Impact of distance between two hubs on the network coherence of tree networks[*]

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Abstract: We study the impact of the distance between two hubs on network coherence defined by Laplacian eigenvalues. Network coherence is a measure of the extent of consensus in a linear system with additive noise. To obtain an exact determination of coherence based on the distance, we choose a family of tree networks with two hubs controlled by two parameters. Using the tree's regular structure, we obtain analytical expressions of the coherences with regard to network parameters and the network size. We then demonstrate that a shorter distance and a larger difference in the degrees of the two hubs lead to a higher coherence. With the same network size and distance, the best coherence occurs in the tree with the largest difference in the hub's degrees. Finally, we establish a correlation between network coherence and average path length and find that they behave linearly.

Key words: Consensus; Coherence; Distance; Average path length

1 Introduction

Complex networks are currently attracting an enormous amount of attention in the scientific community and have proved to be a powerful tool for characterizing complex systems (Newman, 2003; Gao W et al., 2022a, 2022b). Among the network models, deterministic networks (Comellas et al., 2000; Barabási et al., 2001; Comellas and Sampels, 2002) are based on recursive techniques and can provide exact results for the studied network metrics, e.g., degree distribution (Liu et al., 2022), random walks (Gao L et al., 2021; Zaman, 2022; Yu XD et al., 2023; Zaman and Ullah, 2023), spanning trees (Li et al., 2020; Zaman et al., 2022), and synchronizability (Zhu et al., 2021). In the studies of deter-

ministic network models, fractal networks have been widely studied, such as Apollonian networks (Andrade et al., 2005), Koch networks (Zhang ZZ et al., 2009), Sierpiński networks (Imran et al., 2017), and recursive trees (Peng et al., 2014).

As a typical collective behavior in multi-agent systems, consensus problems have gained increasing interest because of wide applications in diverse fields (Olfati-Saber and Murray, 2004; Ren et al., 2007; Yu WW et al., 2010; Lu and Liu, 2019; Hu X et al., 2021; Zhang LZ et al., 2022). Consensus describes the process by which groups of agents reach agreement on a common value or state. For example, the directions of a group of unmanned aerial vehicles (Rao and Ghose, 2014) are consistent in a formation problem. During the consensus of multi-agent systems, they may be subject to external disturbances (Wang and Liu, 2009). In this setting, the agents could not reach consensus. To characterize the system resistance to the disturbances, Patterson and Bamieh (2014) introduced the concept of network

coherence. It was proved that coherence is wholly determined by the Laplacian spectrum of the underlying network. Therefore, network coherence allows investigation of the interplay between network structures and the consensus resistance to the noise (Liu et al., 2021). For example, the fractal dimension of Vicsek fractals is related to the scaling of network coherence, but this result does not hold for a family of recursive trees (Sun et al., 2014).

Because the eigenvalues of the Laplacian matrix are often dominated by the network topology (Grone et al., 1990), it is difficult to obtain exact network coherence results quantified by the sum of the reciprocals of all nonzero Laplacian eigenvalues. For some deterministic networks (e.g., symmetric star topology networks (Gao HP et al., 2022)), great effort has been made to obtain analytical solutions of the network coherence. Hong et al. (2020) studied the relationship between coherence and the number of initial nodes and showed that a smaller number of initial nodes leads to better coherence. Dai et al. (2017) studied the first-order network coherence of weighted Cayley networks by the mean first-passage time and showed that the scalings of coherence regarding the network size obey four types of laws based on the weight factor. Later, Jing et al. (2021) chose a family of ring-trees networks and obtained analytical network coherence solutions using the recursive relations of the Laplacian eigenvalues. Yi et al. (2022) showed that the second-order consensus dynamics with random additive disturbances are related to the concept of biharmonic distance.

It is known that the hub nodes play an important role in dynamic behavior. For example, the hub nodes inhibit the outbreak of epidemics on scale-free networks (Zhang HF et al., 2010). Hu TC et al. (2022) investigated the noise consensus dynamics in a family of tree networks with two hubs and found that a larger difference in the degrees of the hub nodes leads to better coherence. In addition, Chen et al. (2023) showed that the network with one hub displays better leader-follower coherence than networks with two hubs. To our knowledge, few results involve the impact of the distance between two hubs on the studied coherence. Inspired by the above discussions, we choose a family of tree networks with two hubs as our model and investigate the impact of distance on coherence. The aim of the present work is to obtain exact solutions of the coherence

regarding the distance and to further study the relationship between coherence and average path length in an analytical form.

