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Robust wideband waveform design with constant modulus and discrete phase constraints for distributed precision jamming

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Abstract: Distributed precision jamming (DPJ) is a novel blanket jamming concept in electronic warfare, which delivers the jamming resource to the opponent equipment precisely and ensures that friendly devices are not affected. Robust jamming performance and low hardware burden on the jammers are crucial for practical DPJ implementation. To achieve these goals, we study the robust design of wideband constant modulus (CM) discrete phase waveform for DPJ, where the worst-case combined power spectrum (CPS) of both the opponent and friendly devices are considered in the objective function, and the CM discrete phase constraints are attached to design the wideband waveform. Specifically, the resultant mathematical model is a large-scale minimax multiobjective optimization problem (MOP) with CM and discrete phase constraints. To tackle the challenging MOP, we transform it into a single-objective minimization problem using the Lp-norm and Pareto framework. For the approximation problem, we propose the Riemannian conjugate gradient for CM discrete phase constraints (RCG-CMDPC) algorithm with low computational complexity, which leverages the complex circle manifold and a projection method to satisfy the CM discrete phase constraints within the RCG framework. Numerical examples demonstrate the superior robust DPJ effectiveness and computational efficiency compared to other competing algorithms.

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Key words: Wideband waveform design; Constant modulus; Discrete phase; Riemannian conjugate gradient; Distributed precision jamming
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1 Introduction

For most electronic devices, the quality of service (QoS) highly depends on the signal-to-interference-plus-noise ratio (SINR) in practice (Wang et al., 2018; Gao et al., 2020; Geng et al., 2023). Therefore, blanket jamming is usually used in electronic warfare to degrade the QoS of these devices (Pärlin et al., 2021; Kelsey, 2023; Wu et al., 2023). Conventional blanket jamming is typically implemented by a single jammer, where a high power is required for jamming effectiveness but results in low utilization efficiency (Tai et al., 2016). Meanwhile, the beamforming technique is also usually used to enhance the transmit gain by directing the jamming energy toward two-dimensional (2D) directions. However, due to the insufficient dimensionality in the spatial domain, it may cause unexpected injuries to adjacent friendly devices in the three-dimensional (3D) area of interest (Li et al., 2008; Blunt et al., 2016). To overcome these problems, a novel blanket jamming concept, namely, distributed precision jamming (DPJ), is proposed by the US Defense Advanced Research Project Agency (Fox, 2009). This method requires that the opponent equipment is jammed precisely in the 3D spatial domain, and the adjacent friendly devices are not affected by the jammers. Compared to the transmit beamforming technique, in this technique, a group of distributed

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aerial platforms is adopted to improve the coherent synthesis efficiency. More importantly, DPJ exhibits excellent jamming energy performance within the desired 3D area rather than along specific 2D directions (Song et al., 2016).

The existing research on DPJ falls into two distinct categories from the perspective of the transmitted signal model. One category adopts the narrowband signal model and regards the combined energy control performance as the main design metric. Chen et al. (2020) proposed an efficient method to design the single snapshot constant modulus (CM) transmit signals, and the grating lobes in the area of interest are mitigated to some extent. Yang et al. (2022) jointly optimized the CM signals and the location of distributed platforms to improve spatial energy control performance. Zhang et al. (2022) proposed a method to endow the narrowband waveform with jamming characteristics by fitting the combined waveform with white noise, and Yang et al. (2023) further improved the algorithm's performance in terms of both jamming and computational efficiency. Other research works, including those by Yang et al. (2021), Xu et al. (2022), and Zhang et al. (2022), designed the narrowband transmitted waveform to improve the DPJ performance based on previous works, and interested readers may refer to them.

The other category uses the wideband transmitted waveform for the jamming tasks, where the combined power spectrum (CPS) is considered as the design metric. Zhang et al. (2023) obtained the cross-spectral density matrices (CSDMs) of the waveform by maximizing the average CPS of the opponent equipment, and then the available wideband CM waveform was recovered by synthesizing the CSDMs and suppressing the CPS of friendly devices. To improve the robust performance and computational efficiency, Yang et al. (2024) considered the worst-case CPS performance in DPJ. In this work, they maximized the minimum CPS of the opponent's equipment, and the CPS of friendly devices was constrained under a specific threshold. Compared to narrowband waveforms, wideband waveforms are generally preferred because narrowband waveforms require precise prior frequency knowledge of the opponent equipment to execute spot jamming. Meanwhile, most previous works design the CM continuous phase waveform, which makes it possible for the hardware component to operate in a saturation regime. From a practical point of view, CM waveforms should be designed with the discrete phase due to the limited number of bits in the direct digital synthesizer (DDS).

Compared to the narrowband jamming waveform, the advantages of the wideband jamming waveform lie in the following two aspects. Due to detection errors on the working frequency of the opponent equipment, the wideband jamming waveform can cover the working frequency of the opponent equipment better, which has better robust jamming performance especially when the prior knowledge of reconnaissance is imprecise. On the other hand, it is difficult for the antijamming system to suppress all the jamming frequency components from the view of the opponent equipment, making the jamming effect more significant than the narrowband jamming waveform. Even if the opponent equipment adopts the frequency-hopping waveform, the wideband jamming waveform can cover the range of frequency hopping, whereas the narrowband jamming waveform is invalid for this scenario. Therefore, the waveform for DPJ should be wideband to cover the potential working frequency of the opponent equipment. Note that the SINR is usually considered as the design metric to evaluate the antijamming effect from the view of opponent equipment. However, the opponent equipment and the jammers are noncooperative, which means that this metric cannot be obtained by the jammer to evaluate the jamming effect and design the waveform. The DPJ system usually accounts for the combined energy directly to evaluate the jamming effect, which is proportional to the SINR. In this case, the worst-case CPS of both the opponent and friendly devices is used to measure the robust jamming performance. Moreover, the wideband waveforms are required to be with CM and discrete phase to alleviate the hardware burden. Considering the numerous spatial frequency discrete points, our problem boils down to a large-scale minimax multiobjective optimization problem (MOP) with CM and discrete phase constraints, which requires an efficient algorithm with low computational complexity to generate real-time waveforms.

Though many methods have designed the waveform with CM and discrete phase constraints in the field of MIMO radar waveform design, most of

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them are oriented toward the narrowband signal model, where the problem size is much smaller than that for the wideband design. Zhao et al. (2017) used the majorization–minimization (MM) framework to design the low-autocorrelation CM discrete phase sequence. Mohammad et al. (2021) proposed a sequential rank-one constraint relaxation method to further minimize the autocorrelation function sidelobes by jointly designing the discrete phase transmit sequence and receive filter; however, the computational complexity of solving the convex problem at each iteration is too high to tackle the large-scale optimization problem. Lu et al. (2021) designed CM narrowband finite alphabet phase code by maximizing the signal-to-noise ratio, whereby a coordinate descent (CD) algorithm was proposed to improve the computational efficiency. Yang et al. (2022) proposed an accelerated CD algorithm to design the narrowband CM discrete phase waveform for the low sidelobe beam pattern. Bu et al. (2021) proposed an inexact alternating direction penalty method framework to tackle optimization problems with quartic terms, where extra variables are introduced to split the original problem into two subproblems. However, the aforementioned methods are not suitable for our problem: the MM framework poses a high computational burden in addressing the large-scale optimization problem, while the other algorithms do not apply to the minimax MOP.

In this paper, we study the wideband CM discrete phase waveform design for DPJ. To guarantee robust jamming performance, we optimize the worst-case CPS of both the opponent and friendly devices to ensure DPJ effectiveness. The major contributions of this paper are summarized as follows:

- **A more practical wideband CM discrete phase waveform design model:** Different from previous works on the narrowband CM discrete phase waveform design (Zhao et al., 2017; Bu et al., 2021; Lu et al., 2021; Mohammad et al., 2021; Yang et al., 2022), we use the wideband signal model to design the CM discrete phase waveform for DPJ, which is more suitable for jamming and has lower requirement on the hardware components in practice (Yang et al., 2024). To guarantee robust jamming performance, we optimize the worst-case CPS performance of both the opponent and friendly devices, whereby the formulated problem boils down to a large-scale minimax MOP with CM and discrete phase constraints.
- **A novel low-complexity algorithm to tackle the optimization problem with CM and discrete phase constraints:** To tackle the resultant nonconvex NP-hard problem, we simplify the problem using the Lp-norm, and the approximation problem aims to minimize the p-power terms with CM and discrete phase constraints. Unlike the current manifold optimization, which can only deal with the problem with CM constraints (Absil et al., 2008; Manton, 2020), we propose a novel Riemannian conjugate gradient for CM discrete phase constraints (RCG-CMDPC) algorithm to address our large-scale approximation problem, where the algorithm is developed within the complex circle manifold and a projection method to satisfy both the CM and discrete phase constraints.
- **Numerical simulation insights and detailed validation:** The performance of the proposed RCG-CMDPC algorithm is demonstrated under a typical DPJ scenario. Numerical examples are implemented to show that RCG-CMDPC exhibits lower computational complexity and better jamming efficiency compared with the state-of-the-art MM algorithm, which makes it possible to generate real-time wideband CM discrete phase waveform for DPJ. Moreover, we analyze the effect of the hyperparameters on the performance of the RCG-CMDPC algorithm and give the Pareto curve of the worst-case CPSs between the opponent and friendly devices.

