)(r_1 + r) - r_2 r}, \\ b_2 = 1 + \dfrac{2r_0(r + 2r_1)}{2(r_1 + r_2)(r_1 + r) - r_2 r}. \end{cases} \tag{8}$$

Eqs. (1) and (5) are the first of their kind, and show an innovation in theory and methodology. These two conclusions are derived from the RT theory, which involves building equivalent models and recursive equations.

3 Equivalent models and recursive equations

According to the structural features of Fig. 2, assuming that the equivalent resistance between the two nodes of A_n and B_n at the left end of the network is R_n, the equivalent resistance between the two nodes at the left end of $n-1$ network will be R_{n-1}. Since the ideal diode D has zero forward resistance (short circuit) and infinite reverse resistance (open circuit), when the current is input from A_n and output from B_n, we can simplify Fig. 2 into a simple model shown in Fig. 3.

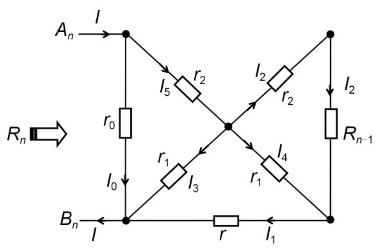

Fig. 3 Equivalent model of a two-terminal circuit network with triangular structure

Next, we establish the relationship between R_n and R_{n-1} using Kirchhoff's law. We suppose that the constant current I is input at node A_n and output at node B_n. Meanwhile, other branch currents are defined in Fig. 3.

From Fig. 3, according to Kirchhoff's law, the circuit current equations can be written as

$$\begin{cases} I_3 r_1 + I_5 r_2 - I_0 r_0 = 0, \\ I_1 r + I_4 r_1 - I_3 r_1 = 0, \\ I_2 (R_{n-1} + r_2) - I_4 r_1 = 0. \end{cases} \tag{9}$$

The node current equations can be written as

$$I_0 + I_1 + I_3 = I,\ \ I_1 = I_2 + I_4,\ \ I_0 + I_5 = I. \tag{10}$$

Then, we substitute Eq. (10) into Eq. (9), eliminate I_3, I_4, I_5, and simplify the equations as

$$\begin{cases} r_1 I_1 + (r_1 + r_2 + r_0) I_0 = (r_1 + r_2) I, \\ (r + r_1 + r_1) I_1 + r_1 I_0 - r_1 I_2 = r_1 I, \\ r_1 I_1 - (R_{n-1} + r_2 + r_1) I_2 = 0. \end{cases} \tag{11}$$

By solving Eq. (11), a current relationship is obtained:

$$\frac{I_0}{I} = \frac{\left[(r + 2r_1)(R_{n-1} + r_2 + r_1) - r_1^2\right] \dfrac{r_1 + r_2}{R_{n-1} + r_2 + r_1} - r_1^2}{\left[(r + 2r_1)(R_{n-1} + r_2 + r_1) - r_1^2\right] \dfrac{r_1 + r_2 + r_0}{R_{n-1} + r_2 + r_1} - r_1^2}. \tag{12}$$

Then, Eq. (12) is simplified, and can be written as

$$\frac{I_0}{I} = \frac{R_{n-1} + a_1}{b_1 R_{n-1} + a_1 + r_0}, \tag{13}$$

where a_1 and b_1 are given by Eq. (4). Since $R_n = U/I = I_0 r_0 / I$, using Eq. (13), one can obtain

$$R_n = \frac{R_{n-1} + a_1}{b_1 R_{n-1} + a_1 + r_0} r_0. \tag{14}$$

Eq. (14) is a simple recurrence formula that we are looking for.

The second condition is assuming that the current is input at node B_n and output at node A_n; in this case, the diode D has zero resistance (short circuit). Similar to the analysis above, we establish an equivalent model in Fig. 4, where other branch currents are defined. The relationship between R_n and R_{n-1} is established using Kirchhoff's law.