2 Preliminaries

2.1 Network models

To study the effect of the distance between two hubs on coherence, we introduce a parameter d measuring the distance between two hubs. The asymmetry of the trees is changed by the hub's degrees. In the following, we propose a family of tree networks with two hubs where the distance between the two hubs remains unchanged by adjusting the degrees of the two hubs. The detailed operation of this tree network model is summarized in Fig. 1.
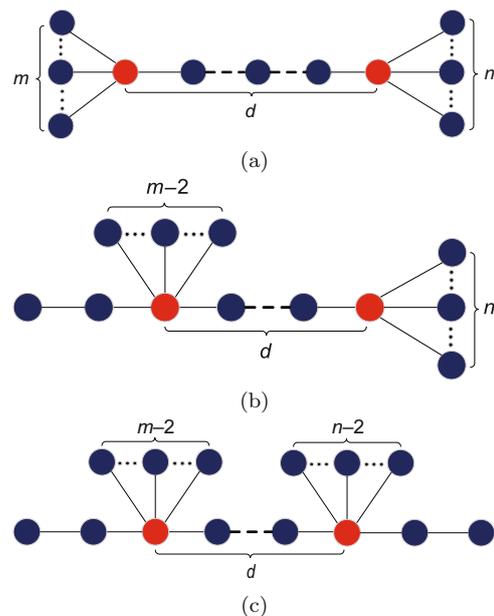

Fig. 1 A family of tree networks with the same distance d between two hubs: (a) T_1; (b) T_2; (c) T_3. The parameters m and n control the tree's asymmetry, and the network size is $N = m + n + d + 1$

2.2 Network coherence

Patterson and Bamieh (2014) proposed the concept of network coherence, which characterizes the extent of consensus resistance to external noise. The governing equation of a consensus system with additive noise is given by

$$\dot{x}(t) = -Lx(t) + \eta(t), \tag{1}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_N(t))^{\mathrm{T}}$ is the system's state, N is a positive integer denoting the network size, and $L = (L_{ij})_{N \times N}$ is the Laplacian matrix where the elements are defined as follows: off-diagonal elements $L_{ij} = -1$ if node i is adjacent to node j and 0 otherwise, and the diagonal elements L_{ii} are the nodes' degrees. $\eta(t)$ is a zero-mean, unit-variance white noise process.

When there is no noise in system (1), the system will asymptotically converge to consensus. Conversely, system (1) does not reach consensus due to the existence of noise terms. To measure the extent of consensus around the average of current node states, Patterson and Bamieh (2014) introduced the concept of network coherence, which is defined by the mean steady-state variance of the deviation from the average of all node states, that is,

$$H := \lim_{t \to \infty} \frac{1}{N} \sum_{i=1}^{N} \mathrm{var} \left\{ x_i(t) - \frac{1}{N} \sum_{k=1}^{N} x_k(t) \right\}.$$

It has been shown that the coherence H is completely determined by the Laplacian eigenvalues (Xiao et al., 2007; Bamieh et al., 2012), i.e.,

$$H = \frac{1}{2N} \sum_{i=2}^{N} \frac{1}{\lambda_i}, \tag{2}$$

where λ_i $(i = 1, 2, \ldots, N)$ are the Laplacian eigenvalues, and $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_N$.

Lemma 1 Let $p(x) = \alpha_n x^n + \ldots + \alpha_1 x + \alpha_0$ be a real polynomial of degree $n \geq 2$ with $\alpha_0 \neq 0$. We have

$$\sum_{k=1}^{n} \frac{1}{\rho_k} = -\frac{\alpha_1}{\alpha_0},$$

where ρ_k $(k = 1, 2, \ldots, n)$ are the roots of $p(x)$ (Karayannakis and Aivalis, 2018).

3 Exact calculations of network coherence

In this section, we present a method for obtaining the analytical expressions of network coherence defined in Eq. (2) of trees T_1, T_2, and T_3 in Fig. 1. H_i $(i = 1, 2, 3)$ represents the coherence of tree T_i. According to the structure of T_1, its Laplacian matrix L_1 reads as

$$L_1 = \begin{pmatrix} A_{(d+1) \times (d+1)} & B_{(d+1) \times (m+n)} \\ \left(B_{(d+1) \times (m+n)} \right)^{\mathrm{T}} & I_{(m+n) \times (m+n)} \end{pmatrix},$$

where I is an identity matrix of order $m + n$ and

$$A_{(d+1) \times (d+1)} = \begin{pmatrix} m+1 & -1 & \cdots & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & \cdots & -1 & n+1 \end{pmatrix},$$

$$B_{(d+1) \times (m+n)} = \begin{pmatrix} -1 & \cdots & -1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -1 & \cdots & -1 \end{pmatrix}.$$