Notations: Vectors and matrices are denoted by lowercase and uppercase boldface letters, respectively. The i-th element of vector $\mathbf{x}$ is represented by x_i. $\mathbb{C}^N$ and $\mathbb{R}^N$ denote the sets of N-dimensional complex and real fields, respectively. $\|\mathbf{x}\|$ denotes the Euclidean norm of vector $\mathbf{x}$, and $\arg(x)$ represents the argument of complex number x. $\mathrm{Tr}(\mathbf{X})$, $\mathrm{rank}(\mathbf{X})$, and $\lambda_{\max}(\mathbf{X})$ are the trace, rank, and the maximum eigenvalue of matrix $\mathbf{X}$, respectively. Superscripts $(\cdot)^*$, $(\cdot)^T$, and $(\cdot)^H$ denote the complex conjugate, transpose, and conjugate transpose. $\mathbf{I}_M$ and $\mathbf{0}_M$ denote the $M \times M$ identity matrix and $M \times 1$ zero

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vector, respectively. □ and ⊗ represent the Hadamard and Kronecker products, respectively.

2 Problem formulation

2.1 Signal model

For the scenario of interest in this paper, the DPJ system consists of M drones as shown in Fig. 1, where the location of the m-th drone is denoted as $\mathbf{r}_m$, $m = 1, 2, \ldots, M$, with M denoting the number of drones in the DPJ system. The drones are distributed randomly in an ultrasparse manner, which means that the distance between any two drones is much greater than the half-wavelength. Suppose that the opponent equipment and friendly devices are located within the area of interest Ω, the opponent region Ω_{O} is assumed as a circle that involves the opponent equipment, and the w-th friendly device is located in the w-th circle friendly region Ω_{F}^{w}, $w = 1, 2, \ldots W$ with W being the number of friendly devices. To avoid prior information errors on the location of these devices and the ensuing need for robustness, the DPJ system should achieve high CPS in Ω_{O} for the blanket jamming action and control the CPS in Ω_{F}^{w} at a low level.

Let each jammer of the DPJ system be equipped with a single antenna, and let the m-th jammer radiate a distinct baseband signal $x_m(t)$ with bandwidth B, where the spectral support is in the interval [–B/2, B/2]. Thus, the combined signal at any point $\boldsymbol{\sigma}_z$ in the area of interest Ω can be expressed as follows (Zhang *et al*., 2023):

$$s_z(t) = \sum_{m=1}^{M} x_m\left(t - \tau_m(\boldsymbol{\sigma}_z)\right) \frac{e^{j2\pi f_0\left(t-\tau_m(\boldsymbol{\sigma}_z)\right)}}{\left\|\mathbf{r}_m - \boldsymbol{\sigma}_z\right\|_2}, \tag{1}$$

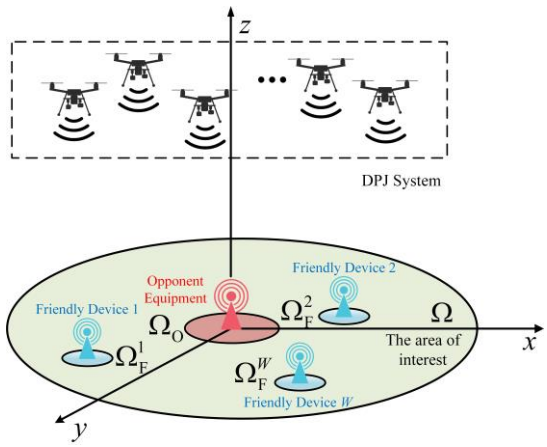

Fig. 1 The blanket jamming action implemented by the DPJ system consisting of M drones.

where f_0 is the carrier frequency, $\tau_m(\boldsymbol{\sigma}_z) = \left\|\mathbf{r}_m - \boldsymbol{\sigma}_z\right\|_2 / c$ is the propagation time, and c is the speed of light. Note that these distributed UAVs are assumed to be calibrated perfectly, i.e., the localization and synchronization errors are not taken into account, which is a typical assumption in the research on the distributed platform (Liu *et al*., 2022; Zeng *et al*., 2023; Tan *et al*., 2024). When adopting digital signal-processing techniques, the sampled baseband signal is represented by $x_m(n) = x_m(t)\big|_{t=nT_s}$ $n = 1, 2, \ldots, N$, where $T_s = 1/B$ is the sampling interval, n denotes the particular sample number, and N is the total number of samples. Thus, the discrete Fourier transform (DFT) of $x_m(n)$ is given by Eq. (2):

$$y_m(q) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} x_m(n) e^{-j2\pi\frac{(n-1)q}{N}}, \; q \in \left[-\frac{N}{2}, \frac{N}{2} - 1\right]. \tag{2}$$

To facilitate the following derivation, the sampled baseband signal $\left\{x_m(n) \middle| n = 1, 2, \ldots, N\right\}$ is vectorized by $\mathbf{x}_m = \left[x_m(1), x_m(2), \ldots, x_m(N)\right]^T$, and thus $y_m(q)$ can be rewritten as follows:

$$y_m(q) = \mathbf{x}_m^T \mathbf{f}_q, \tag{3}$$

where $\mathbf{f}_q = \left[1, e^{-j2\pi q/N}, \ldots, e^{-j2\pi(N-1)q/N}\right]^T / \sqrt{N}$. In the same manner, by implementing N-point DFT on the sampled form of Eq. (1), we deconstruct the steering vector of the spatial point $\boldsymbol{\sigma}_z$ with respect to the discrete frequency grid frequency q for the wideband signal as follows (Yang *et al*., 2024):

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$$\mathbf{a}_{\boldsymbol{\sigma}_z,q}=\left[\frac{e^{j2\pi(qB/N+f_0)\tau_1(\boldsymbol{\sigma}_z)}}{\left\|\mathbf{r}_1-\boldsymbol{\sigma}_z\right\|_2},\ldots,\frac{e^{j2\pi(qB/N+f_0)\tau_M(\boldsymbol{\sigma}_z)}}{\left\|\mathbf{r}_M-\boldsymbol{\sigma}_z\right\|_2}\right]^T. \quad (4)$$

Therefore, based on Eqs (1)–(4), the CPS at spatial frequency point $(\boldsymbol{\sigma}_z,q)$ is given by Eq. (5):

$$P_{\boldsymbol{\sigma}_z,q}=\left|\mathbf{a}_{\boldsymbol{\sigma}_z,q}{}^H\mathbf{y}_q\right|^2=\left|\mathbf{a}_{\boldsymbol{\sigma}_z,q}{}^H\mathbf{F}_q\mathbf{x}\right|^2=\mathbf{x}^H\boldsymbol{\Pi}_{\boldsymbol{\sigma}_z,q}\mathbf{x}, \quad (5)$$

where $\mathbf{y}_q=\left[y_1(q),y_2(q),\ldots,y_M(q)\right]^T$, $\mathbf{F}_q=\left(\mathbf{f}_q^T\otimes\mathbf{I}_M\right)\in\mathbb{C}^{M\times MN}$, $\mathbf{x}=\left[\mathbf{x}_1^T,\mathbf{x}_2^T,\ldots,\mathbf{x}_M^T\right]^T\in\mathbb{C}^{MN}$ and $\boldsymbol{\Pi}_{\boldsymbol{\sigma}_z,q}=\mathbf{F}_q^H\mathbf{a}_{\boldsymbol{\sigma}_z,q}\mathbf{a}_{\boldsymbol{\sigma}_z,q}^H\mathbf{F}_q$ is defined as the steering matrix of the spatial frequency point $(\boldsymbol{\sigma}_z,q)$.

2.2 Design metric and waveform constraints

To guarantee robust wideband DPJ effect, we optimize the minimum and maximum CPSs among all the spatial frequency points of the opponent equipment and friendly devices, and the corresponding design metrics are formulated as follows:

$$\begin{cases} f_1(\mathbf{x})=\min\limits_{\boldsymbol{\sigma}_\mathrm{O}\in\Omega_\mathrm{O},q_\mathrm{O}\in\Xi_\mathrm{O}}\mathbf{x}^H\boldsymbol{\Pi}_{\boldsymbol{\sigma}_\mathrm{O},q_\mathrm{O}}\mathbf{x} \\ \qquad=\max\limits_{\boldsymbol{\sigma}_\mathrm{O}\in\Omega_\mathrm{O},q_\mathrm{O}\in\Xi_\mathrm{O}}-\mathbf{x}^H\boldsymbol{\Pi}_{\boldsymbol{\sigma}_\mathrm{O},q_\mathrm{O}}\mathbf{x} \\ f_2(\mathbf{x})=\max\limits_{\boldsymbol{\sigma}_\mathrm{F}\in\Omega_\mathrm{F},q_\mathrm{F}\in\Xi_\mathrm{F}}\mathbf{x}^H\boldsymbol{\Pi}_{\boldsymbol{\sigma}_\mathrm{F},q_\mathrm{F}}\mathbf{x}, \end{cases} \quad (6)$$

where $\Omega_\mathrm{F}=\Omega_\mathrm{F}^1\cup\Omega_\mathrm{F}^2\cup\cdots\cup\Omega_\mathrm{F}^\mathrm{W}$ is the union of all the W friendly regions, and $\boldsymbol{\sigma}_\mathrm{O}$ and $\boldsymbol{\sigma}_\mathrm{F}$ represent the discrete spatial points in Ω_O and Ω_F, respectively. Meanwhile, q_O and q_F denote the discrete frequency points in the frequency bands Ξ_O and Ξ_F of opponent and friendly devices, respectively.