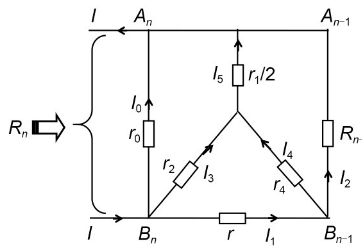

Fig. 4 Equivalent model of a two-terminal circuit network with Y circuits

From Fig. 4, according to Kirchhoff's law, the circuit current equations can be written as

$$\begin{cases} I_3 r_1 + I_5 (r_2/2) - I_0 r_0 = 0, \\ I_1 r + I_4 r_1 - I_3 r_1 = 0, \\ I_4 r_1 + I_5 (r_2/2) - I_2 R_{n-1} = 0. \end{cases} \tag{15}$$

From Fig. 4, the node current equations can be written as

$$\begin{cases} I_0 + I_1 + I_3 = I,\ I_1 = I_2 + I_4, \\ I_5 = I - I_0 - I_2. \end{cases} \tag{16}$$

We substitute Eq. (16) into Eq. (15) and eliminate I_3, I_4, I_5, to simplify the equations as

$$\begin{cases} \left(r_1 + r_0 + \frac{1}{2}r_2\right)I_0 + r_1 I_1 + \frac{1}{2}r_2 I_2 = \left(r_1 + \frac{1}{2}r_2\right)I, \\ r_1 I_0 + \left(r + 2r_1\right)I_1 - r_1 I_2 = r_1 I, \\ \frac{1}{2}r_2 I_0 - r_1 I_1 + \left(r_1 + \frac{1}{2}r_2 + R_{n-1}\right)I_2 = \frac{1}{2}r_2 I. \end{cases} \tag{17}$$

By solving Eq. (17), a current relationship can be obtained:

$$\frac{I_0}{I} = \frac{P_{n-1}(r_1 + r_2) - (r_1 + r_2 + R_{n-1})\left(r_1 + r_2 + \frac{r_2 r}{2r_1}\right)}{P_{n-1}(r_0 + r_1 + r_2) - (r_1 + r_2 + R_{n-1})\left(r_1 + r_2 + \frac{r_2 r}{2r_1}\right)}, \tag{18}$$

where $P_{n-1} = r_1 + r + r_2 + \frac{r_2 r}{2r_1} + R_{n-1}\left(\frac{r}{r_1} + 2\right)$. Further simplifying the above equation can yield

$$\frac{I_0}{I} = \frac{R_{n-1} + a_2}{b_2 R_{n-1} + a_2 + r_0}, \tag{19}$$

where a_2 and b_2 are given by Eq. (8). As $R_n = U/I = I_0 r_0/I$, substituting Eq. (19) into this equation yields

$$R_n = \frac{R_{n-1} + a_2}{b_2 R_{n-1} + a_2 + r_0} r_0. \tag{20}$$

In this way, we have deduced Eq. (20), a simple recurrence formula that is classed as a nonlinear difference equation. It is interesting to note that Eq. (14) is similar to Eq. (20) in structure, so we can deal with them together. The work that follows is to seek the general and special solution to Eqs. (14) and (20).

4 Transformation and derivation

According to the structural similarity between Eqs. (14) and (20), we rewrite the two recursive equations as

$$R_n = \frac{R_{n-1} + a}{bR_{n-1} + a + r_0} r_0. \tag{21}$$

In this way, we need only to study the solution to Eq. (21) to prove Eqs. (1) and (5). Here, we use the variable substitution method described in Tan ZZ (2011); that is, supposing that there is a sequence $\{x_n\}$, we use the following transformation relationship:

$$R_n = \frac{x_{n+1}}{x_n} - \frac{a + r_0}{b}. \tag{22}$$

The initial term can be specified as x_0=1. On using Eq. (22), one has

$$x_0 = 1, x_1 = R_0 + \frac{a + r_0}{b}. \tag{23}$$

By substituting Eq. (22) and its recurrence formula R_{n-1} into Eq. (21) and simplifying it, we can obtain

$$x_{n+1} = \frac{a + 2r_0}{b} x_n + r_0 \frac{ab - a - r_0}{b^2} x_{n-1}. \tag{24}$$

Assuming that α and β are two roots of the characteristic equation of Eq. (24), and solving the eigenvalue of Eq. (24), we can obtain the values of Eqs. (3) and (7). Therefore, from Eq. (24), we can obtain

$$x_{n+1} = (\alpha + \beta)x_n - \alpha\beta x_{n-1}. \tag{25}$$

According to the method offered in Tan ZZ (2011) to solve Eq. (25), we can deduce

$$x_n = \frac{1}{\alpha - \beta}\left[\left(x_1 - \beta x_0\right)\alpha^n - \left(x_1 - \alpha x_0\right)\beta^n\right]. \tag{26}$$