To obtain the solution to Eq. (2), we need to calculate the characteristic polynomial $P_1(\lambda)$ for L_1. Using the elementary row operations on this determinant, we transform $P_1(\lambda)$ into a tridiagonal determinant and further expand $P_1(\lambda)$ by row expansion of this determinant. The detailed operations are as follows:

$$\begin{aligned} P_1(\lambda) &= |\lambda I - L_1| \\ &= (\lambda - 1)^{m+n} \overline{A}_{(d+1) \times (d+1)} \\ &= (\lambda - 1)^{m+n} \Big[a_1 b_1 U_{d-1}(\lambda) \\ &\quad - (a_1 + b_1) U_{d-2}(\lambda) + U_{d-3}(\lambda) \Big] \\ &= (\lambda - 1)^{m+n-2} \Big\{ \Big[\lambda^2 - (m+2)\lambda + 1 \Big] \\ &\quad \cdot \Big[\lambda^2 - (n+2)\lambda + 1 \Big] U_{d-1}(\lambda) \\ &\quad - \Big[2\lambda^2 - (m+n+4)\lambda + 2 \Big] (\lambda - 1) U_{d-2}(\lambda) \\ &\quad + (\lambda - 1)^2 U_{d-3}(\lambda) \Big\}, \end{aligned}$$

where $a_1 = \lambda - m - 1 - \frac{m}{\lambda - 1}$, $b_1 = \lambda - n - 1 - \frac{n}{\lambda - 1}$, and

$$\overline{A}_{(d+1) \times (d+1)} = \begin{vmatrix} a_1 & 1 & \cdots & 0 & 0 \\ 1 & \lambda - 2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda - 2 & 1 \\ 0 & 0 & \cdots & 1 & b_1 \end{vmatrix}.$$

The tridiagonal determinant $U_k(\lambda)$ of order k reads as

$$U_k(\lambda) = \begin{vmatrix} \lambda - 2 & 1 & \cdots & 0 & 0 \\ 1 & \lambda - 2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda - 2 & 1 \\ 0 & 0 & \cdots & 1 & \lambda - 2 \end{vmatrix}. \tag{3}$$

According to Eq. (3), we obtain a recursive relation in the form below:

$$U_k(\lambda) = (\lambda - 2)U_{k-1}(\lambda) - U_{k-2}(\lambda),\ k \geq 3, \quad (4)$$

where the initial conditions are $U_1(\lambda) = \lambda - 2$ and $U_2(\lambda) = \lambda^2 - 4\lambda + 3$.

Using Eq. (4) yields

$$
\begin{aligned}
U_k(0) &= -2U_{k-1}(0) - U_{k-2}(0) = (-1)^k(k+1), \\
U_k(1) &= U_{k-1}(0) - 2U_{k-1}(1) - U_{k-2}(1) \\
&= (-1)^{k+1}\frac{k(k+1)(k+2)}{6}, \\
U_k(2) &= U_{k-1}(1) - 2U_{k-1}(2) - U_{k-2}(2) \\
&= (-1)^k\frac{(k-1)k(k+1)(k+2)(k+3)}{120},
\end{aligned}
$$

where $U_k(0)$, $U_k(1)$, and $U_k(2)$ are the coefficients of the constant, first-order, and second-order terms of λ, respectively.

Because there is a zero eigenvalue of the Laplacian matrix L_1, we need to introduce a new polynomial $\overline{P}_1(\lambda) = \frac{P_1(\lambda)}{\lambda}$ to obtain all the nonzero eigenvalues. Based on Lemma 1, we determine the coefficients to obtain an exact solution of network coherence. The coefficients are

$$
\begin{aligned}
\overline{P}_1(0) &= (-1)^{m+n+d}(m+n+d+1) \\
&= (-1)^{m+n+d}N, \\
\overline{P}_1(1) &= (-1)^{m+n+d-1}\Big[\frac{1}{6}(d-2)(d-1)d \\
&\quad +\frac{1}{2}(d-1)d(m+n+2) + (m+1)(n+1)d \\
&\quad +m+n+2(d+1) + N(m+n-2)\Big].
\end{aligned}
$$