Next, we further give the constraints that the waveform $\mathbf{x}$ should satisfy. To avoid waveform distortion when the amplifiers work at the saturation condition (Aubry *et al*., 2020; Zhang *et al*., 2020; Fan *et al*., 2024) and to adhere to the sufficient utilization of the limited jamming resource (Qiu *et al*., 2023; Yang *et al*., 2023; Shi *et al*., 2024), we impose the CM constraint on the waveform $\mathbf{x}$, which is given by Eq. (7):

$$\left|x_i\right|=1,\ i=1,2,\ldots,MN. \quad (7)$$

For the phase part of waveform $\mathbf{x}$, a limited number of bits are available in the DDS (Zhang *et al*., 2020; Song *et al*., 2022), and thus the finite alphabet, i.e., discrete phase, constraint is enforced on $\mathbf{x}$ as follows:

$$\arg(x_i)\in\Psi_V=\frac{2\pi}{V}\left[0,1,\ldots,V-1\right],\ i=1,2,\ldots,MN, \quad (8)$$

where $V=2^v$ is the cardinality of the finite alphabet, and v is the number of bits in the DDS. In general, as v increases, it allows for more precise phases and a higher degree of freedom in the waveform $\mathbf{x}$. However, the DDS incurs a heavier hardware burden and higher cost on the jammer. It needs to be emphasized that the actual DDS cannot generate the waveform with an arbitrary (continuous) phase. Meanwhile, directly performing the quantization of continuous phase waveform devised by the above metric and CM constraints does not guarantee a desirable solution, which may trap the final waveform into local optimality or even cause severe performance loss in practice (Fan *et al*., 2021; Mani *et al*., 2021; Yang *et al*., 2022). Thus, it is of great significance to attach the waveform with discrete phase constraints as presented in Eq. (8).

2.3 Optimization problem

Based on the design metrics and waveform constraints, we formulate the minimax MOP as follows:

$$\mathrm{P}\begin{cases} \min\limits_{\mathbf{x}}\ \max\limits_{\boldsymbol{\sigma}_\mathrm{O}\in\Omega_\mathrm{O},q_\mathrm{O}\in\Xi_\mathrm{O}}-\mathbf{x}^H\boldsymbol{\Pi}_{\boldsymbol{\sigma}_\mathrm{O},q_\mathrm{O}}\mathbf{x} \\ \min\limits_{\mathbf{x}}\ \max\limits_{\boldsymbol{\sigma}_\mathrm{F}\in\Omega_\mathrm{F},q_\mathrm{F}\in\Xi_\mathrm{F}}\mathbf{x}^H\boldsymbol{\Pi}_{\boldsymbol{\sigma}_\mathrm{F},q_\mathrm{F}}\mathbf{x} \\ \text{s.t.}\ \left|x_i\right|=1,\ i=1,2,\ldots,MN \\ \qquad \arg(x_i)\in\Psi_V,\ i=1,2,\ldots,MN \end{cases} \quad (9)$$

It is seen that P is a nonconvex, nonsmooth, and NP-hard problem due to the objective function and constraints, which makes it challenging to obtain the solution directly. Meanwhile, the abundant discrete points in the spatial and frequency domains result in a large-scale optimization problem, which causes great trouble in real-time waveform generation.

3 Algorithm design

In this section, we propose the RCG-CMDPC algorithm to obtain the wideband CM discrete phase waveform for robust DPJ effect. First, we use L_p ap-

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proximation and the Pareto optimization framework to transform the original minimax MOP P into a single-objective minimization problem with CM and discrete phase constraints. Then, the transformed problem is tackled by our proposed RCG-CMDPC algorithm, wherein the complex circle manifold is used, and the corresponding retraction methods are modified to satisfy the CM discrete phase constraints within the RCG framework. Finally, we provide the computational complexity analysis of the proposed RCG-CMDPC algorithm.

3.1 The problem transformation on P

Inspired by the research on designing waveforms with low autocorrelation sidelobes (Song *et al*., 2016), we approximate the design metrics $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ using the L$_p$-norm to simplify the original problem P. The inner term is required to be positive when applying the L$_p$-norm approximation. Thus, we transform the first term regarding $f_1(\mathbf{x})$ to a strictly positive term $C-f_1(\mathbf{x})$ by introducing the upper bound C of $f_1(\mathbf{x})$, where the maximization on $-f_1(\mathbf{x})$ is equivalent to that on $C-f_1(\mathbf{x})$. In this case, we have the expression of C as follows:

$$C=\max_{\boldsymbol{\sigma}_{\mathrm{O}}\in\Omega_{\mathrm{O}},q_{\mathrm{O}}\in\Xi_{\mathrm{O}}}\lambda_{\max}(\boldsymbol{\Pi}_{\boldsymbol{\sigma}_{\mathrm{O}},q_{\mathrm{O}}})MN\geq\max_{\boldsymbol{\sigma}_{\mathrm{O}}\in\Omega_{\mathrm{O}},q_{\mathrm{O}}\in\Xi_{\mathrm{O}}}\mathbf{x}^H\boldsymbol{\Pi}_{\boldsymbol{\sigma}_{\mathrm{O}},q_{\mathrm{O}}}\mathbf{x}. \tag{10}$$

The inequality holds true by the fact that $\mathbf{x}^H\boldsymbol{\Pi}_{\boldsymbol{\sigma}_{\mathrm{O}},q_{\mathrm{O}}}\mathbf{x}\leq\mathbf{x}^H\lambda_{\max}(\boldsymbol{\Pi}_{\boldsymbol{\sigma}_{\mathrm{O}},q_{\mathrm{O}}})\mathbf{I}_{MN}\mathbf{x}=\lambda_{\max}(\boldsymbol{\Pi}_{\boldsymbol{\sigma}_{\mathrm{O}},q_{\mathrm{O}}})MN$. As discussed by Yang *et al*. (2024), the constant C has the closed-form expression as follows:

$$C=\max_{\boldsymbol{\sigma}_{\mathrm{O}}\in\Omega_{\mathrm{O}}}\sum_{m=1}^{M}\frac{MN^2}{\left\|\mathbf{r}_m-\boldsymbol{\sigma}_{\mathrm{O}}\right\|^2}. \tag{11}$$

With the constant C at hand, the original problem P is recast as follows:

$$\mathrm{P}\begin{cases}\min\limits_{\mathbf{x}}\ \max\limits_{\boldsymbol{\sigma}_{\mathrm{O}}\in\Omega_{\mathrm{O}},q_{\mathrm{O}}\in\Xi_{\mathrm{O}}}C-\mathbf{x}^H\boldsymbol{\Pi}_{\boldsymbol{\sigma}_{\mathrm{O}},q_{\mathrm{O}}}\mathbf{x}\\ \min\limits_{\mathbf{x}}\ \max\limits_{\boldsymbol{\sigma}_{\mathrm{F}}\in\Omega_{\mathrm{F}},q_{\mathrm{F}}\in\Xi_{\mathrm{F}}}\mathbf{x}^H\boldsymbol{\Pi}_{\boldsymbol{\sigma}_{\mathrm{F}},q_{\mathrm{F}}}\mathbf{x}\\ \text{s.t.}\ \ |x_i|=1,\ i=1,2,\ldots,MN\\ \qquad \arg(x_i)\in\Psi_V,\ i=1,2,\ldots,MN.\end{cases} \tag{12}$$

where the equivalence between Eqs (9) and (12) results from the identical solution for the maximization of $-\mathbf{x}^H\boldsymbol{\Pi}_{\boldsymbol{\sigma}_{\mathrm{O}},q_{\mathrm{O}}}\mathbf{x}$ and $C-\mathbf{x}^H\boldsymbol{\Pi}_{\boldsymbol{\sigma}_{\mathrm{O}},q_{\mathrm{O}}}\mathbf{x}$. By using the L$_p$-norm on the inner maximization operation, we obtain the approximated MOP as follows:

$$\mathrm{P}'\begin{cases}\min\limits_{\mathbf{x}}\left(\sum\limits_{\boldsymbol{\sigma}_{\mathrm{O}}\in\Omega_{\mathrm{O}}}\sum\limits_{q_{\mathrm{O}}\in\Xi_{\mathrm{O}}}\left(C-\mathbf{x}^H\boldsymbol{\Pi}_{\boldsymbol{\sigma}_{\mathrm{O}},q_{\mathrm{O}}}\mathbf{x}\right)^p\right)^{1/p}\\ \min\limits_{\mathbf{x}}\left(\sum\limits_{\boldsymbol{\sigma}_{\mathrm{F}}\in\Omega_{\mathrm{F}}}\sum\limits_{q_{\mathrm{F}}\in\Xi_{\mathrm{F}}}\left(\mathbf{x}^H\boldsymbol{\Pi}_{\boldsymbol{\sigma}_{\mathrm{F}},q_{\mathrm{F}}}\mathbf{x}\right)^p\right)^{1/p}\\ \text{s.t.}\ \ |x_i|=1,\ i=1,2,\ldots,MN\\ \qquad \arg(x_i)\in\Psi_V,\ i=1,2,\ldots,MN\end{cases} \tag{13}$$