Substituting the initial term Eq. (23) into Eq. (26) yields

$$x_n = \frac{1}{\alpha - \beta}\left[\left(R_0 + \frac{a + r_0}{b} - \beta\right)\alpha^n - \left(R_0 + \frac{a + r_0}{b} - \alpha\right)\beta^n\right]. \tag{27}$$

To further simplify Eq. (27), from Eqs. (24) and (25), we can obtain

$$\alpha + \beta - \frac{a + r_0}{b} = \frac{r_0}{b} = \lambda. \tag{28}$$

By substituting Eq. (28) into Eq. (27), we can obtain

$$x_n = \frac{1}{\alpha - \beta}\left[\left(R_0 + \alpha - \lambda\right)\alpha^n - \left(R_0 + \beta - \lambda\right)\beta^n\right]. \tag{29}$$

Substituting Eq. (29) and its recurrence x_n into Eq. (22) gives

$$R_n = \frac{\left(R_0 + \alpha - \lambda\right)\alpha^{n+1} - \left(R_0 + \beta - \lambda\right)\beta^{n+1}}{\left(R_0 + \alpha - \lambda\right)\alpha^n - \left(R_0 + \beta - \lambda\right)\beta^n} - \frac{a + r_0}{b}. \tag{30}$$

By further simplifying it with function $F_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, we can obtain

$$R_n = \frac{(R_0 - \lambda)F_{n+1} + F_{n+2}}{(R_0 - \lambda)F_n + F_{n+1}} - \frac{a + r_0}{b}. \tag{31}$$

Eq. (31) is a general equivalent resistance formula, containing two different conclusions to be proved. For example, when $F_n = F_n^{(1)}$, $a=a_1$, $b=b_1$, and $\lambda=\lambda_1$, Eq. (1) can be obtained from Eq. (31); when $F_n = F_n^{(2)}$, $a=a_2$, $b=b_2$, and $\lambda=\lambda_2$, Eq. (5) can be obtained from Eq. (31).

Obviously, the above unified derivation is a theoretical and methodological innovation; it is meaningful because it has solved a profound equivalent resistance problem. Since all calculation processes are precise and rigorous, and all calculation equations are self-consistent, the conclusions drawn are necessarily correct.

5 Special cases

The network model in Fig. 2 contains five different resistor elements and n ideal diodes. Since the resistor elements are arbitrary (the resistor value can be zero or infinite), this multi-parameter network has many special cases. Several of these special cases are given below.

Case 1: In the network of Fig. 2, when $r_1=\infty$, from Eqs. (2) and (4), we can obtain

$$a_1 = r + 2r_2,\ b_1 = 1,\ \lambda_1 = r_0. \tag{32}$$

Then, using Eq. (1), the equivalent resistance is obtained:

$$R(A_n \to B_n) = \frac{(R_0 - r_0)F_{n+1}^{(1)} + F_{n+2}^{(1)}}{(R_0 - r_0)F_n^{(1)} + F_{n+1}^{(1)}} - \left(r + r_0 + 2r_2\right), \tag{33}$$

where $F_n^{(1)} = (\alpha_1^n - \beta_1^n)/(\alpha_1 - \beta_1)$, and

$$\begin{cases} \alpha_1 = \dfrac{1}{2}\left(r + 2r_2 + 2r_0 + \sqrt{(r + 2r_2 + 4r_0)(r + 2r_2)}\right), \\ \beta_1 = \dfrac{1}{2}\left(r + 2r_2 + 2r_0 - \sqrt{(r + 2r_2 + 4r_0)(r + 2r_2)}\right). \end{cases} \tag{34}$$