Then, expression (2) of network coherence is

$$
\begin{aligned}
H_1 &= \frac{1}{2N}\sum_{i=2}^{N}\frac{1}{\lambda_i} = -\frac{1}{2N}\frac{\overline{P}_1(1)}{\overline{P}_1(0)} \\
&= \frac{\phi_1 - d(m - \frac{N-d-1}{2})^2}{2N^2}, \quad (5)
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_1 &= \frac{1}{6}(d-2)(d-1)d + \frac{1}{2}(d-1)d(N-d+1) \\
&\quad +\frac{1}{4}d(N-d-1)^2 \\
&\quad +(N-d)(N+d-1) - (N-1).
\end{aligned}
$$

For tree T_2, we reduce the degree of the leftmost hub to prevent the distance from changing. In the

same way, we obtain coherence H_2 in the following form:

$$H_2 = \frac{\phi_2 - d(m - \frac{N-d-1}{2})^2}{2N^2}, \quad (6)$$

where

$$
\begin{aligned}
\phi_2 &= \frac{1}{6}(d-2)(d-1)d + \frac{1}{2}(d-1)d(N-d+1) \\
&\quad +\frac{1}{4}d(N-d-1)^2 \\
&\quad +(N-d)(N+d-1) - 2.
\end{aligned}
$$

Further, we reduce the degrees of both hubs to form tree T_3. Its Laplacian matrix is

$$
L_3 = \begin{pmatrix} C_{(d+5)\times(d+5)} & M_{(d+5)\times(m+n-4)} \\ \left(M_{(d+5)\times(m+n-4)}\right)^{\mathrm{T}} & I_{(m+n-4)\times(m+n-4)} \end{pmatrix},
$$

where

$$
C_{(d+5)\times(d+5)} =
\begin{pmatrix}
1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & -1 & m & -1 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & -1 & n & -1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 1
\end{pmatrix},
$$

and

$$
M_{(d+5)\times(m+n-4)} =
\begin{pmatrix}
0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
-1 & \cdots & -1 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & -1 & \cdots & -1 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0
\end{pmatrix}.
$$

The corresponding characteristic polynomial $P_3(\lambda)$ for matrix L_3 becomes

$$
\begin{aligned}
P_3(\lambda) &= |\lambda I - L_3| \\
&= (\lambda - 1)^{m+n-4}\Big\{\{a_2 b_2 \gamma \\
&\quad -(\lambda-1)^2\big[(\lambda-2)(a_2+b_2) - 1\big] \\
&\quad +(a_2+b_2)(\lambda-1)\}U_{d-1}(\lambda) \\
&\quad -\{(a_2+b_2)\gamma - \big[2(\lambda-1)^2(\lambda-2) \\
&\quad -2(\lambda-1)\big]\}U_{d-2}(\lambda) + \gamma U_{d-3}(\lambda)\Big\},
\end{aligned}
$$

where $\gamma = (\lambda-1)^2(\lambda-2)^2 - 2(\lambda-1)(\lambda-2) + 1$, $a_2 = \lambda - m - \frac{m-2}{\lambda-1}$, and $b_2 = \lambda - n - \frac{n-2}{\lambda-1}$.

Finally, we obtain a solution of coherence H_3 as follows:

$$H_3 = \frac{\phi_3 - d(m - \frac{N-d-1}{2})^2}{2N^2}, \qquad (7)$$

where

$$\begin{aligned}
\phi_3 &= \frac{1}{6}(d-2)(d-1)d + \frac{1}{2}(d-1)d(N-d+1) \\
&+ \frac{1}{4}d(N-d-1)^2 \\
&+ (N-d)(N+d-1) + N - 5.
\end{aligned}$$

4 Robustness of network coherence

This section presents our study of the robustness of coherences H_1, H_2, and H_3 regarding distance d and network parameters m and n. First, we plot the distributions of coherences H_1, H_2, and H_3 against distance d between two hubs in Fig. 2, which shows that the network coherence is better with a smaller distance. According to Eqs. (5)–(7), we further obtain

$$H_3 - H_2 = H_2 - H_1 = \frac{N-3}{2N^2}.$$

It then follows that $H_3 > H_2 > H_1$, becacuse $N = m + n + d + 1 > 3$, meaning that the coherence of tree T_1 is the best.

Second, we investigate the effect of parameters m and n on the coherence with a fixed network size N and distance d. Expression (5) gives

$$\begin{aligned}
H_1 &= \frac{\phi_1 - d(m - \frac{N-d-1}{2})^2}{2N^2} \\
&= \frac{\phi_1 - d(\frac{m-n}{2})^2}{2N^2}.
\end{aligned}$$

Because $\phi_1 > 0$, coherence H_1 achieves the maximum value $\frac{\phi_1}{2N^2}$ when $m = n$. On the other hand, H_1 reaches the minimum when $|m - n|$ is the maximum. It then shows that the larger the difference between the degrees of the two hubs, the higher the coherence. In Fig. 3, we plot the coherence change regarding parameter m, verifying the effectiveness of the above analysis.