where the Lp-norm tends to be the original maximization (L$_\infty$-norm) problem when $p\to\infty$. As $(\cdot)^{1/p}$ is a monotonic function in the domain of definition, the power of $1/p$ in the two objective functions can be ignored, and the approximated MOP is equivalently represented by the following expression:

$$\mathrm{P}'\begin{cases}\min\limits_{\mathbf{x}}\sum\limits_{\boldsymbol{\sigma}_{\mathrm{O}}\in\Omega_{\mathrm{O}}}\sum\limits_{q_{\mathrm{O}}\in\Xi_{\mathrm{O}}}\left(C-\mathbf{x}^H\boldsymbol{\Pi}_{\boldsymbol{\sigma}_{\mathrm{O}},q_{\mathrm{O}}}\mathbf{x}\right)^p\\ \min\limits_{\mathbf{x}}\sum\limits_{\boldsymbol{\sigma}_{\mathrm{F}}\in\Omega_{\mathrm{F}}}\sum\limits_{q_{\mathrm{F}}\in\Xi_{\mathrm{F}}}\left(\mathbf{x}^H\boldsymbol{\Pi}_{\boldsymbol{\sigma}_{\mathrm{F}},q_{\mathrm{F}}}\mathbf{x}\right)^p\\ \text{s.t.}\ \ |x_i|=1,\ i=1,2,\ldots,MN\\ \qquad \arg(x_i)\in\Psi_V,\ i=1,2,\ldots,MN\end{cases} \tag{14}$$

To proceed, we apply the Pareto framework to transform P' to the following single-objective minimization problem:

$$\tilde{\mathrm{P}}\begin{cases}\min\limits_{\mathbf{x}}\ f_\rho(\mathbf{x})=\rho\sum\limits_{\boldsymbol{\sigma}_{\mathrm{O}}\in\Omega_{\mathrm{O}}}\sum\limits_{q_{\mathrm{O}}\in\Xi_{\mathrm{O}}}\left(C-\mathbf{x}^H\boldsymbol{\Pi}_{\boldsymbol{\sigma}_{\mathrm{O}},q_{\mathrm{O}}}\mathbf{x}\right)^p\\ \qquad\qquad+(1-\rho)\sum\limits_{\boldsymbol{\sigma}_{\mathrm{F}}\in\Omega_{\mathrm{F}}}\sum\limits_{q_{\mathrm{F}}\in\Xi_{\mathrm{F}}}\left(\mathbf{x}^H\boldsymbol{\Pi}_{\boldsymbol{\sigma}_{\mathrm{F}},q_{\mathrm{F}}}\mathbf{x}\right)^p\\ \text{s.t.}\ \ |x_i|=1,\ i=1,2,\ldots,MN\\ \qquad \arg(x_i)\in\Psi_V,\ i=1,2,\ldots,MN.\end{cases} \tag{15}$$

where $\rho\in[0,1]$ is the Pareto weight that is determined by the practical trade-offs between two objectives.

3.2 The proposed RCG-CMDPC algorithm for $\tilde{\mathrm{P}}$

In this subsection, we propose a novel RCG-CMDPC algorithm to tackle the transformed minimization problem $\tilde{\mathrm{P}}$ within the RCG framework. The geometrical structure of the complex circle

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manifold makes it possible to avoid some saddle points and achieve considerable convergence speed, and interested readers may refer to Manton (2020). In essence, manifold optimization is a gradient-based method in a particular space, where the original constrained problem is considered an unconstrained problem within the specific manifold.

3.2.1 Complex circle manifold and Riemannian metric

To make the description clear, we present the geometric interpretations of the complex circle manifold and the projection to the CM discrete phase constraints in Fig. 2, where the blue spherical surface denotes the complex circle manifold, and the red points are the discrete phases. As the modulus and phase parts for each x_i are independent, we give the set of points satisfying the CM constraints consisting of the proposed manifold M as follows:

$$\mathrm{M} = \left\{ \mathbf{x} \in \mathbb{C}^{MN} \,\middle|\, |x_i| = 1,\ i = 1, 2, \ldots, MN \right\}. \tag{16}$$

For the discrete phase constraints, we quantize the solution at each iteration by a projection method, which is illustrated in the following section, and thus the manifold optimization can be implemented by temporarily leaving out the discrete phase constraints. In this case, the constrained problem $\tilde{\mathrm{P}}$ is viewed as an unconstrained problem over the manifold M.

Compared with the Euclidean space, M is a particular space endowed with a local Euclidean structure, namely, any point $\mathbf{x} \in \mathrm{M}$ only can be implemented with Euclidean operation locally around itself. Suppose that $\mathbf{x}^{(l)}$ is the obtained point at the l-th iteration; we introduce the tangent space $\mathrm{T}_{\mathbf{x}^{(l)}}\mathrm{M}$, which contains all the tangent vectors of $\mathbf{x}^{(l)}$, and we represent $\mathrm{T}_{\mathbf{x}^{(l)}}\mathrm{M}$ by the light yellow surface in Fig. 2. To update $\mathbf{x}^{(l+1)}$ within the RCG framework, we need to specify a Riemannian metric for M, which defines the product operation in the manifold. One common choice is the following Euclidean inner product (Absil *et al*., 2008):

$$\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = \mathrm{Re}\left\{ \boldsymbol{\xi}^H \boldsymbol{\eta} \right\}, \tag{17}$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are the tangent vectors on $\mathrm{T}_{\mathbf{x}^{(l)}}\mathrm{M}$.

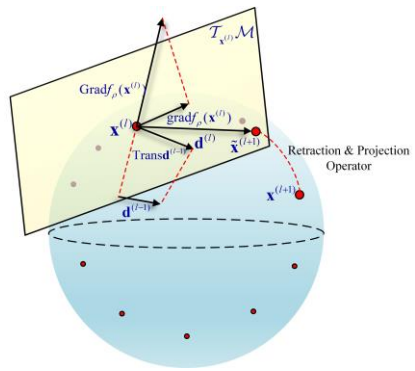

Fig. 2 The geometric interpretations of the complex circle manifold M, where the red points appear on the spherical surface of the complex circle and possess discrete phases.

3.2.2 Computation of the Riemannian gradient

To implement Euclidean operation on the current point $\mathbf{x}^{(l)}$, the corresponding Riemannian gradient $\mathrm{grad}f_\rho(\mathbf{x}^{(l)})$ is defined as a tangent vector through $\mathbf{x}^{(l)}$, which is the unique element with the fastest decrease of $f_\rho(\mathbf{x}^{(l)})$ such that

$$\left\langle \boldsymbol{\xi}, \mathrm{grad}f_\rho(\mathbf{x}^{(l)}) \right\rangle = \mathrm{D}f_\rho(\mathbf{x}^{(l)})[\boldsymbol{\xi}], \tag{18}$$

where $\mathrm{D}f_\rho(\mathbf{x}^{(l)})[\boldsymbol{\xi}]$ is the directional derivative of $f_\rho(\mathbf{x}^{(l)})$ at $\mathbf{x}^{(l)}$ in the direction $\boldsymbol{\xi}$. Therefore, the Riemannian gradient $\mathrm{Grad}f_\rho(\mathbf{x}^{(l)})$ is regarded as the orthogonal projection of its Euclidean gradient $\mathrm{Grad}f_\rho(\mathbf{x}^{(l)})$ to $\mathrm{T}_{\mathbf{x}^{(l)}}\mathrm{M}$, as shown in Fig. 2. As discussed by Absil *et al*. (2008), the generalized projection operator of a point $\mathbf{u}$ to the tangent space of $\mathbf{v}$ is given by Eq. (19):

$$\mathrm{Proj}_{\mathrm{T}_{\mathbf{v}}\mathrm{M}}(\mathbf{u}) = \mathbf{u} - \mathbf{v} \cdot \mathrm{Re}\left\{ \mathbf{u} \cdot \mathbf{v}^* \right\}. \tag{19}$$

Thus, the Riemannian gradient $\mathrm{grad}f_\rho(\mathbf{x}^{(l)})$ is represented by

$$\begin{aligned} \mathrm{grad}f_\rho(\mathbf{x}^{(l)}) &= \mathrm{Proj}_{\mathrm{T}_{\mathbf{x}^{(l)}}\mathrm{M}}\left(\mathrm{Grad}f_\rho(\mathbf{x}^{(l)})\right) \\ &= \mathrm{Grad}f_\rho(\mathbf{x}^{(l)}) - \mathbf{x}^{(l)} \cdot \mathrm{Re}\left\{ \mathrm{Grad}f_\rho(\mathbf{x}^{(l)}) \cdot \mathbf{x}^{(l)*} \right\}. \end{aligned} \tag{20}$$