When $r_1=\infty$, from Eqs. (6) and (8), we can obtain

$$a_2 = r,\ b_2 = 1,\ \lambda_2 = r_0. \tag{35}$$

Then, the equivalent resistance Eq. (5) is simlified to

$$R(B_n \to A_n) = \frac{(R_0 - r_0)F_{n+1}^{(2)} + F_{n+2}^{(2)}}{(R_0 - r_0)F_n^{(2)} + F_{n+1}^{(2)}} - (r + r_0), \tag{36}$$

where $F_n^{(2)} = (\alpha_2^n - \beta_2^n)/(\alpha_2 - \beta_2)$, and

$$\begin{cases} \alpha_2 = \dfrac{1}{2}\left(r + 2r_0 + \sqrt{(r + 4r_0)r}\right), \\ \beta_2 = \dfrac{1}{2}\left(r + 2r_0 - \sqrt{(r + 4r_0)r}\right). \end{cases} \tag{37}$$

Case 2: In the network of Fig. 2, when $r_1=0$, from Eqs. (2) and (4), we can obtain

$$a_1 = r_2,\ b_1 = 1 + \frac{r_0}{r_2}, \tag{38}$$

$$\lambda_1 = \frac{r_0 r_2}{r_2 + r_0} = r_0 // r_2. \tag{39}$$

Substituting them into Eq. (3) yields

$$\alpha_1 = \frac{r_2(r_2 + 2r_0)}{r_2 + r_0},\ \beta_1 = 0. \tag{40}$$

Therefore, $F_n^{(1)}=\alpha_1^{n-1}$, then substituting it into Eq. (1) and simplifying Eq. (1), we have

$$R(A_n \to B_n) = \alpha_1 - r_2 = \frac{r_2 r_0}{r_2 + r_0}. \tag{41}$$

This is an interesting result: the equivalent resistance is independent of n, which is completely consistent with the actual situation, since the circuit network in this case is as shown in Fig. 5.

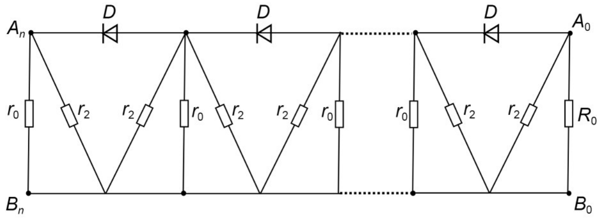

Fig. 5 An *n*-order resistor network with diodes, which contains three different resistor elements and an ideal diode at the upper boundary

When r_1=0, from Eqs. (6) and (8), we can obtain

$$a_2=0, b_2=1+\frac{2r_0}{r_2}, \lambda_2=\frac{r_0r_2}{r_2+2r_0}. \tag{42}$$

Substituting them into Eq. (7) yields

$$\alpha_2=\beta_2=\frac{r_0}{b_2}=\frac{r_0r_2}{r_2+2r_0}=\lambda_2. \tag{43}$$

Because α_2=β_2, if taking the limit, we can obtain

$$F_n^{(2)}=\lim_{\alpha_2\to\beta_2}\frac{\alpha_2^n-\beta_2^n}{\alpha_2-\beta_2}=n\alpha_2^{n-1}. \tag{44}$$

Substituting Eq. (44) into Eq. (5) gives

$$R(B_n\to A_n)=\frac{(R_0-\lambda_2)(n+1)+(n+2)\alpha_2}{(R_0-\lambda_2)n\alpha_2^{-1}+(n+1)} -\frac{r_0r_2}{r_2+2r_0}. \tag{45}$$

Then, simplifying Eq. (45) yields

$$R(B_n\to A_n)=\frac{R_0\alpha_2}{nR_0+\alpha_2}, \tag{46}$$

where α_2=$r_0r_2/(r_2+2r_0)$ is given by Eq. (43). Eq. (46) is an interesting concise result, in full agreement with the actual situation, since the circuit network in this case is as shown in Fig. 5.

Case 3: When n=0, one can verify the correctness of Eqs. (1) and (5) in this simple case; for example,

$$R_0(A_0\to B_0)=\frac{(R_0-\lambda_1)F_1^{(1)}+F_2^{(1)}}{(R_0-\lambda_1)F_0^{(1)}+F_1^{(1)}}-\frac{a_1+r_0}{b_1}, \tag{47}$$

$$R_0(B_0\to A_0)=\frac{(R_0-\lambda_2)F_1^{(2)}+F_2^{(2)}}{(R_0-\lambda_2)F_0^{(2)}+F_1^{(2)}}-\frac{a_2+r_0}{b_2}. \tag{48}$$