Finally, we show the phase diagrams of coherences H_1, H_2, and H_3 in a two-dimensional space (m, n) in Fig. 4. In summary, a larger difference of parameters m and n leads to a better coherence.
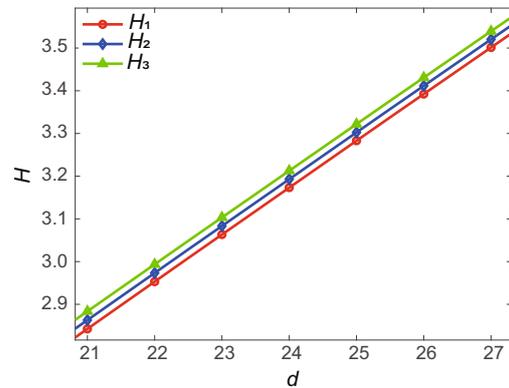

Fig. 2   Distributions of coherences $H_1, H_2,$ and H_3 against distance d with $m = 30$ and $n = 20$
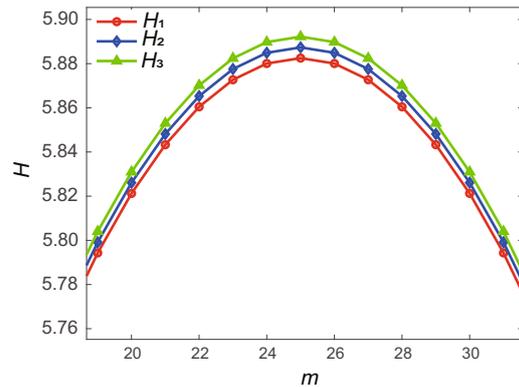

Fig. 3   Distributions of coherences $H_1, H_2,$ and H_3 against parameter m with $N = 100$, $d = 49$, and $n = 50 - m$

5 Relationship between coherence and average path length

We identify a relationship between the coherence and average path length. In network science, the average path length (APL) is defined as the average distance between any two nodes, which is a measure of the efficiency of information transfer on a network. It is defined as

$$\text{APL} = \frac{2}{N(N-1)} \sum_{i \leq j} d_{ij},$$

where d_{ij} is the shortest distance between two nodes. Using the tree's regular structure, the exact
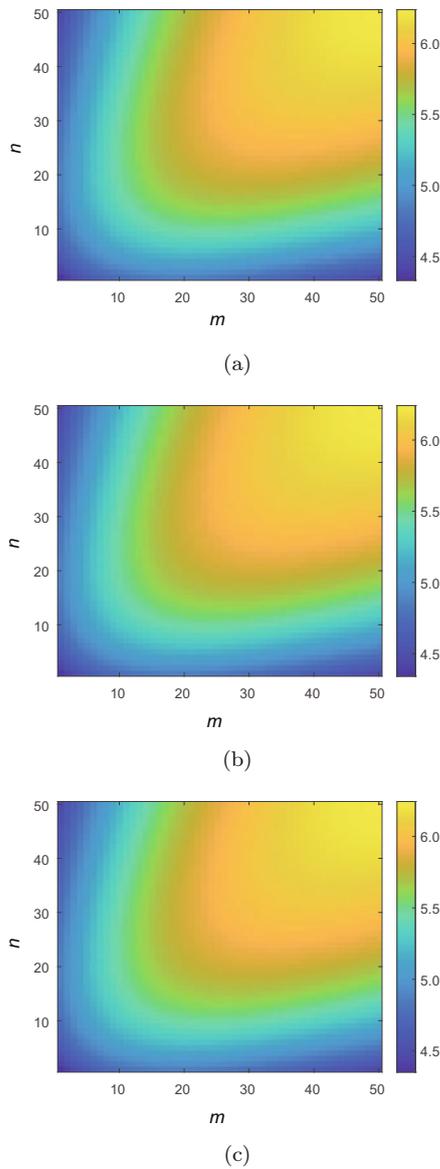
(a)

(b)

(c)