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where $\mathrm{Grad}f_\rho(\mathbf{x}^{(l)})$ is derived as follows:

$$\mathrm{Grad}f_\rho(\mathbf{x}^{(l)}) = -2\rho \sum_{\boldsymbol{\sigma}_O \in \Omega_O} \sum_{q_O \in \Xi_O} p(C - \mathbf{x}^{(l)H} \mathbf{\Pi}_{\boldsymbol{\sigma}_O, q_O} \mathbf{x}^{(l)})^{p-1} \left(\mathbf{\Pi}_{\boldsymbol{\sigma}_O, q_O} \mathbf{x}^{(l)}\right)$$
$$+2(1-\rho) \sum_{\boldsymbol{\sigma}_V \in \Omega_V} \sum_{q_V \in \Xi_V} p\left(\mathbf{x}^{(l)H} \mathbf{\Pi}_{\boldsymbol{\sigma}_V, q_V} \mathbf{x}^{(l)}\right)^{p-1} \left(\mathbf{\Pi}_{\boldsymbol{\sigma}_V, q_V} \mathbf{x}^{(l)}\right). \tag{21}$$

3.2.3 The RCG framework and projection to the CM discrete phase constraints

To proceed, we adopt the RCG framework to update $\mathbf{x}^{(l+1)}$. Similar to the conjugate gradient algorithm in Euclidean space, the RCG algorithm combines with the last descent direction $\mathbf{d}^{(l-1)}$ for the current update direction $\mathbf{d}^{(l)}$, which is given as follows:

$$\mathbf{d}^{(l)} = -\mathrm{grad}f_\rho(\mathbf{x}^{(l)}) + \mu^{(l)} \mathrm{Trans}\mathbf{d}^{(l-1)}. \tag{22}$$

Compared to the common conjugate gradient algorithm in Euclidean space, the last descent direction $\mathbf{d}^{(l-1)}$ in the RCG algorithm needs be projected to $\mathrm{T}_{\mathbf{x}^{(l)}}\mathrm{M}$ for the local Euclidean operations around $\mathbf{x}^{(l)}$. Thus, $\mathrm{Trans}\mathbf{d}^{(l-1)}$ in Eq. (22) is expressed as follows:

$$\mathrm{Trans}\mathbf{d}^{(l-1)} = \mathrm{Proj}_{\mathrm{T}_{\mathbf{x}^{(l)}}\mathrm{M}}(\mathbf{d}^{(l-1)})$$
$$= \mathbf{d}^{(l-1)} - \mathbf{x}^{(l)} \square \mathrm{Re}\{\mathbf{d}^{(l-1)} \square \mathbf{x}^{(l)*}\}, \tag{23}$$

and $\mu^{(l)}$ is the Polak–Ribière parameter for the generalized conjugate gradient algorithm such that (Boumal, 2024)

$$\mu^{(l)} = \frac{\left\langle \mathrm{grad}f_\rho(\mathbf{x}^{(l)}), \mathrm{grad}f_\rho(\mathbf{x}^{(l)}) - \mathrm{Proj}_{\mathrm{T}_{\mathbf{x}^{(l)}}\mathrm{M}}\left(\mathrm{grad}f_\rho(\mathbf{x}^{(l-1)})\right)\right\rangle}{\left\langle \mathrm{grad}f_\rho(\mathbf{x}^{(l-1)}), \mathrm{grad}f_\rho(\mathbf{x}^{(l-1)})\right\rangle}. \tag{24}$$

For the clear derivation, we introduce the following updated intermediate variable to represent the updated vector on the tangent space $\mathrm{T}_{\mathbf{x}^{(l)}}\mathrm{M}$:

$$\tilde{\mathbf{x}}^{(l+1)} = \mathbf{x}^{(l)} + \zeta^{(l)} \mathbf{d}^{(l)} \tag{25}$$

where $\zeta^{(l)}$ is the step size of the descent direction, which is determined by the backtracking Armijo line-search method, and the corresponding expression is given subsequently.

In what follows, we introduce the retraction operation and the projection method, which ensure that the updated point satisfies the CM discrete phase constraints. Differing from the retraction that only operates the modulus of $\mathbf{x}^{(l)}$ in the complex circle manifold, we define the retraction and projection operator $\mathrm{R}(\cdot)$, which projects both the modulus and the phase parts of $\mathbf{x}^{(l)}$, and the updated $\mathbf{x}^{(l+1)}$ is expressed as follows:

$$\begin{cases} \mathbf{x}^{(l+1)} = \mathrm{R}\left(\tilde{\mathbf{x}}^{(l+1)}\right) = \left[e^{j\hat{\varphi}_1}, e^{j\hat{\varphi}_2}, ..., e^{j\hat{\varphi}_{MN}}\right]^T \\ \hat{\varphi}_i = \dfrac{2\pi}{V}\left\lfloor \dfrac{\arg(\hat{x}_i^{(l+1)})}{2\pi / V} \right\rfloor \end{cases} \tag{26}$$

with $\lfloor \cdot \rfloor$ denoting the biggest integer not larger than itself. This retraction and projection method ensures that $\tilde{\mathbf{x}}^{(l+1)}$ is projected to satisfy the CM discrete phase constraints at each iteration within the RCG framework.

Then, the expression of the parameter $\zeta^{(l)}$ in Eq. (25) is as follows (Absil *et al*., 2008):

$$\begin{cases} \zeta^{(l)} = \alpha^k \beta \\ f_\rho(\mathbf{x}^{(l)}) - f_\rho\left(\Re(\tilde{\mathbf{x}}^{(l+1)})\right) \geq -\delta \left\langle \mathrm{grad}f_\rho(\mathbf{x}^{(l)}), \zeta^{(l)} \mathbf{d}^{(l)} \right\rangle \end{cases} \tag{27}$$

where $\alpha = 0.5$, $\beta = 1/\left\|\mathrm{grad}f_\rho \mathbf{x}^{(l)}\right\|$, and $\delta = 0.5$ are a group of classic parameters in the Armijo line-search method, and k is the smallest nonnegative integer such that Eq. (27) is satisfied.

Finally, we summarize the RCG-CMDPC algorithm for our problem $\tilde{\mathrm{P}}$ in Algorithm 1, and it terminates when the absolute variation of $f_\rho(\mathbf{x})$ satisfies the stopping criterion.

3.3 Complexity analysis of the proposed RCG-CMDPC algorithm

Though most of the update procedures in Algorithm 1 involve projection operations, the computational complexity remains linear with $O(MN)$ due to the selected Riemannian metric. However, the computational complexity of $\mathrm{grad}f_\rho(\mathbf{x}^{(l)})$ and $\zeta^{(l)}$ is high due to the costly computation of $\mathrm{Grad}f_\rho(\mathbf{x}^{(l)})$ and $f_\rho(\mathbf{x}^{(l)})$, respectively. To alleviate the computational burden on the algorithm, we give an equivalent

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computation of $f_\rho(\mathbf{x}^{(l)})$ as follows:

$$\begin{aligned} f_\rho(\mathbf{x}^{(l)}) &= f_\rho(\mathbf{X}^{(l)}) \\ &= \rho \sum_{\boldsymbol{\sigma}_{\mathrm{O}} \in \Omega_{\mathrm{O}}} \sum_{q_{\mathrm{O}} \in \Xi_{\mathrm{O}}} \left(C - \mathbf{f}_{q_{\mathrm{O}}}^H \mathbf{X}^{(l)H} \mathbf{a}_{\boldsymbol{\sigma}_{\mathrm{O}}, q_{\mathrm{O}}} \mathbf{a}_{\boldsymbol{\sigma}_{\mathrm{O}}, q_{\mathrm{O}}}^H \mathbf{X}^{(l)} \mathbf{f}_{q_{\mathrm{O}}} \right)^p \\ &\quad + (1-\rho) \sum_{\boldsymbol{\sigma}_{\mathrm{F}} \in \Omega_{\mathrm{F}}} \sum_{q_{\mathrm{F}} \in \Xi_{\mathrm{F}}} \left(\mathbf{f}_{q_{\mathrm{F}}}^H \mathbf{X}^{(l)H} \mathbf{a}_{\boldsymbol{\sigma}_{\mathrm{F}}, q_{\mathrm{F}}} \mathbf{a}_{\boldsymbol{\sigma}_{\mathrm{F}}, q_{\mathrm{F}}}^H \mathbf{X}^{(l)} \mathbf{f}_{q_{\mathrm{F}}} \right)^p \end{aligned} \tag{28}$$

where $\mathbf{X}^{(l)} \in \mathbb{C}^{M \times N}$ is the matrix form of $\mathbf{x}^{(l)}$ by reshaping every N elements of $\mathbf{x}^{(l)}$ as a row of $\mathbf{X}^{(l)}$ in order, while $\mathbf{f}_{q_{\mathrm{O}}}$ and $\mathbf{f}_{q_{\mathrm{F}}}$ satisfy $\mathbf{F}_{q_{\mathrm{O}}} = \mathbf{f}_{q_{\mathrm{O}}}^T \otimes \mathbf{I}_M$ and $\mathbf{F}_{q_{\mathrm{F}}} = \mathbf{f}_{q_{\mathrm{F}}}^T \otimes \mathbf{I}_M$, respectively, as presented in Section 2. Compared with the original form, the computational complexity of $f_\rho(\mathbf{x}^{(l)})$ is reduced from $O\left((U_{\mathrm{O}}Q_{\mathrm{O}} + U_{\mathrm{F}}Q_{\mathrm{F}})M^2N\right)$ to $O\left((U_{\mathrm{O}}Q_{\mathrm{O}} + U_{\mathrm{F}}Q_{\mathrm{F}})MN\right)$ significantly, where U_{O} and U_{F} are the numbers of discrete spatial points in Ω_{O} and Ω_{F}, while Q_{O} and Q_{F} are the numbers of discrete frequency points in Ξ_{O} and Ξ_{F}, respectively.