Since $F_0^{(s)}=0$, $F_1^{(s)}=1\,(s=1,2)$, $F_2^{(s)}=\alpha_s+\beta_s=(a_s+2r_0)/b_s$, then

$$R_0(A_0\to B_0)=(R_0-\lambda_1)+F_2^{(1)}-\frac{a_1+r_0}{b_1}=R_0, \tag{49}$$

$$R_0(B_0\to A_0)=(R_0-\lambda_2)+F_2^{(2)}-\frac{a_2+r_0}{b_2}=R_0. \tag{50}$$

Eqs. (49) and (50) show that the general results, i.e., Eqs. (1) and (5), are applicable to the case of n=0, and the actual circuit shows that $R_0(A_0, B_0)$=R_0.

Case 4: When n=1, one can verify the correctness of Eqs. (1) and (5) in this simple case; for example,

$$R_1(A_1\to B_1)=\frac{(R_0-\lambda_1)F_2^{(1)}+F_3^{(1)}}{(R_0-\lambda_1)F_1^{(1)}+F_2^{(1)}}-\frac{a_1+r_0}{b_1}, \tag{51}$$

$$R_1(B_n\to A_n)=\frac{(R_0-\lambda_2)F_{n+1}^{(2)}+F_{n+2}^{(2)}}{(R_0-\lambda_2)F_n^{(2)}+F_{n+1}^{(2)}}-\frac{a_2+r_0}{b_2}. \tag{52}$$

Since $F_1^{(s)}=1\,(s=1,2)$, $F_2^{(s)}=\alpha_s+\beta_s$=$(a_s+2r_0)/b_s$, and

$$F_3^{(s)}=(\alpha_s+\beta_s)^2-\alpha_s\beta_s=\left(\frac{a_s+2r_0}{b_s}\right)^2 +r_0\frac{a_sb_s-a_s-r_0}{b_s^2}, \tag{53}$$

substituting them into Eqs. (51) and (52) yields

$$R_1(A_1\to B_1)=\frac{R_0+a_1}{b_1R_0+a_1+r_0}r_0, \tag{54}$$

$$R_1(B_1\to A_1)=\frac{R_0+a_2}{b_2R_0+a_2+r_0}r_0, \tag{55}$$

where a_1 and b_1 are given by Eq. (4), and a_2 and b_2 are given by Eq. (8).

It is found that, if n=1 is substituted into Eqs. (14) and (20), Eqs. (54) and (55) can also be derived. Obviously, Eqs. (54) and (55) are completely consistent with the results obtained by the actual circuit calculation, which proves that the conclusion is correct when n=1.

6 Discussion and summary

In this paper, an n-order resistor network model with X circuits and diodes (Fig. 2) is proposed which has not been studied previously. The recursion-transform (RT) method is used to evaluate the equivalent resistance of this new resistor network. The equivalent models given by Eqs. (14) and (20) are established for the forward and reverse resistance of the circuit network (due to the unidirectional conductivity of the diode), respectively. Then, a unified difference model given by Eq. (21) is established by the similarity of the structure. The general solution of the unified difference equation is given by establishing a variable substitution method given by Eq. (31). The general formula of equivalent resistance is expressed as a function $F_n=(\alpha^n-\beta^n)/(\alpha-\beta)$, and a highly concise result is obtained. Since the general equivalent resistance is given in this paper, based on its formula, a series of special equivalent resistance conclusions are derived. If the variable substitution $r_i=a+\mathrm{j}b$ is used for resistance elements, it can be seen that the research methods and conclusions of this paper are also applicable to the study of complex impedance networks in Fig. 1.

The innovative research in this paper has obtained two new equivalent resistance formulae, which establish a new theoretical tool for future research on the resistor network model. The research methods and innovative ideas in this paper have theoretical and practical significance for future research-based teaching and scientific exploration.

Contributors

Xiaoyan LIN designed the research. Xiaoyan LIN and Zhizhong TAN processed the data. Xiaoyan LIN drafted the paper. Zhizhong TAN helped organize the paper. Xiaoyan LIN and Zhizhong TAN revised and finalized the paper.

Compliance with ethics guidelines

Xiaoyan LIN and Zhizhong TAN declare that they have no conflict of interest.

Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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