Fig. 4 Phase diagrams for coherences H_1 (a), H_2 (b), and H_3 (c) in a two-dimensional space (m, n) with $d = 49$

expressions of APL are given by

$$
\begin{aligned}
\mathrm{APL}_1 &= \frac{2}{N(N-1)}\Big[\frac{1}{2}\sum_{k=1}^{d} k(k+1) \\
&\quad + \frac{1}{2}(d+1)(d+2)(m+n) \\
&\quad + (m-1)m + (n-1)n + (d+2)mn\Big] \\
&= \frac{2}{N(N-1)}\Big[\xi_1 - d\Big(\frac{m-n}{2}\Big)^2\Big],
\end{aligned}
$$
(8)

$$
\mathrm{APL}_2 = \frac{2}{N(N-1)}\Big[\xi_2 - d\Big(\frac{m-n}{2}\Big)^2\Big], \tag{9}
$$

$$
\mathrm{APL}_3 = \frac{2}{N(N-1)}\Big[\xi_3 - d\Big(\frac{m-n}{2}\Big)^2\Big], \tag{10}
$$

where

$$
\begin{aligned}
\xi_1 &= \frac{1}{6}d(d+1)(d+2) + \frac{1}{4}d(N-d-1)^2 \\
&\quad + \frac{1}{2}(d+1)(d+2)(N-d-1) \\
&\quad + (N-d-2)(N-d-1), \\
\xi_2 &= \frac{1}{6}(d+2)(d+3)(d+4) + \frac{1}{4}d(N-d-1)^2 \\
&\quad + \frac{1}{2}(d+1)(d+2)(N-d-3) \\
&\quad + (N-d-3)(N-d+1), \\
\xi_3 &= \frac{1}{6}(d+4)(d+5)(d+6) + \frac{1}{4}d(N-d-1)^2 \\
&\quad + \frac{1}{2}(d+1)(d+2)(N-d-5) \\
&\quad + (N-d-1)(N-d) - 4(d+5).
\end{aligned}
$$

From the expressions of the coherences (Eqs. (5)–(7)) and the average path lengths (Eqs. (8)–(10)), we find that the coherence value increases with the increase of the average path length at a linear rate. When the parameters are the same $(m = n)$, the average path length is the largest, which means that the coherence is the worst. Conversely, when there is a larger difference in the hub's degrees (i.e., $|m - n|$), the average path lengths become smaller, showing that the trees display better coherence. In Fig. 5, we plot the linear relationship between coherence H_1 and average path length APL_1. Similar results hold for trees T_2 and T_3.
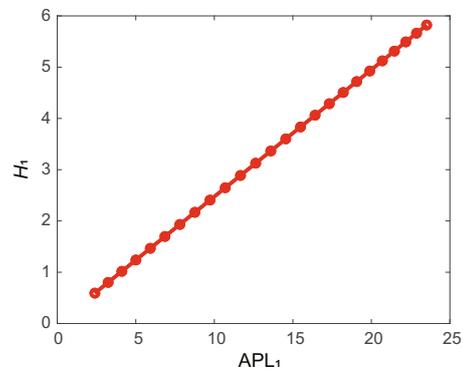

Fig. 5 Relationship between coherence H_1 and average path length APL_1 with $N = 100, m = 30$, and $n = 69 - d$

6 Conclusions

In this study, we investigated the impact of the distance between two hubs on network coherence and chose a family of tree networks as an example. The main feature of this kind of tree is that the degrees of two hubs are adjusted using two parameters to maintain the same distance between the hubs. We then obtained the analytical solutions of the coherence based on the tree's structure and found that the coherence robustness is significantly affected by the distance and the degrees of the hubs. A smaller distance between two hubs and a larger difference between the degrees will result in a higher coherence. We also studied the relationship between the coherence and average path length and showed that the tree displays higher coherence when the average distance is smaller. Because the Laplacian eigenvalues are related to the network topology, how to obtain generalization results for real-world networks is an interesting problem with technical challenges, leaving some related work to be carried out in the future.

Contributors

Weigang SUN guided the research. Daquan LI, Weigang SUN, and Hongxiang HU contributed to the conceptualization and methodology of the study. Daquan LI and Weigang SUN performed the formal analysis and drafted the paper. Weigang SUN and Hongxiang HU revised and finalized the paper.

Compliance with ethics guidelines

Daquan LI, Weigang SUN, and Hongxiang HU declare that they have no conflict of interest.

Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

References

Andrade JS, Herrmann HJ, Andrade RFS, et al., 2005. Apollonian networks: simultaneously scale-free, small world, euclidean, space filling, and with matching graphs. *Phys Rev Lett*, 94(1):018702.
https://doi.org/10.1103/PhysRevLett.94.018702

Bamieh B, Jovanovic MR, Mitra P, et al., 2012. Coherence in large-scale networks: dimension-dependent limitations of local feedback. *IEEE Trans Autom Contr*, 57(9):2235-2249.
https://doi.org/10.1109/TAC.2012.2202052

Barabási AL, Ravasz E, Vicsek T, 2001. Deterministic scale-free networks. *Phys A Stat Mech Appl*, 299(3-4):559-564. https://doi.org/10.1016/S0378-4371(01)00369-7

Chen J, Sun WG, Wang J, 2023. Topology design for leader-follower coherence in noisy asymmetric networks. *Phys Scr*, 98(1):015215.
https://doi.org/10.1088/1402-4896/aca9a3

Comellas F, Sampels M, 2002. Deterministic small-world networks. *Phys A Stat Mech Appl*, 309(1-2):231-235.
https://doi.org/10.1016/S0378-4371(02)00741-0

Comellas F, Ozón J, Peters JG, 2000. Deterministic small-world communication networks. *Inform Process Lett*, 76(1-2):83-90.
https://doi.org/10.1016/S0020-0190(00)00118-6

Dai MF, Wang XQ, Zong Y, et al., 2017. First-order network coherence and eigentime identity on the weighted Cayley networks. *Fractals*, 25(5):1750049.
https://doi.org/10.1142/S0218348X17500499

Gao HP, Zhu J, Chen X, et al., 2022. Coherence analysis of symmetric star topology networks. *Front Phys*, 10:876994. https://doi.org/10.3389/fphy.2022.876994

Gao L, Peng JH, Tang CM, 2021. Optimizing the first-passage process on a class of fractal scale-free trees. *Fract Fract*, 5(4):184.
https://doi.org/10.3390/fractalfract5040184

Gao W, Yan L, Li YF, et al., 2022a. Network performance analysis from binding number prospect. *J Amb Intell Human Comput*, 13:1259-1267.
https://doi.org/10.1007/s12652-020-02553-3

Gao W, Chen YJ, Zhang YQ, 2022b. Viewing the network parameters and *H*-factors from the perspective of geometry. *Int J Intell Syst*, 37(10):6686-6728.
https://doi.org/10.1002/int.22859

Grone R, Merris R, Sunder V, 1990. The Laplacian spectrum of a graph. *SIAM J Matr Anal Appl*, 11(2):218-238.
https://doi.org/10.1137/0611016

Hong MD, Sun WG, Liu SY, et al., 2020. Coherence analysis and Laplacian energy of recursive trees with controlled initial states. *Front Inform Technol Electron Eng*, 21(6):931-938.
https://doi.org/10.1631/FITEE.1900133

Hu TC, Li LL, Wu YQ, et al., 2022. Consensus dynamics in noisy trees with given parameters. *Mod Phys Lett B*, 36(7):2150608.
https://doi.org/10.1142/S0217984921506089

Hu X, Zhang ZF, Li CD, 2021. Consensus of multi-agent systems with dynamic join characteristics under impulsive control. *Front Inform Technol Electron Eng*, 22(1):120-133. https://doi.org/10.1631/FITEE.2000062

Imran M, Hafi S, Gao W, et al., 2017. On topological properties of Sierpinski networks. *Chaos Sol Fract*, 98:199-204.
https://doi.org/10.1016/j.chaos.2017.03.036

Jing T, Yang L, Sun WG, 2021. Exact calculations of network coherence in weighted ring-trees networks and recursive trees. *Phys Scr*, 96(8):085217.
https://doi.org/10.1088/1402-4896/ac0277

Karayannakis D, Aivalis CJ, 2018. Reciprocal Vieta-type formulas and some applications. *J Discr Math Sci Cryptogr*, 21(1):35-39.
https://doi.org/10.1080/09720529.2015.1132045

Li QS, Zaman S, Sun WT, et al., 2020. Study on the normalized Laplacian of a penta-graphene with applications. *Int J Quant Chem*, 120(9):e26154. https://doi.org/10.1002/qua.26154

Liu JB, Bao Y, Zheng WT, et al., 2021. Network coherence analysis on a family of nested weighted *n*-polygon networks. *Fractals*, 29(8):2150260. https://doi.org/10.1142/S0218348X21502601

Liu JB, Bao Y, Zheng WT, 2022. Analyses of some structural properties on a class of hierarchical scale-free networks. *Fractals*, 30(7):2250136. https://doi.org/10.1142/S0218348X22501365