Algorithm 1 The proposed RCG-CMDPC algorithm for tackling $\tilde{\mathrm{P}}$

Input: Initial point $\mathbf{x}^{(0)} \in \mathrm{M}$.

Output: The optimal solution $\mathbf{x}^{(\mathrm{opt})} \in \mathrm{M}$.

Initialize $l = 1$.
Compute $\mathbf{d}^{(0)} = -\mathrm{grad} f_\rho(\mathbf{x}^{(0)})$, the step size $\zeta^{(0)}$.
Obtain the intermediate variable $\tilde{\mathbf{x}}^{(1)} = \mathbf{x}^{(0)} + \zeta^{(0)}\mathbf{d}^{(0)}$, and update $\mathbf{x}^{(1)} = \mathrm{R}\,(\tilde{\mathbf{x}}^{(0)})$.
Repeat
  Compute the Riemannian gradient $\mathrm{grad} f_\rho(\mathbf{x}^{(l)})$ using Eqs (20) and (21).
  Calculate the Polark-Ribière parameter $\mu^{(l)}$ using Eq. (24).
  Update $\mathrm{Trans}\mathbf{d}^{(l-1)}$ and $\mathbf{d}^{(l)}$ using Eqs (22) and (23).
  Compute the step size $\zeta^{(l)}$ by the Armijo's back-tracking method in Eq. (27).
  Update $\tilde{\mathbf{x}}^{(l+1)} = \mathbf{x}^{(l)} + \zeta^{(l)}\mathbf{d}^{(l)}$.
  Implement retraction and update $\mathbf{x}^{(l+1)} = \mathrm{R}\left(\tilde{\mathbf{x}}^{(l+1)}\right)$ using Eq. (26).
  Let $l = l + 1$.
until $\left| f_\rho(\mathbf{x}^{(l+1)}) - f_\rho(\mathbf{x}^{(l)}) \right| / f_\rho(\mathbf{x}^{(l)}) \le \varepsilon$, then output $\mathbf{x}^{(\mathrm{opt})} = \mathbf{x}^{(l)}$.

In the same manner, we rewrite the Euclidean gradient $\mathrm{Grad} f_\rho(\mathbf{x}^{(l)})$ as the following equivalent form:

$$\begin{aligned} &\mathrm{Grad} f_\rho(\mathbf{X}^{(l)}) = \\ &-2\rho \sum_{\boldsymbol{\sigma}_{\mathrm{O}} \in \Omega_{\mathrm{O}}} \mathbf{F}_{q_{\mathrm{O}}}^H \left(\sum_{q_{\mathrm{O}} \in \Xi_{\mathrm{O}}} \begin{array}{c} p(C - \mathbf{f}_{q_{\mathrm{O}}}^H \mathbf{X}^{(l)H} \mathbf{a}_{\boldsymbol{\sigma}_{\mathrm{O}}, q_{\mathrm{O}}} \mathbf{a}_{\boldsymbol{\sigma}_{\mathrm{O}}, q_{\mathrm{O}}}^H \mathbf{X}^{(l)} \mathbf{f}_{q_{\mathrm{O}}})^{p-1} \\ \left((\mathbf{a}_{\boldsymbol{\sigma}_{\mathrm{O}}, q_{\mathrm{O}}}^H \mathbf{X}^{(l)} \mathbf{f}_{q_{\mathrm{O}}}) \mathbf{a}_{\boldsymbol{\sigma}_{\mathrm{O}}, q_{\mathrm{O}}} \right) \end{array} \right) \\ &+2(1-\rho) \sum_{\boldsymbol{\sigma}_{\mathrm{F}} \in \Omega_{\mathrm{F}}} \mathbf{F}_{q_{\mathrm{F}}}^H \left(\sum_{q_{\mathrm{F}} \in \Xi_{\mathrm{F}}} \begin{array}{c} p\left(\mathbf{f}_{q_{\mathrm{F}}}^H \mathbf{X}^{(l)H} \mathbf{a}_{\boldsymbol{\sigma}_{\mathrm{F}}, q_{\mathrm{F}}} \mathbf{a}_{\boldsymbol{\sigma}_{\mathrm{F}}, q_{\mathrm{F}}}^H \mathbf{X}^{(l)} \mathbf{f}_{q_{\mathrm{F}}} \right)^{p-1} \\ \left((\mathbf{a}_{\boldsymbol{\sigma}_{\mathrm{F}}, q_{\mathrm{F}}}^H \mathbf{X}^{(l)} \mathbf{f}_{q_{\mathrm{F}}}) \mathbf{a}_{\boldsymbol{\sigma}_{\mathrm{F}}, q_{\mathrm{F}}} \right) \end{array} \right). \end{aligned} \tag{29}$$

where the computational complexity is highly decreased from $O\left((U_{\mathrm{O}}Q_{\mathrm{O}} + U_{\mathrm{F}}Q_{\mathrm{F}})(M^2N^2 + M^2N)\right)$ to $O\left((U_{\mathrm{O}}Q_{\mathrm{O}} + U_{\mathrm{F}}Q_{\mathrm{F}})MN + (U_{\mathrm{O}} + U_{\mathrm{F}})M^2N\right)$.

To sum up, we give the computational com-

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plexity of the proposed RCG-CMDPC algorithm in Table 1, while – as customary – disregarding the coefficients of high-order terms.

Table 1 The computational complexity summary of the RCG-CMDPC algorithm

Computation	Complexity
$\mathrm{Grad} f_\rho(\mathbf{x}^{(l)})$	$O\begin{pmatrix}(U_\mathrm{O}Q_\mathrm{O}+U_\mathrm{F}Q_\mathrm{F})MN \\ +(U_\mathrm{O}+U_\mathrm{F})M^2N\end{pmatrix}$
$f_\rho(\mathbf{x}^{(l)})$	$O\left((U_\mathrm{O}Q_\mathrm{O}+U_\mathrm{F}Q_\mathrm{F})MN\right)$
$\mathrm{grad} f_\rho(\mathbf{x}^{(l)})$	$O\left(MN\right)$
$\mu^{(l)}$	$O\left(MN\right)$
$\mathrm{Trans}\mathbf{d}^{(l-1)}$	$O\left(MN\right)$
$\mathbf{d}^{(l)}$	$O\left(MN\right)$
$\zeta^{(l)}$	$O\left(MN\right)$
$\mathbf{x}^{(l+1)}$	$O\left(MN\right)$

4 Numerical examples

In this section, we demonstrate the performance of our proposed RCG-CMDPC algorithm by implementing multiple numerical examples. First, we describe the setup experimental scenario and give the detailed parameters in Table 2. Then, we verify the effectiveness of the RCG-CMDPC algorithm under a typical DPJ scenario, and thereafter, the performance of our proposed algorithm is compared with that of a competing algorithm. Finally, we analyze the effect of the hyperparameters on the algorithm performance and present the Pareto curve of the RCG-CMDPC algorithm.

4.1 Setting of the experimental scenario

In the implemented experiment, we consider a typical scenario, which is consistent with the most advanced DPJ research (Yang *et al*., 2024) for the convenience of subsequent comparison. The area of interest Ω is set as a 100 m × 100 m square, wherein all the opponent equipment and three friendly devices are located. We set the center of Ω_O as the origin, i.e., (0 m, 0 m, 0 m), and the corresponding radius is $r_\mathrm{O}=5\ \mathrm{m}$. The radii of Ω_F^1, Ω_F^2, and Ω_F^3 are all set as $r_\mathrm{F}=2\ \mathrm{m}$, while their centers are located at (–20 m, 10 m, 0 m), (10 m, –20 m, 0 m), and (20 m, 25 m, 0 m), respectively. Within the area of interest Ω, the spatial grids are discretized as 1 m × 1 m. The DPJ system is deployed above Ω at a distance *d*= 2000 m, and the drones are assumed to be distributed randomly within a horizontal circle of radius 30 m, which is the same as in the work of Yang *et al*. (2024). The carrier frequency is set as $f_0=1\ \mathrm{GHz}$, and the bandwidth and the length of the transmitted waveform are *B* = 100 MHz and *N* = 100, respectively. The potential working frequencies of the opponent and friendly devices are Ξ_O = [950 MHz, 1050 MHz] and Ξ_F = [965 MHz, 1015 MHz]. Unless stated otherwise, we set *p* = 30 for our RCG-CMDPC algorithm, and the stopping criterion is $\varepsilon=10^{-5}$. The experiments are conducted on a PC with an i7-10510U CPU and 16 GB RAM. For a clear understanding of our experiments, the above-stated main settings are listed in Table 2.

4.2Verification of the effectiveness

In this subsection, we verify the effectiveness of our proposed RCG-CMDPC algorithm. Meanwhile, we compare the MM algorithm proposed by Yang *et al*. (2024) with our algorithm in terms of DPJ performance and computational efficiency, which is the latest achievement of the wideband waveform design for DPJ. Note that the MM algorithm of Yang *et al*. (2024) cannot be directly applied to our approximation problem $\tilde{\mathrm{P}}$ because the MM algorithm was proposed to tackle the $\mathrm{L_p}$-norm approximation problem with CM continuous phase and the maximum CPS of friendly devices was considered as a constraint therein. In this case, we extend this MM algorithm to tackle our CM discrete-phase constrained problem $\tilde{\mathrm{P}}$, where the objective function $f_\rho(\mathbf{x})$ is majorized as discussed by Yang *et al*. (2024), and the solution with CM and discrete phase constraints at each iteration is obtained as outlined by Bu *et al*. (2021).

Table 2 The main settings of the typical DPJ scenario considered in this work

Symbol	Value	Meaning
Ω	100 m × 100 m	The range of the area of interest

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Ω_{O}	(0 m, 0 m, 0 m)	The center of opponent equipment
Ω_{F}^{1}	(-20 m, 10 m, 0 m)	The center of friendly device #1
Ω_{F}^{2}	(10 m, -20 m, 0 m)	The center of friendly device #2
Ω_{F}^{3}	(20 m, 25 m, 0 m)	The center of friendly device #3
r_{O}	5 m	The radius of opponent region
r_{F}	2 m	The radius of three friendly region
d	2000 m	The distance from DPJ system to the area of interest
f_0	1 GHz	The carrier frequency of transmitted waveform
B	100 MHz	The bandwidth of baseband jamming waveform
N	100	The length of sampled waveform
Ξ_{O}	[950 MHz, 1050 MHz]	The potential working frequency of opponent equipment
Ξ_{F}	[965 MHz, 1015 MHz]	The working frequency of three friendly devices
p	30	The L_p-norm approximation parameter
ε	10^{-5}	The stopping criterion of the RCG-CMDPC algorithm

As shown in Figs 3 and 4, we verify the DPJ effectiveness by the CPS distribution by adopting the MM and RCG-CMDPG algorithms for two cases, where Case 1 is the typical DPJ scenario, and the working frequency of friendly devices is set the same as that of the opponent equipment in Case 2. For these two cases, we set $\rho = 10^{-10}$ in $\tilde{\mathrm{P}}$, and the effect of different V values on the final performance is also verified in these figures. It is seen that the two algorithms achieve the desirable jamming effect, where the red color represents that the corresponding spatial frequency grid has a high CPS level, and the blue color denotes a low CPS level. From the comparison, we find that the RCG-CMDPC algorithm performs better both on the minimum and the average CPSs of the opponent equipment than the MM algorithm, while the maximum CPSs of friendly devices using these two algorithms seem the same. The detailed CPS indicators of these two algorithms are shown in Table 3, where $\mathrm{CPS}_{\mathrm{O}}^{\min}$ and $\mathrm{CPS}_{\mathrm{O}}^{\mathrm{ave}}$ denote the minimum and average CPSs in Ω_{O}, respectively, while $\mathrm{CPS}_{\mathrm{O}}^{\mathrm{ave}}$ and $\mathrm{CPS}_{\mathrm{F}}^{\mathrm{ave}}$ represent the maximum and average CPSs in Ω_{F}, respectively. We see that our proposed algorithm shows better performance on the CPS in Ω_{O}, and the MM algorithm has only a bit lower CPS in Ω_{F}, which can even be seen as the same as that of the RCG-CMDPC algorithm.

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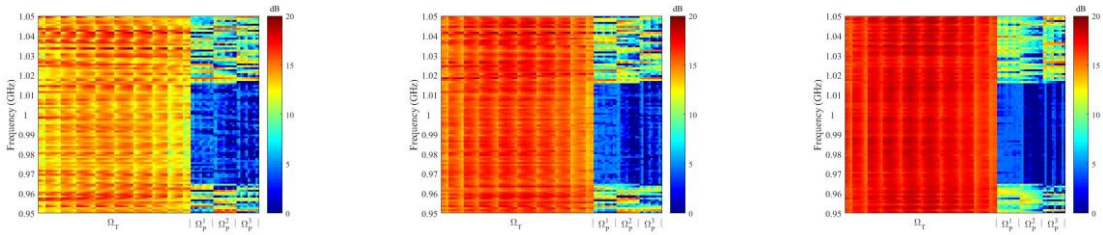
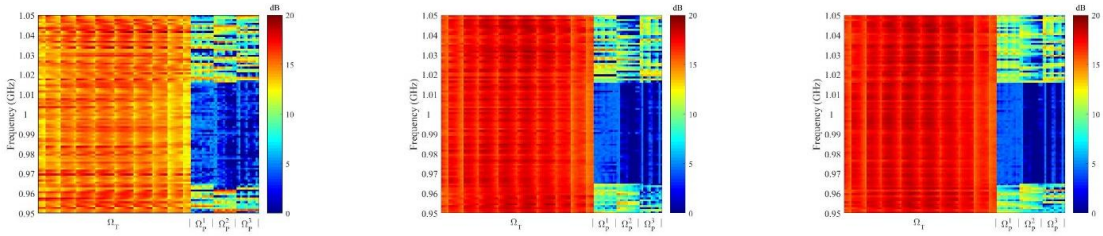


(a) MM algorithm (V = 256; Discrete Phase)

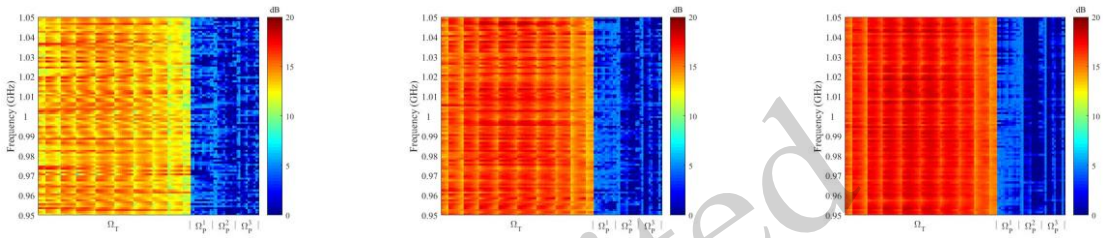
(b) MM algorithm (V = 1024; Discrete Phase)

(c) MM algorithm (V = ∞; Continuous Phase)

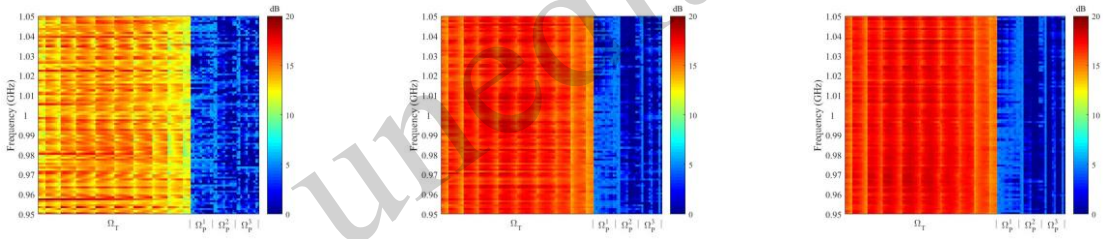


(d) RCG-CMDPC algorithm (V = 256; Discrete Phase) (e) RCG-CMDPC algorithm (V = 1024; Discrete Phase) (f) RCG-CMDPC algorithm (V = ∞; Continuous Phase)

Fig. 3 The CPS distribution in Ω_O and Ω_F with the adoption of different algorithms for Case 1. CPS, combined power spectrum; MM, majorization–minimization; RCG-CMDPC, Riemannian conjugate gradient for CM discrete phase constraints.



(a) MM algorithm (V = 256; Discrete Phase) (b) MM algorithm (V = 1024; Discrete Phase) (c) MM algorithm (V = ∞; Continuous Phase)



(d) RCG-CMDPC algorithm (V = 256; Discrete Phase) (e) RCG-CMDPC algorithm (V = 1024; Discrete Phase) (f) RCG-CMDPC algorithm (V = ∞; Continuous Phase)

Fig. 4 The CPS distribution in Ωo and Ωf with the adoption of different algorithms for Case 2. CPS, combined power spectrum; MM, majorization–minimization; RCG-CMDPC, Riemannian conjugate gradient for CM discrete phase constraints.