Lu MB, Liu L, 2019. Leader-following consensus of second-order nonlinear multi-agent systems subject to disturbances. *Front Inform Technol Electron Eng*, 20(1):88-94. https://doi.org/10.1631/FITEE.1800611

Newman MEJ, 2003. The structure and function of complex networks. *SIAM Rev*, 45(2):167-256. https://doi.org/10.1137/S003614450342480

Olfati-Saber R, Murray RM, 2004. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Trans Autom Contr*, 49(9):1520-1533. https://doi.org/10.1109/TAC.2004.834113

Patterson S, Bamieh B, 2014. Consensus and coherence in fractal networks. *IEEE Trans Contr Netw Syst*, 1(4):338-348. https://doi.org/10.1109/TCNS.2014.2357552

Peng JH, Xiong J, Xu GA, 2014. Analysis of diffusion and trapping efficiency for random walks on non-fractal scale-free trees. *Phys A Stat Mech Appl*, 407:231-244. https://doi.org/10.1016/j.physa.2014.04.017

Rao S, Ghose D, 2014. Sliding mode control-based autopilots for leaderless consensus of unmanned aerial vehicles. *IEEE Trans Contr Syst Technol*, 22(5):1964-1972. https://doi.org/10.1109/TCST.2013.2291784

Ren W, Beard RW, Atkins EM, 2007. Information consensus in multivehicle cooperative control. *IEEE Contr Syst Mag*, 27(2):71-82. https://doi.org/10.1109/MCS.2007.338264

Sun WG, Ding QY, Zhang JY, et al., 2014. Coherence in a family of tree networks with an application of Laplacian spectrum. *Chaos*, 24(4):043112. https://doi.org/10.1063/1.4897568

Wang L, Liu ZX, 2009. Robust consensus of multi-agent systems with noise. *Sci China Ser F Inform Sci*, 52(5):824-834. https://doi.org/10.1007/s11432-009-0082-0

Xiao L, Boyd S, Kim SJ, 2007. Distributed average consensus with least-mean-square deviation. *J Parall Distrib Comput*, 67(1):33-46. https://doi.org/10.1016/j.jpdc.2006.08.010

Yi YH, Yang BJ, Zhang ZB, et al., 2022. Biharmonic distance-based performance metric for second-order noisy consensus networks. *IEEE Trans Inform Theory*, 68(2):1220-1236. https://doi.org/10.1109/TIT.2021.3127272

Yu WW, Chen GR, Cao M, 2010. Some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems. *Automatica*, 46(6):1089-1095. https://doi.org/10.1016/j.automatica.2010.03.006

Yu XD, Zaman S, Ullah A, et al., 2023. Matrix analysis of hexagonal model and its applications in global mean-first-passage time of random walks. *IEEE Access*, 11:10045-10052. https://doi.org/10.1109/ACCESS.2023.3240468

Zaman S, 2022. Spectral analysis of three invariants associated to random walks on rounded networks with 2*n*-pentagons. *Int J Comput Math*, 99(3):465-485. https://doi.org/10.1080/00207160.2021.1919303

Zaman S, Ullah A, 2023. Kemeny's constant and global mean first passage time of random walks on octagonal cell network. *Math Methods Appl Sci*, 46(8):9177-9186. https://doi.org/10.1002/mma.9046

Zaman S, Koam ANA, Khabyah AA, et al., 2022. The Kemeny's constant and spanning trees of hexagonal ring network. *CMC-Comput Mater Contin*, 73:6347-6365. https://doi.org/10.32604/cmc.2022.031958

Zhang HF, Zhang J, Zhou CS, et al., 2010. Hub nodes inhibit the outbreak of epidemic under voluntary vaccination. *New J Phys*, 12(2):023015. https://doi.org/10.1088/1367-2630/12/2/023015

Zhang LZ, Li YY, Lou JG, et al., 2022. Bipartite asynchronous impulsive tracking consensus for multi-agent systems. *Front Inform Technol Electron Eng*, 23(10):1522-1532. https://doi.org/10.1631/FITEE.2100122

Zhang ZZ, Zhou SG, Xie WL, et al., 2009. Standard random walks and trapping on the Koch network with scale-free behavior and small-world effect. *Phys Rev E*, 79(6):061113. https://doi.org/10.1103/PhysRevE.79.061113

Zhu J, Huang D, Jiang HJ, et al., 2021. Synchronizability of multi-layer variable coupling windmill-type networks. *Mathematics*, 9(21):2721. https://doi.org/10.3390/math9212721