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Table 3 The CPS indicators in Ωo and Ω_F for Cases 1 and 2

V	Algorithm	CPS_O^{min} (dB) (Case 1/2)	CPS_F^{max} (dB) (Case 1/2)	CPS_O^{ave} (dB) (Case 1/2)	CPS_F^{ave} (dB) (Case 1/2)
256	MM	8.63/7.87	6.49/6.39	14.37/13.97	3.11/**2.91**
(Discrete Phase)	RCG-CMDPC	**11.20/8.79**	**6.30/6.29**	**15.08/14.24**	**3.02**/3.00
1024	MM	12.92/12.78	5.68/5.70	15.97/15.93	3.04/2.70
(Discrete Phase)	RCG-CMDPC	**14.46/13.64**	**5.02/5.32**	**17.01/16.39**	**2.50/2.55**
∞	MM	14.85/14.01	4.84/5.18	17.23/16.76	**2.31/2.33**
(Continuous Phase)	RCG-CMDPC	**15.03/14.12**	**4.82/5.17**	**17.37/16.81**	2.35/2.77

To further demonstrate the superiority of our proposed RCG-CMDPC algorithm, we give the convergence curves of the MM and RCG-CMDPC algorithms under the typical DPJ scenario, where the variation of $f_\rho(\mathbf{x})$ versus CPU time is given in Fig. 5. We observe that these curves decrease monotonically, which illustrates that both the MM and our proposed RCG-CMDPC algorithm converge well. From the comparison, we see that our proposed RCG-CMDPC algorithm has much better computational efficiency than the MM algorithm, with a decrease of about one order of magnitude in CPU time. The RCG-CMDPC algorithm obtains the corresponding discrete phase waveform efficiently, which makes it possible to generate real-time wideband waveform for DPJ. This superiority lies in the observation that the highest order of computational complexity of our algorithm is M^2N, while that of the MM algorithm is M^3N^3. Moreover, for the same algorithm, it is observed that the final objective decreases with increasing V. The reason is that a higher value of V provides a wider solution space for the approximation problem $\tilde{\mathrm{P}}$, and the discrete phase constraints are equivalent, to be consistent with the continuous phase scenario when $V = \infty$. From the above comparison, we conclude that the proposed algorithm achieves better CPS performance than the previous MM algorithm, and the computational efficiency is highly improved.

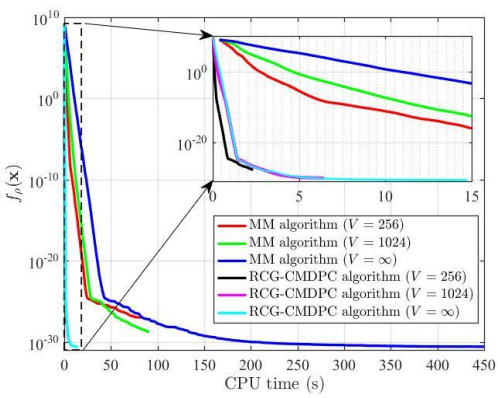

Fig. 5 The convergence curves of the MM and RCG-CMDPC algorithms.

4.3 Performance analysis

To proceed, we analyze the effect of L_p-norm adoption on the performance of our proposed RCG-CMDPC algorithm. As shown in Fig. 6, we implement this experiment when V= 1024, which means that each jammer is equipped with a 10-bit DDS. To investigate the effect of value of p on the CPS performance and convergence of the algorithm, we set p = 10, 20, 30, 40, and 50. From the curves, we see that the minimum CPS in Ω_O, i.e., the blue curve, increases slightly with p increasing, while the maximum CPS in Ω_F tends to decrease as p increases. However, the convergence performance deteriorates severely when p is large, whereby the CPU time grows sharply in the figure. This phenomenon indicates that the CPS performance improvement is accompanied by the sacrifice of convergence and computational performance when adopting the L_p-norm approximation. It is explained by the fact that the L_p-norm with larger p values approximates the original (L_∞-norm) problem better.

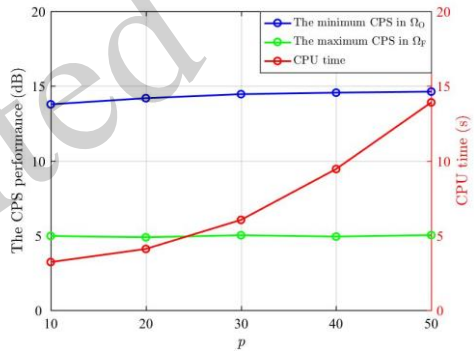

Fig. 6 The effect of value of p on the RCG-CMDPC algorithm performance.

In what follows, we further analyze the effect of V on the convergence of our proposed RCG-CMDPC algorithm. Fig. 7 depicts the variation of the objective function $f_\rho(\mathbf{x})$ versus the iteration index under different V values. From the comparison, it is seen that the final objective decreases with V increasing, where this value is far lower when V is >128. Meanwhile, the iteration index at the convergence highly increases, which means that the higher V value slows down the convergence of our RCG-CMDPC algorithm. It is explained by the fact that the higher V value results in a wider solution space, allowing the algorithm to obtain a better solution but inevitably requiring more iteration steps to the convergence. Specifically, we further give the analysis of the effect of V on the final CPS performance, where the minimum and maximum CPSs in Ω_O and Ω_F are shown by the curves in Fig. 8. These two CPS performances improve significantly when V grows no larger than 256. Moreover, when V

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is >1024, the achieved CPS performances in the discrete phase scenarios tend to be consistent with that in the continuous phase scenario.

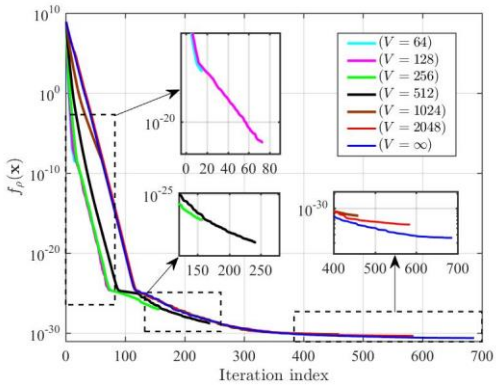

Fig. 7 The variation of $f_\rho(\mathbf{x})$ versus iteration index under different *V* values.

Finally, we illustrate the impact of the parameter ρ on the CPS performances achieved using our proposed RCG-CMDPC algorithm. Specifically, the Pareto curves are obtained under three scenarios, where $V = 256$ and 1024 for the discrete phase scenario, and $V = \infty$ for the continuous phase scenario. Here, we set $\rho \in \{0, \rho_1, \rho_2, ..., \rho_{10}, 1\} \subseteq [0,1]$, where $\rho_i = 10^{-55+5i}, i = 1,2,...,10$. As marked in Fig. 9, when $\rho = 1$ and 0, the obtained wideband waveform only considers Ω_O and Ω_F, respectively. Specifically, the higher the value of ρ, the better is the minimum CPS in Ω_O and the worst is the maximum CPS in Ω_F because the weight of this performance becomes higher in the formulated problem. This phenomenon is consistent with the Pareto theory as expected, which is a classical feature of biobjective Pareto curves.

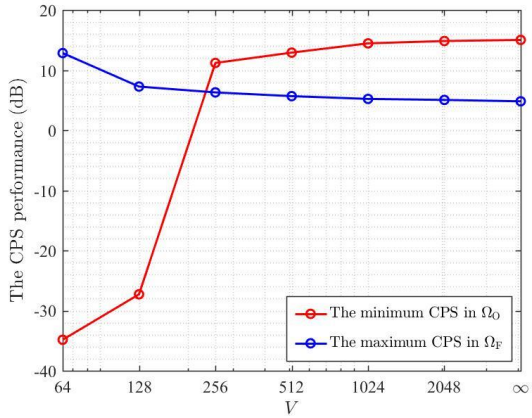

Fig. 8 The effect of V on the CPS performance of the RCG-CMDPC algorithm.

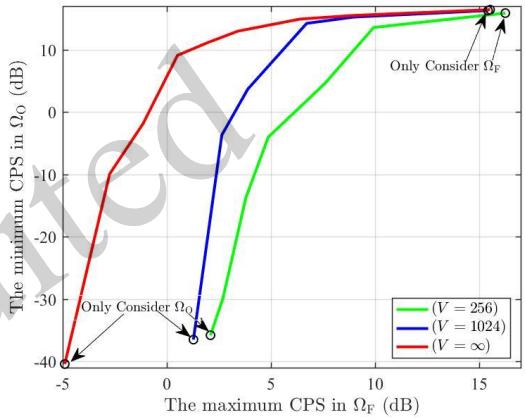

Fig. 9 Performance trade-off between the CPS in Ω_O and Ω_F under different V values. CPS, combined power spectrum.

5 Conclusion

In this paper, we propose an efficient algorithm to design a robust wideband CM discrete phase waveform for DPJ, which aims to provide a robust DPJ effect and low hardware burden on the jammers. The corresponding design model is formulated as a minimax MOP, which is transformed as $\tilde{P}$ using the Lp-norm and Pareto framework. To tackle the challenging problem $\tilde{P}$, we develop an efficient RCG-CMDPC algorithm within the RCG framework, whereby a novel retraction and projection method is provided to let the solution satisfy the CM discrete phase constraints. The proposed RCG-CMDPC algorithm achieves a lower computational complexity than the competing MM algorithm in theory. The numerical examples show that the proposed algorithm has better CPS performance and computational effi-

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ciency than the MM algorithm, which makes it possible to design the real-time wideband CM discrete phase waveform for DPJ. Possible future works may include the robust wideband waveform design against the localization and synchronization errors of distributed UAVs.

Contributors

Qingsong ZHOU and Zhongping YANG designed the research. Jialong Qian, Chao HUANG, Qinxian CHEN, Yibo XU, and Zhengkai WEI processed the data. Qingsong ZHOU and Jialong QIAN drafted the manuscript. Zhongping YANG helped organize the manuscript. Qingsong ZHOU, Jialong QIAN, and Zhongping YANG revised and finalized the paper.

Conflict of interest

All the authors declare that they have no conflict of interest.

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